BEAD SLIDING AND SCHUR INEQUALITIES

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ABSTRACT. We analyze a simple game of beads on a rod and relate it to some classical convex inequalities.

1. INTRODUCTION

We consider distributions (or configurations) of n beads on the real semiaxis $[\mu, \infty)$, where μ is a fixed real number. Any bead in such a distribution is capable of sliding to the right (in the positive direction) but not allowed to slide to the left. We indicate such a distribution of beads by a vector

$$\vec{A} = (A_1, \dots, A_n), \quad \mu \leq A_1 < A_2 < \cdots < A_n,$$

where the coordinates A_i indicate the positions of the beads. The i-th bead is the bead located at A_i. A distribution is called *monotone* if

$$A_1 - \mu \leq A_2 - A_1 \leq \cdots \leq A_n - A_{n-1}. \tag{M}$$

In other words, as we move along the rod, the beads are further and further apart. We denote by $\mathcal{B}_n = \mathcal{B}_n(\mu)$ the collection of monotone distributions of n beads on the semiaxis $[\mu, \infty)$.

We will indicate the elements of $\mathcal{B}_n(\mu)$ using capital letters $\vec{A}$, $\vec{B}$ etc. A configuration $\vec{A} \in \mathcal{B}_n(\mu)$ can be identified with a point in $\mathbb{R}^n$, and the inequalities (M) show that we can indentify $\mathcal{B}_n(\mu)$ with a convex subset of $\mathbb{R}^n$. To a configuration $\vec{A}$ we associate the vector of differences $\vec{a} := \Delta\vec{A}$,

$$\vec{a} = (a_1, \dots, a_n), \quad a_1 := A_1 - \mu, \dots, a_k := A_k - A_{k-1}, \forall k = 2, \dots, n.$$

We have a natural partial order on $\mathcal{B}_n(\mu)$

$$\vec{A} \leq \vec{B} \Longleftrightarrow A_k \leq B_k, \quad \forall k = 1, \dots, n.$$

Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ denote the canonical basis of $\mathbb{R}^n$. Given a bead distribution $\vec{A} \in \mathcal{B}_n(\mu)$ we define a *monotone bead slide* to be a transformation $\vec{A} \mapsto \vec{A}' = \vec{A} + \delta\boldsymbol{e}_k$, where $\delta \geq 0$, $1 \leq k \leq n$ and the distribution $\vec{A}'$ is monotone. Intuitively, this means that we slide to the right by a distance δ the k-th bead of the distribution $\vec{A}$. The monotonicity of the move means that the resulting distribution of beads continues to be monotone.

We define a new partial order relation $\preceq$ on $\mathcal{B}_n(\mu)$ by declaring $\vec{A} \preceq \vec{B}$ if the distribution $\vec{B}$ can be obtained from $\vec{A}$ via a finite sequence of monotone bead slides. When $\vec{A} \preceq \vec{B}$ we say that *we can slide the distribution* $\vec{A}$ to the distribution $\vec{B}$.

The partial order $\preceq$ can be given a very simple geometric interpretation. If we think of $\mathcal{B}_n(\mu)$ as a closed convex set in $\mathbb{R}^n$ and $\vec{A}, \vec{B} \in \mathcal{B}_n(\mu)$, then $\vec{A} \preceq \vec{B}$ if and only if we can travel from $\vec{A}$ to $\vec{B}$, *inside* $\mathcal{B}_n(\mu)$, along a positive zig-zag, i.e., a continuous path consisting of finitely many segments parallel to the coordinate axes and oriented in the positive directions of the axes.

The first goal of this note is to investigate when we can monotonically slide one mononote distribution of beads to another monotone distribution. Clearly, if we can slide $\vec{A}$ to $\vec{B}$, then $\vec{A} \leq \vec{B}$.

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Remark 1.1. The converse implication is true if $n = 1, 2$, but false if $n \geq 3$. Indeed if $n \geq 3$, and $\vec{B} \in \mathcal{B}_n(\mu)$ is an equidistant distribution, i.e.,

$$B_1 - \mu = B_2 - B_1 = \cdots = B_n - B_{n-1}$$

then there is no distribution $\vec{A} \prec \vec{B}$. To see this observe that there is no distribution $\vec{A}$ such that $\vec{B}$ is obtained from $\vec{A}$ by a single admissible bead slide. □

The example presented in the above remark is essentially the only obstruction to sliding one configuration to another. This is the content of the next theorem which is the key technical result of this paper.

Theorem 1.2. *Let $\mu \in \mathbb{R}$, fix $\vec{B} \in \mathcal{B}_n(\mu)$ and set*

$$b_1 := B_1 - \mu, \;\; b_k := B_k - B_{k-1}, \;\; \forall k \geq 2.$$

Then the following statements are equivalent

(a) $b_k > b_{k-2}$, $\forall k \geq 3$.
(b) *If $\vec{A} \in \mathcal{B}_n(\mu)$ satisfies $\vec{A} \leq \vec{B}$, then $\vec{A} \preceq \vec{B}$.*

□

Remark 1.3. The condition (a) signifies that no four consecutive beads of the the distribution $\vec{B}$ are equidistant. □

The second goal of this note is to show that Theorem 1.2 implies the Schur majorization inequalities. We refer to [2, 2.19-20] and [3, Chap. 13] for an in depth discussion of these inequalities and their surprising connections with other ares of mathematics.

Theorem 1.4 (Schur majorization)**.** *Suppose $b_1 \geq \cdots \geq b_n$ is a nonincreasing sequence of real numbers and $g : \mathbb{R} \to \mathbb{R}$ is a C^1, convex function, i.e. , g' is nondecreasing. Then for any sequence $a_1, \dots, a_n$ satisfying*

$$a_1 + \cdots + a_k \geq b_1 + \cdots + b_k, \;\; k = 1, \dots, n-1,$$

and

$$a_1 + \cdots + a_n = b_1 + \cdots + b_n,$$

we have

$$g(a_1) + \cdots + g(a_n) \geq g(b_1) + \cdots + g(b_n). \qquad \square$$

Remark 1.5. The differentiability assumption on g is not needed. We included it only because it leads to more transparent proofs. □

The Schur majorization inequalities are very powerful, yet they are not as widely known. To illustrate their strength we discuss some problems from mathematical competitions that follow immediately from Schur's inequalities.

2. Proof of Theorem 1.2

Define $\lambda_n : \mathcal{B}_n(\mu) \to [0,\infty)$ by setting

$$\lambda_n(\vec{A}) := a_n - a_1 = (a_k - a_{n-1}) + \cdots (a_2 - a_1) + a_1,$$

where we recall that

$$a_1 = A_1 - \mu, \;\; a_k = A_k - A_{k-1}, \;\; k > 1.$$

Clearly $\lambda_n(\vec{A}) = 0$ if and only the beads described by the distribution $\vec{A}$ are equidistant, i.e.,

$$A_n - A_{n-1} = \cdots = A_2 - A_1 = A_1 - \mu.$$

In other words, $\lambda_n(\vec{A})$ measures how far is $\vec{A}$ from an equidistant distribution. We first prove the implication (a) $\Rightarrow$ (b), i.e.,

$$\vec{A} \leq \vec{B} \in \mathcal{B}_n \;\text{ and }\; b_k > b_{k-2}, \; \forall k \geq 3 \Longrightarrow \vec{A} \preceq \vec{B}. \tag{S_n}$$

We argue by induction on n. The cases $n = 1$ and $n = 2$ are trivial. Observe next that the assumption $b_k > b_{k-2}$ for all $k \geq 2$ implies $\lambda(\vec{B}) > 0$. We have the following key estimate.

Lemma 2.1. *If $\vec{A}, \vec{B} \in \mathcal{B}_{n+1}(\mu)$ are such that $\vec{A} \leq \vec{B}$ and $A_{n+1} = B_{n+1}$ then*

$$\lambda_{n+1}(\vec{A}) \geq \frac{1}{n}\lambda_{n+1}(\vec{B}). \tag{2.1}$$

Proof. For $k = 2, \dots, n+1$ we set

$$\alpha_k := a_k - a_{k-1}, \;\; \beta_k := b_k - b_{k-1}.$$

From the equalities

$$\lambda_{n+1}(\vec{A}) = \sum_{k=2}^{n+1} \alpha_k, \;\; \lambda_{n+1}(\vec{B}) = \sum_{k=2}^{n+1} \beta_k, a_k = a_1 + \sum_{i=2}^{k} \alpha_i, \;\; b_k = a_1 + \sum_{i=2}^{k} \beta_i,$$

and

$$(n+1)a_1 + \sum_{k=2}^{n+1}(n-k+1)\alpha_k = A_{n+1} - \mu = B_{n+1} - \mu = (n+1)b_1 + \sum_{k=2}^{n+1}(n-k+1)\beta_k,$$

we conclude that

$$\sum_{k=2}^{n+1}(n-k+1)\alpha_k = (n+1)(b_1 - a_1) + \sum_{k=2}^{n+1}(n-k+1)\beta_k \geq \sum_{k=2}^{n+1}(n-k+1)\beta_k.$$

Since $\alpha_k, \beta_k \geq 0$ we deduce

$$n\lambda_{n+1}(\vec{A}) = n\sum_{k=2}^{n+1}\alpha_k \geq \sum_{k=2}^{n+1}(n-k+1)\alpha_k \geq \sum_{k=2}^{n+1}(n-k+1)\beta_k \geq \sum_{k=2}^{n+2}\beta_k = \lambda_{n+1}(\vec{B}).$$

$\square$

Consider two distributions $\vec{A}, \vec{B} \in \mathcal{B}_{n+1}(\mu)$, $\vec{A} \leq \vec{B}$. We slide the last bead of $\vec{A}$ until it reaches the position of the last bead of $\vec{B}$.

$$\vec{A} \mapsto \vec{A}' := \vec{A} + \big(B_{n+1} - A_{n+1}\big)\boldsymbol{e}_{n+1}$$

Clearly this slide is monotone. This shows that it suffices to prove (S_{n+1}) only in the special case $A_{n+1} = B_{n+1}$. To prove the implication (S_{n+1}) we will rely on the following simple observation.

Lemma 2.2. *Assume that the implication (S_k) holds for every $k \leq n$. If $\vec{A}, \vec{B} \in \mathcal{B}_{n+1}(\mu)$ are two distributions such that $\vec{A} \leq \vec{B}$, and $A_k = B_k$ for some $k \leq n$ then $\vec{A} \preceq \vec{B}$.*

Proof. Note that

$$(A_1, \dots, A_k) \leq (B_1, \dots, B_k) \;\text{ and }\; (A_{k+1}, \dots, A_{n+1}) \leq (B_{k+1}, \dots, B_{n+1}).$$

According to S_k, we can slide the first k-beads of the distribution $\vec{A}$ to the first k beads of the distribution $\vec{B}$. Using S_{n-k+1} we can then slide the last $(n-k+1)$ beads of the distribution $\vec{A}$ to the last $(n-k+1)$ beads of the distribution $\vec{B}$. □

Using the above observations we deduce that the implication S_{n+1} is a consequence of the following result.

Lemma 2.3. *Assume that the implication (S_k) holds for every $k \leq n$. If $\vec{A}, \vec{B} \in \mathcal{B}_{n+1}(\mu)$ are two distributions such that $\vec{A} \leq \vec{B}$ and $A_{n+1} = B_{n+1}$ then we can slide $\vec{A}$ to a configuration $\vec{C} \in \mathcal{B}_{n+1}(\mu)$ that* crosses *$\vec{B}$, i.e.,*

(a) $\vec{C} \leq \vec{B}$,
(b) $C_{n+1} = B_{n+1}$,
(c) $C_k = B_k$ *for some* $k \leq n$.

Proof. Define

$$\mathcal{B}_{n+1}(\vec{B}) := \big\{ \vec{T} \in \mathcal{B}_{n+1}(\mu);\;\; \vec{T} \leq \vec{B},\;\; T_{n+1} = B_{n+1} \big\}.$$

Note that $\vec{A} \in \mathcal{B}_{n+1}(\vec{B})$. We define a $\vec{B}$-move, to be a monotone bead slide on a configuration $\vec{T} \in \mathcal{B}_{n+1}(\vec{B})$ that produces another configuration in $\mathcal{B}_{n+1}(\vec{B})$. We need to prove that by a sequence of $\vec{B}$-moves starting with $\vec{A}$ we can produce a configuration $\vec{C} \in \mathcal{B}_{n+1}(\vec{B})$ that crosses $\vec{B}$. We argue by contradiction so that we will work under the following assumption.

$$\textit{We cannot produce crossing configurations via any sequence of } \vec{B}\textit{-moves starting with } \vec{A}. \tag{†}$$

We will prove that (†) implies the existence of a sequence of configurations $\vec{A}_\nu \in \mathcal{B}_{n+1}(\vec{B})$, $\nu \geq 1$, such that

$$\lim_{\nu \to \infty} \lambda_{n+1}(\vec{A}_\nu) = 0.$$

In view of the assumption $\lambda(\vec{B}) > 0$ this sequence contradicts the inequality (2.1).

Denote by $\mathcal{B}_{n+1}(\vec{A}, \vec{B})$ the set of configurations in $\mathcal{B}_{n+1}(\vec{B})$ that can be obtained from $\vec{A}$ by a sequence of $\vec{B}$-moves. We will produce a real number $\kappa \in (0,1)$ and a map

$$\mathcal{T} : \mathcal{B}_{n+1}(\vec{A}, \vec{B}) \to \mathcal{B}_{n+1}(\vec{A}, \vec{B})$$

such that

$$\lambda\big(\mathcal{T}(\vec{X})\big) \leq \kappa \lambda(\vec{X}),\;\; \forall \vec{X} \in \mathcal{B}_{n+1}(\vec{A}, \vec{B}).$$

The sequence

$$\vec{A}_\nu := \mathcal{T}^\nu(\vec{A}) = \underbrace{\mathcal{T} \circ \cdots \circ \mathcal{T}}_{\nu}(\vec{A})$$

will then yield the sought for contradiction. We begin by constructing maps

$$\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n : \mathcal{B}_{n+1}(\mu) \to \mathcal{B}_{n+1}(\mu)$$

so that for any $k = 1, \dots, n$ and any $\vec{X} \in \mathcal{B}_{n+1}(\mu)$ we have

$$\mathcal{M}_k(\vec{X}) = \Big(X_1, \dots, x_{k-1}, \frac{1}{2}(X_{k-1} + X_{k+1}), X_{k+1}, \dots, X_{n+1} \Big),$$

where for uniformity we set $X_0 = \mu$. In other words, $\mathcal{M}_k(\vec{X})$ is obtained from $\vec{X}$ by sliding the k-th bead of $\vec{X}$ to the midpoint of the interval (X_{k-1}, X_k). In the new configuration the beads $(k-1)$, k and $(k+1)$ are equidistant.

Now define

$$\mathcal{T} : \mathcal{B}_{n+1}(\mu) \to \mathcal{B}_{n+1}(\mu), \quad \mathcal{T} = \mathcal{M}_1 \circ \mathcal{M}_2 \circ \cdots \circ \mathcal{M}_n.$$

Note that

$$\mathcal{M}_n(X_1, \dots, X_{n+1}) = \Big(X_1, \dots, X_{n-1}, \frac{1}{2}(X_{n-1} + X_{n+1}), X_{n+1} \Big).$$

The configuration $\mathcal{M}_{n-1} \circ \mathcal{M}_n(\vec{X})$ differs from $\mathcal{M}_n(X)$ only at the $(n-1)$-th component which is

$$\frac{1}{2} X_{n-2} + \frac{1}{4} X_{n-1} + \frac{1}{4} X_{n+1}.$$

The $(n-k)$-th component of $\mathcal{M}_{n-k} \circ \cdots \circ \mathcal{M}_n(\vec{X})$ is

$$\frac{1}{2} X_{n-k-1} + \frac{1}{4} X_{n-k} + \cdots + \frac{1}{2^{k+1}} X_{n-1} + \frac{1}{2^{k+1}} X_{n+1}.$$

The first component of $\vec{Y} := \mathcal{T}(\vec{X})$ is

$$Y_1 = \frac{1}{2} X_0 + \frac{1}{4} X_1 + \cdots + \frac{1}{2^n} X_{n-1} + \frac{1}{2^n} X_{n+1}.$$

If we set

$$x_1 := X_1 - X_0 = X_1 - \mu, \;\; x_2 := X_2 - X_1, \dots, x_{n+1} := X_{n+1} - X_n,$$

then we deduce

$$\begin{aligned} Y_1 = \frac{1}{2^n} X_{n+1} + \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} X_k &= \frac{1}{2^n} X_{n+1} + \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} \Big(\mu + \sum_{i=1}^{k} x_i \Big) \\ &= \frac{1}{2^n} \Big(\mu + \sum_{i=1}^{n+1} x_i \Big) + (1 - \frac{1}{2^n}) \mu + \sum_{k=1}^{n-1} \frac{1}{2^{k+1}} \sum_{i=1}^{k} x_i \\ = \mu + \frac{1}{2^n} \sum_{i=1}^{n+1} x_i &+ \Big(\sum_{k=1}^{n-1} \frac{1}{2^{k+1}} \Big) x_1 + \Big(\sum_{k=2}^{n-1} \frac{1}{2^{k+1}} \Big) x_2 + \cdots + \frac{1}{2^n} x_{n-1} \\ &= \mu + \frac{1}{2} x_1 + \frac{1}{4} x_2 + \cdots + \frac{1}{2^{n-1}} x_{n-1} + \frac{1}{2^n} x_n + \frac{1}{2^n} x_{n+1}. \end{aligned}$$

Observe that

$$\lambda_{n+1}(X) = x_{n+1} - x_1, \;\; \lambda_{n+1}(Y) = y_{n+1} - y_1 = Y_{n+1} - Y_n - Y_1 + Y_0.$$

We have

$$\begin{aligned} \lambda_{n+1}(Y) &= X_{n+1} - \frac{1}{2}(X_{n+1} + X_{n-1}) - Y_1 + \mu \\ &= \sum_{i=1}^{n+1} x_i - \frac{1}{2} \Big(\sum_{i=1}^{n+1} x_i + \sum_{i=1}^{n-1} x_i \Big) - \Big(\frac{1}{2^n} x_{n+1} + \sum_{k=1}^{n} \frac{1}{2^k} x_k \Big) \\ &= \frac{1}{2}(x_{n+1} + x_n) - \Big(\frac{1}{2^n} x_{n+1} + \sum_{k=1}^{n} \frac{1}{2^k} x_k \Big) \\ &\leq \big(1 - \frac{1}{2^n} \big) x_{n+1} - \sum_{k=1}^{n} \frac{1}{2^k} x_k = \sum_{k=1}^{n} \frac{1}{2^k} (x_{n+1} - x_k) \end{aligned}$$

$$\leq \Big(\sum_{k=1}^{n}\frac{1}{2^k}\Big)(x_{n+1}-x_1) = \big(1-\frac{1}{2^n}\big)\lambda_{n+1}(X).$$

Hence

$$\lambda_{n+1}\big(\mathcal{T}(\vec{X})\big) \leq (1-2^{-n})\lambda_{n+1}(\vec{X}),\ \ \forall \vec{X}\in \mathcal{B}_{n+1}(\mu). \tag{2.2}$$

To conclude the proof it suffices to show that

$$\mathcal{M}_k(\vec{X}) \in \mathcal{B}(\vec{B}),\ \ \forall \vec{X}\in\mathcal{B}(\vec{A},\vec{B}),\ \ k=1,\dots,n. \tag{2.3}$$

Let $\vec{X}=(X_1,\dots,X_{n+1})\in\mathcal{B}(\vec{A},\vec{B})$ and set $\vec{Y}=\mathcal{M}_k(\vec{X})$. Then

$$Y_i = \begin{cases} X_i, & i\neq k\\ \frac{1}{2}(X_{k-1}+X_{k+1}), & i=k.\end{cases}$$

To prove that $\vec{Y}\in\mathcal{B}(\vec{A},\vec{B})$ we have to prove that $Y_k\leq B_k$. If this were not the case, then $Y_k>B_k$. Since $X_k<B_k$, we deduce $(B_k-X_k)<(Y_k-X_k)$. This implies that sliding the k-th bead of $\vec{X}$ by the distance (B_k-X_k) is a monotone slide and it is obviously a $\vec{B}$-move since the resulting configuration $\vec{X}'$ is in $\mathcal{B}_{n+1}(\vec{B})$. Clearly, the configuration $\vec{X}'$ crosses $\vec{B}$ since $X_k'=B_k$. This contradicts the assumption (†) and finishes the proof of Lemma 2.3 and of the implication (a) ⇒ (b). □

To prove the converse implication (b) ⇒ (a) we argue by induction. The cases $n=1,2$ are trivial, while the case $n=3$ follows from Remark 1.1.

For the inductive step suppose $\vec{A}\prec\vec{B}$ in $\mathcal{B}_{n+1}(\mu)$, $\forall\vec{A}<\vec{B}$. Then

$$(B_1,\dots,A_n)\prec(B_1,\dots,B_n)\in\mathcal{B}_n(\mu),\ \ \forall(A_1,\dots,A_n)<(B_1,\dots,B_n),$$

and the inductive assumption implies

$$b_k>b_{k-2},\ \ \forall 2\leq k\leq n.$$

To prove that $b_{n+1}>b_{n-1}$ we argue by contradiction. Suppose $b_{n+1}=b_{n-1}$ so that

$$b_{n+1}=b_n=b_{n-1}.$$

The condition $b_n>b_{n-2}$ implies that $b_{n-2}<b_{n-1}$. Consider the bead distribution $\vec{C}\in\mathcal{B}_{n+1}(\mu)$ described by

$$C_k=B_k,\ \ \forall k\leq n-2,$$
$$C_{n-1}=C_{n-2}+b_{n-2}=B_{n-2}+b_{n-2}<B_{n-1},$$
$$C_n=C_{n-1}+b_{n-1}<B_n,\ \ C_{n+1}=C_n+b_n<B_n.$$

Then $\vec{C}<\vec{B}$, yet arguing as in Remark 1.1 we see that $\vec{C}\not\prec\vec{B}$. This contradiction completes the proof of Theorem 1.2.

3. THE SCHUR INEQUALITIES

The partial order $\preceq$ on $\mathcal{B}_n(\mu)$ is a binary relation and thus can be identified with a subset of $\mathcal{B}_n(\mu)\times\mathcal{B}_n(\mu)$. We denote by $\preceq_t$ its (topological) closure in $\mathcal{B}_n(\mu)\times\mathcal{B}_n(\mu)$.

Corollary 3.1. *The binary relation $\preceq_t$ is a partial order relation. More precisely*

$$\vec{A}\preceq_t\vec{B}\Longleftrightarrow\vec{A}\leq\vec{B}.$$

Proof. Clearly $\vec{A} \preceq_t \vec{B} \Longrightarrow \vec{A} \leq \vec{B}$. Conversely, suppose $\vec{A} \leq \vec{B}$. For every $\varepsilon > 0$ we define

$$\vec{B}(\varepsilon) = \big(B_1(\varepsilon), \dots, B_n(\varepsilon) \big),$$

where $B_k(\varepsilon) = 2^k\varepsilon$. Then

$$B_{k+1}(\varepsilon) - B_k(\varepsilon) = b_{k+1} + 2^k\varepsilon > b_k + 2^{k-1}\varepsilon = B_k(\varepsilon) - B_{k-1}(\varepsilon).$$

Theorem 1.2 implies that $\vec{A} \prec \vec{B}(\varepsilon)$. Letting $\varepsilon \to 0$ we deduce $\vec{A} \preceq_t \vec{B}$. □

The above corollary can be used to produce various interesting inequalities. For simplicity we set $\mathcal{B}_n := \mathcal{B}_n(0)$. The bead distributions in $\mathcal{B}_n$ are described by nondecreasing strings of nonnegative numbers

$$\vec{a} = (a_1, \dots, a_n),\ \ 0 \leq a_1 \leq \dots \leq a_n$$

To such a vector we associate the monotone bead distribution

$$\vec{A} = (A_1, \dots, A_n),\ \ A_k = a_1 + \ldots + a_k.$$

The condition $\vec{A} \leq \vec{B}$ in $\mathcal{B}_n$ can then be rewritten as

$$a_1 + \dots + a_k \leq b_1 + \dots + b_k,\ \ \forall k = 1, \dots, n.$$

In this notation, a monotone bead slide is a transformation of the form

$$(a_1, \dots, a_k, a_{k+1}, \dots, a_n) \longmapsto (a_1, \dots, a_k + \delta, a_{k+1} - \delta, \dots, a_n),\ \ 2\delta \leq a_{k+1} - a_k. \tag{3.1}$$

Suppose $f : [0, \infty) \to [0, \infty)$ is a nondecreasing C^1 function. We then get a map $\mathcal{T}_f : \mathcal{B}_n \to \mathcal{B}_n$,

$$(a_1, a_1 + a_2, \dots, a_1 + \dots + a_n) \mapsto \big(f(a_1), f(a_1) + f(a_2), \dots, f(a_1) + \dots + f(a_n) \big).$$

Theorem 3.2. *Suppose $f : [0, \infty) \to [0, \infty)$ is C^1 and nondecreasing. Then the following statements are equivalent.*

(a) *The induced map $\mathcal{T}_f : \mathcal{B}_n \to \mathcal{B}_n$ preserves the order relation $\leq$.*
(b) *The function f is concave, i.e., the derivative f' is nonincreasing.*

Proof. In view of Corollary 3.1 and the continuity of f we deduce that $\mathcal{T}_f$ preserves the order $\leq$ if and only if $\mathcal{T}_f(\vec{A}) \leq \mathcal{T}_f(\vec{B})$ whenever $\vec{B}$ is obtained from $\vec{A}$ via a single monotone bead slide. Using (3.1) we see that this means that for any $0 \leq x \leq y$, $0 \leq \delta \leq \frac{1}{2}(y - x)$ we have

$$f(x + \delta) \geq f(x),\ \ f(x + \delta) + f(y - \delta) \geq f(x) + f(y).$$

The first inequality follows from the fact that f is nondecreasing. The second inequality can be rephrased as

$$\int_x^{x+\delta} f'(t)dt = f(x + \delta) - f(x) \geq f(y) - f(y - \delta) = \int_{y-\delta}^{y} f'(s)ds,$$

for any $x, y, \delta \geq 0$ such that $x \leq x + \delta \leq y - \delta \leq y$. This clearly happens if and only if f' is nonincreasing. □

Remark 3.3. In the above result we can drop the C^1 assumption on f, but the last step in the proof requires a slightly longer and less transparent argument. □

Corollary 3.4. *Suppose $f:[\mu,\infty)\to\mathbb{R}$ is C^1, nondecreasing and concave, and $(y_i)_{1\leq i\leq n}$ is a nondecreasing sequence of real numbers $\mu\leq y_1\leq\cdots\leq y_n$. Then for any numbers $x_1,\dots,x_n\in[\mu,\infty)$ such that*

$$x_1+\cdots+x_k\leq y_1+\cdots+y_k,\;\;\forall k=1,\dots,n$$

we have

$$f(x_1)+\cdots+f(x_n)\leq f(y_1)+\cdots+f(y_n). \tag{3.2}$$

Proof. Denote by (x_k') the increasing rearrangement of the numbers $x_1,\dots,x_n$. Then

$$x_1'+\cdots+x_k'\leq x_1+\cdots+x_k\leq y_1+\cdots+y_k, \forall k=1,\dots,n,$$
$$f(x_1')+\cdots+f(x_n')=f(x_1)+\cdots+f(x_n),$$

so it suffices to prove (2.1) in the special case when the sequence (x_k) is nondecreasing. Define

$$a_k:=x_k-\mu,\;\; b_k:=y_k-\mu,\;\; 1\leq k\leq n,$$
$$A_k=a_1+\cdots+a_k,\;\; B_k=b_1+\cdots+b_k,\;\; 1\leq k\leq n,$$
$$g:[0,\infty)\to[0,\infty),\;\; g(t)=f(t+\mu)-f(\mu).$$

Then $(A_1,\dots,A_n)\leq(B_1,\dots,B_n)\in\mathcal{B}_n$, and the function g is C^1, nondecreasing and concave. It follows that the induced map $\mathcal{T}_g:\mathcal{B}_n\to\mathcal{B}_n$ is order preserving. In particular, we conclude that

$$g(a_1)+\cdots+g(a_n)\leq g(b_1)+\cdots+g(b_n).$$

This clearly implies (3.2). □

Example 3.5. Here is a simple application of the above inequalities to a Romanian Olympiad problem, 1977. Suppose a,b,c,d are nonnegative numbers such that

$$a\leq 1,\;\; a+b\leq 5,\;\; a+b+c\leq 14,\;\; a+b+c+d\leq 30.$$

Then $\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\leq 10$. This follows from Corollary 3.4 applied to the concave increasing function $f(t)=\sqrt{t}$, $t\geq 0$, $n=4$ and $b_k=k^2$, $k=1,2,3,4$. For a different proof we refer to [1, p.178]. □

Corollary 3.6. *Suppose $f:\mathbb{R}\to\mathbb{R}$ is a C^1, concave function and $y_1\leq\cdots\leq y_n$. Then for any sequence $x_1,\dots,x_n$ such that*

$$x_1+\cdots+x_k\leq y_1+\cdots+y_k,\;\;\forall k=1,\dots,n-1,$$

and

$$x_1+\cdots+x_n=y_1+\cdots+y_n \tag{3.3}$$

we have

$$f(x_1)+\cdots+f(x_n)\leq f(y_1)+\cdots+f(y_n). \tag{3.4}$$

Proof. Choose $L>\max\{x_i,y_j;\;\; 1\leq i,j\leq n\}$ and define

$$g:\mathbb{R}\to\mathbb{R},\;\; g(t)=\begin{cases} f(t)-f'(L)t, & t\leq L\\ f(L)-f'(L)L, & t>L.\end{cases}$$

Then g is C^1, nondecreasing and concave and Corollary 3.4 implies that

$$f(x_1)+\cdots+f(x_n)-f'(L)\sum_{k=1}^n x_k\leq f(y_1)+\cdots+f(y_n)-f'(L)\sum_{k=1}^n y_k.$$

The inequality (3.4) now follows by invoking the equality (3.3). ∎

Corollary 3.6 implies the Schur majorization inequalities in Theorem 1.4. Indeed, it suffices to use Corollary 3.6 with the sequences $x_k = -a_k$, $y_j = -b_j$ and $f(t) = -g(-t)$. The Schur majorization inequalities can be strengthened a bit. More precisely, we have the following convex version of Corollary 3.4.

Corollary 3.7. *Suppose $(b_n)_{n\geq 1}$ is a nonincreasing sequence of real numbers and $g : \mathbb{R} \to \mathbb{R}$ is a* nondecreasing *C^1, convex function, i.e. , g' is nondecreasing. Then for any sequence of real numbers $(a_n)_{n\geq 1}$ satisfying*

$$\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} b_k, \;\; \forall n \geq 1,$$

we have

$$\sum_{k=1}^{n} g(a_k) \geq \sum_{k=1}^{n} g(b_k), \;\; \forall n \geq 1. \tag{3.5}$$

Proof. Fix $n \geq 1$. If $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k$, then (3.5) follows from Theorem 1.4. If

$$\sum_{k=1}^{n} a_k > \sum_{k=1}^{n} b_k,$$

then let $a'_n < a_n$ be such that

$$a_1 + \ldots + a_{n-1} + a'_n = b_1 + \cdots + b_n.$$

Since g is nondecreasing we deduce

$$g(a_1) + \cdots + g(a_{n-1}) + g(a_n) \geq g(a_1) + \cdots + g(a_{n-1}) + g(a'_n) \geq g(b_1) + \cdots + g(b_n).$$

∎

Example 3.8. As a simple application of the the Schur inequalities (3.5) we consider the following US Olympiad problem, 1995. Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that

$$\sum_{k=1}^{n} a_k \geq \sqrt{n}, \;\; \forall n \geq 1. \tag{3.6}$$

Then

$$\sum_{k=1}^{n} a_k^2 \geq \frac{1}{4}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right). \tag{3.7}$$

Consider the sequence $b_n := \sqrt{n} - \sqrt{n-1}$, $n \geq 1$. The sequence $(b_n)_{\geq 1}$ is decreasing since the function $h(x) = \sqrt{x}$, $x \geq 0$ is strictly concave. The convex function $g(t) = t^2$ is nondecreasing for $t \geq 0$. The inequality (3.6) shows that the assumptions of Schur's inequalities are satisfies and we deduce

$$\sum_{k=1}^{n} a_k^2 \geq \sum_{k=1}^{n} b_k^2, \;\; \forall n \geq 1.$$

To obtain the inequality (3.7) it suffices to observe that

$$b_k = \frac{1}{\sqrt{k} + \sqrt{k-1}} > \frac{1}{2\sqrt{k}}.$$

For a different proof of (3.7) we refer to [1, p. 180]. ∎

Corollary 3.9. *Suppose that* $(x_n)_{n\geq 1}$ *and* $(y_n)_{n\geq 1}$ *are two sequences of positive real numbers such that*

$$y_1 \geq y_2 \geq \cdots \geq y_n \geq \cdots, \tag{3.8a}$$

$$\prod_{k=1}^{n} x_k \geq \prod_{k=1}^{n} y_k, \quad \forall n \geq 1. \tag{3.8b}$$

Then

$$x_1 + \cdots + x_n \geq y_1 + \cdots + y_n, \quad \forall n \geq 1. \tag{3.9}$$

Proof. The inequalities (3.9) follow from the Schur inequalities applied to the sequences

$$a_n = \log x_n, \quad b_n = \log y_n$$

and the convex increasing function $g(t) = e^t$. □

Example 3.10. Corollary 3.9 has a special case a problem proposed by France at the 20th International Mathematical Olympiad 1978. Suppose that $\phi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is an injective function from the set of positive integers to itself. Then

$$\sum_{k=1}^{n} \frac{\phi(k)}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}, \quad \forall n \geq 1. \tag{3.10}$$

Consider the sequences

$$x_n = \frac{\phi(n)}{n^2}, \quad y_n = \frac{1}{n}, \quad n \geq 1.$$

The sequence $(y_n)_{n\geq 1}$ is obviously decreasing so (3.8a) is satisfied. Note that since ϕ is injective we conclude that

$$\prod_{k=1}^{n} \phi(k) \geq n! \ \text{ so that } \prod_{k=1}^{n} \frac{\phi(k)}{k^2} \geq \frac{1}{n!} = \prod_{k=1}^{n} y_k.$$

This shows that (3.8b) is satisfied. The inequalities (3.10) now follow from Corollary 3.9. For a different proof of (3.10) we refer to [1, p. 180-181]. □

References

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