CRITICAL POINTS OF MULTIDIMENSIONAL RANDOM FOURIER SERIES: CENTRAL LIMITS

LIVIU I. NICOLAESCU

Abstract. We investigate certain families $X^\hbar$, $0 < \hbar \ll 1$, of Gaussian random smooth functions on the m-dimensional torus $\mathbb{T}^m_\hbar := \mathbb{R}^m/(\hbar^{-1}\mathbb{Z})^m$. We show that for any cube $B \subset \mathbb{R}^m$ of size $< 1/2$ and centered at the origin, the number of critical points of $X^\hbar$ in the region $\hbar^{-1}B/(\hbar^{-1}\mathbb{Z})^m \subset \mathbb{T}^m_\hbar$ has mean $\sim c_1\hbar^{-m}$, variance $\sim c_2\hbar^{-m/2}$, $c_1, c_2 > 0$, and satisfies a central limit theorem as $\hbar \searrow 0$.

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Notation

- We set
$$\mathbb{N} := \big\{ n \in \mathbb{Z};\;\; n > 0 \big\},\;\; \mathbb{N}_0 := \big\{ n \in \mathbb{Z};\;\; n \geq 0 \big\}.$$
- $\mathbf{1}_A$ denotes the characteristic function of a subset A of a set S,
$$\mathbf{1}_A : S \to \{0,1\},\;\; \mathbf{1}_A(a) = \begin{cases} 1, & a \in A, \\ 0, & a \in S \setminus A. \end{cases}$$

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- For a topological space X we denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X.
- We will write $N \sim \mathcal{N}(m, v)$ to indicate that N is a normal random variable with mean m and variance v.
- For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ we set

$$|\boldsymbol{x}|_\infty := \max_{1 \leq j \leq m} |x_j|, \;\; (\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^m x_j y_j, \;\; |\boldsymbol{x}| := \sqrt{(\boldsymbol{x}, \boldsymbol{x})}.$$

- We denote by $\mathbb{A}^m$ the *affine lattice*

$$\mathbb{A}^m = \left(\frac{1}{2} + \mathbb{Z}\right)^m.$$

- For any matrix A, we denote by $A^\top$ its transpose, and by $\|A\|$ its norm

$$\|A\| = \sup_{|\boldsymbol{x}|=1} |A\boldsymbol{x}|.$$

- We denote by $\mathbb{1}_m$ the identity operator $\mathbb{R}^m \to \mathbb{R}^m$.
- For any Borel subset $B \subset \mathbb{R}^m$ we denote by $|B|$ its Lebesgue measure.
- We denote by $\boldsymbol{\gamma}$ the canonical Gaussian measure on $\mathbb{R}$

$$\boldsymbol{\gamma}(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx,$$

  and by $\boldsymbol{\Gamma}$ the canonical Gaussian measure on $\mathbb{R}^m$

$$\boldsymbol{\Gamma}(d\boldsymbol{x}) = (2\pi)^{-\frac{m}{2}} e^{-\frac{|x|^2}{2}} d\boldsymbol{x},$$

- If C is a symmetric, nonnegative definite $m \times m$ matrix, we write $N \sim \mathcal{N}(0, C)$ to indicate that N is an $\mathbb{R}^m$-valued Gaussian random vector with mean 0 and covariance form C.
- If $f : \mathbb{R}^m \to \mathbb{R}$ is a twice differentiable function, and $\boldsymbol{x} \in \mathbb{R}^m$, then we denote by $\nabla^2 f(\boldsymbol{x})$ its *Hessian*, viewed as a symmetric operator $\mathbb{R}^m \to \mathbb{R}^m$.

1. The main results

1.1. **The problem.** We begin by recalling the setup in [23]. For any $\hbar > 0$ we denote by $\mathbb{T}^m_\hbar$ m-dimensional torus $\mathbb{R}^m/\mathbb{Z}^m$ with angular coordinates $\theta_1, \dots, \theta_m \in \mathbb{R}/\mathbb{Z}$ equipped with the flat metric

$$g_\hbar := \sum_{j=1}^m \hbar^{-2} (d\theta_j)^2.$$

For a measurable subset $S \subset \mathbb{T}^m$ we denote by $\mathrm{vol}_\hbar(S)$ its volume with respect to the metric $g_\hbar$, and we set $\mathrm{vol} := \mathrm{vol}_\hbar|_{\hbar=1}$. Hence

$$\mathrm{vol}_\hbar(\mathbb{T}^m) = \hbar^{-m}\,\mathrm{vol}(\mathbb{T}^m) = \hbar^{-m}.$$

The eigenvalues of the corresponding Laplacian $\Delta_\hbar = -\hbar^2 \sum_{k=1}^m \partial^2_{\theta_k}$ are

$$\lambda(\boldsymbol{k}, \hbar) = \hbar^2 \lambda(\boldsymbol{k}), \;\; \lambda(\boldsymbol{k}) := \left| 2\pi \boldsymbol{k} \right|^2, \;\; \boldsymbol{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m.$$

For $\boldsymbol{\theta} = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m$ and $\boldsymbol{k} \in \mathbb{Z}^m$ we set

$$\langle \boldsymbol{k}, \boldsymbol{\theta} \rangle := \sum_j k_j \theta_j.$$

Denote by $\prec$ the lexicographic order on $\mathbb{R}^m$. An *orthonormal* basis of $L^2(\mathbb{T}^m_\hbar)$ is given by the functions $(\psi^\hbar_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^m}$, where

$$\psi^\hbar_{\boldsymbol{k}}(\boldsymbol{\theta}) = \hbar^{\frac{m}{2}}\psi_{\boldsymbol{k}}(\boldsymbol{\theta}),\;\; \psi_{\boldsymbol{k}}(\boldsymbol{\theta}) := \begin{cases} 1, & \boldsymbol{k}=0\\ \sqrt{2}\sin 2\pi\langle\boldsymbol{k},\boldsymbol{\theta}\rangle, & \boldsymbol{k}\succ\vec{0},\\ \sqrt{2}\cos 2\pi\langle\boldsymbol{k},\boldsymbol{\theta}\rangle, & \boldsymbol{k}\prec\vec{0}.\end{cases}$$

Fix a nonnegative, even Schwartz function $w\in\mathcal{S}(\mathbb{R})$, set $w_\hbar(t)=w(\hbar t)$ so that

$$w\big(\sqrt{\lambda(\boldsymbol{k},\hbar)}\,\big) = w_\hbar\big(\sqrt{\lambda(\boldsymbol{k})}\,\big).$$

Consider the random function given by the random Fourier series

$$\begin{split} X^\hbar(\boldsymbol{\theta}) &= \sum_{\boldsymbol{k}\in\mathbb{Z}^m} w_\hbar\big(\sqrt{\lambda(\boldsymbol{k},\hbar)}\,\big)^{\frac12} N_{\boldsymbol{k}}\psi^\hbar_{\boldsymbol{k}}(\boldsymbol{\theta}) = \hbar^{\frac{m}{2}}\sum_{\boldsymbol{k}\in\mathbb{Z}^m} w_\hbar\big(\sqrt{\lambda(\boldsymbol{k})}\,\big)^{\frac12} N_{\boldsymbol{k}}\psi_{\boldsymbol{k}}(\boldsymbol{\theta})\\ &= \hbar^{\frac{m}{2}}\sum_{\boldsymbol{k}\in\mathbb{Z}^m} w\big(2\pi\hbar|\boldsymbol{k}|\big)^{\frac12}\,\big(A_{\boldsymbol{k}}\cos(2\pi\langle\boldsymbol{k},\boldsymbol{\theta}\rangle)+B_{\boldsymbol{k}}\sin(2\pi\langle\boldsymbol{k},\boldsymbol{\theta}\rangle)\big), \end{split}\tag{1.1}$$

where the coefficients $A_{\boldsymbol{k}}, B_{\boldsymbol{k}}, N_{\boldsymbol{k}},\ \boldsymbol{k}\in\mathbb{Z}^m$, are independent standard normal random variables.

Note that if $w\equiv 1$ in a neighborhood of 1, then the random function $\hbar^{-m/2}X^\hbar$ converges to a Gaussian white-noise on $\mathbb{T}^m$ and, extrapolating, we can think of the $\hbar\to 0$ limits in this paper as white-noise limits.

The random function $X^\hbar(\boldsymbol{\theta})$ is. a.s. smooth and Morse. For any Borel set $\mathcal{B}\subset\mathbb{T}^m$ we denote by $\boldsymbol{Z}(X^\hbar,\mathcal{B})$ the number of critical points of $X^\hbar$ in $\mathcal{B}$. In [23] we have shown that there exist constants $C=C_m(w)>0$, $S=S_m(w)\geq 0$ such that, for any open set $\mathcal{O}\subset\mathbb{T}^m$,

$$\mathbb{E}\big[\,\boldsymbol{Z}(X^\hbar,\mathcal{O})\,\big]\sim C_m(w)\hbar^{-m}\operatorname{vol}(\mathcal{O})\;\text{ as }\hbar\to 0.$$

$$\boldsymbol{var}\big[\,\boldsymbol{Z}(X^\hbar,\mathcal{O})\,\big]\sim S_m(w)\hbar^{-m}\operatorname{vol}(\mathcal{O})\;\text{ as }\hbar\to 0.$$

In [23] we described the constants $C_m(w)$ and $S_m(w)$ explicitly as certain rather complicated Gaussian integrals, and we conjectured that $S_m(w)$ is actually strictly positive.

In this paper we will show that indeed $S_m(w)>0$, and we will prove a central limit theorem stating that, as $\hbar\to 0$, the random variables

$$\zeta_\hbar(\mathcal{O}) := \left(\frac{\hbar}{\operatorname{vol}(\mathcal{O})}\right)^{\frac{m}{2}}\Big(\boldsymbol{Z}(X^\hbar,\mathcal{O})-\mathbb{E}\big[\,\boldsymbol{Z}(X^\hbar,\mathcal{O})\,\big]\Big)$$

converge in law to a *nondegenerate* normal random variable $\sim\mathcal{N}\big(0,S_m(w)\big)$. Our approach relies on abstract central limit results of the type pioneered by Breuer and Major [7]. This requires placing the the problem within a Gaussian Hilbert space context. To achieve this we imitate the strategy employed by Azaïs and León [5] in a related 1-dimensional problem.

1.2. **The Wiener chaos setup.** Let $\boldsymbol{x}=(x_1,\dots,x_m)$ denote the standard Euclidean coordinates on $\mathbb{R}^m$. For $\boldsymbol{p}_0\in\mathbb{R}^m$ and $R>0$ we set

$$\widehat{B}_R(\boldsymbol{p}_0) := \Big\{\boldsymbol{x}\in\mathbb{R}^m;\;\; |\boldsymbol{x}-\boldsymbol{p}_0|_\infty\leq\frac{R}{2}\Big\},\;\; \widehat{B}_R=\widehat{B}_R(0)=\Big[-\frac{R}{2},\frac{R}{2}\Big]^m.$$

For $r\in(0,1)$ and denote by B_r the image of the cube $\widehat{B}_r$ in the quotient $\mathbb{R}^m/\mathbb{T}^m$. Thus, B_r is a cube on the torus centered at 0. We identify the tangent space $T_0\mathbb{T}^m_\hbar$ with $\mathbb{R}^m$ and we denote by $\exp_\hbar$ the exponential map $\exp_\hbar:\mathbb{R}^m\to\mathbb{T}^m_\hbar$ defined by the metric $g_\hbar$. In the coordinates $\boldsymbol{x}$ on $\mathbb{R}^m$ and $\boldsymbol{\theta}$ on $\mathbb{T}^m$, this map is decribed by $\boldsymbol{\theta}=\hbar\boldsymbol{x} \bmod \mathbb{Z}^m$. Using this map we obtain by pullback a $(\hbar^{-1}\mathbb{Z})^m$-periodic random function on $\mathbb{R}^m$,

$$Y^\hbar(\boldsymbol{x}) := (\exp_\hbar^* X^\hbar)(\boldsymbol{x}) = \hbar^{m/2}\sum_{\boldsymbol{k}\in\mathbb{Z}^m} w\big(2\pi\hbar|\boldsymbol{k}|\big)^{\frac{1}{2}}\,\big(A_{\boldsymbol{k}}\cos(2\pi\hbar\langle\boldsymbol{k},\boldsymbol{x}\rangle)+B_{\boldsymbol{k}}\sin(2\pi\hbar\langle\boldsymbol{k},\boldsymbol{x}\rangle)\big).$$

We denote by $\boldsymbol{Z}(Y^\hbar,\mathcal{B})$ the number of critical points of $Y^\hbar$ in the Borel set $\mathcal{B}\subset\mathbb{R}^m$. Note that

$$\boldsymbol{Z}\big(\,X^\hbar, B_r\big)=\boldsymbol{Z}\big(\,Y^\hbar,\widehat{B}_{\hbar^{-1}r}\,\big). \tag{1.2}$$

To investigate $\boldsymbol{Z}\big(\,Y^\hbar,\widehat{B}_{\hbar^{-1}\ell_0}\,\big)$ it is convenient to give an alternate description to the random function $Y^\hbar$.

A simple computation shows that the covariance kernel of $Y^\hbar$ is

$$\begin{aligned}\mathcal{K}^\hbar(\boldsymbol{x},\boldsymbol{y}) &= \hbar^m\sum_{\boldsymbol{k}\in\mathbb{Z}^m} w_\hbar(2\pi\hbar|\boldsymbol{k}|)\cos 2\pi\hbar\langle\boldsymbol{k},\boldsymbol{y}-\boldsymbol{x}\rangle\\ &= \hbar^m\sum_{\boldsymbol{k}\in\mathbb{Z}^m} w(2\pi\hbar|\boldsymbol{k}|)\exp\big(-2\pi\hbar\boldsymbol{i}\langle\boldsymbol{k},\boldsymbol{\tau}\rangle\,\big).\end{aligned} \tag{1.3}$$

Define

$$\phi:\mathbb{R}^m\to\mathbb{C},\;\;\phi(\boldsymbol{\xi}):=e^{-\boldsymbol{i}\langle\boldsymbol{\xi},\boldsymbol{\tau}\rangle}w(|\boldsymbol{\xi}|).$$

Using the Poisson formula [14, §7.2] we deduce that for any $a>0$ we have

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^m}\phi\left(\frac{2\pi}{a}\boldsymbol{k}\right)=\Big(\frac{a}{2\pi}\Big)^m\sum_{\boldsymbol{\nu}\in\mathbb{Z}^m}\widehat{\phi}(a\boldsymbol{\nu}),$$

where for any $u=u(\boldsymbol{\xi})\in\mathcal{S}(\mathbb{R}^m)$ we denote by $\widehat{u}(\boldsymbol{x})$ its Fourier transform

$$\widehat{u}(\boldsymbol{x})=\int_{\mathbb{R}^m}e^{-\boldsymbol{i}\langle\boldsymbol{\xi},\boldsymbol{x}\rangle}u(\boldsymbol{\xi})|d\boldsymbol{\xi}|. \tag{1.4}$$

If we let $a=\hbar^{-1}$, then we deduce

$$\mathcal{K}^\hbar(\boldsymbol{x},\boldsymbol{y})=\frac{1}{(2\pi)^m}\sum_{\boldsymbol{\nu}\in\mathbb{Z}^m}\widehat{\phi}\left(\,\hbar^{-1}\boldsymbol{\nu}\,\right).$$

Define $V:\mathbb{R}^m\to\mathbb{R}$ by

$$V(\boldsymbol{x}):=\frac{1}{(2\pi)^m}\widehat{w}(\boldsymbol{x})=\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}e^{-\boldsymbol{i}\langle\boldsymbol{\xi},\boldsymbol{x}\rangle}w(|\boldsymbol{\xi}|)\,|d\boldsymbol{x}|. \tag{1.5}$$

Then

$$\widehat{\phi}(\boldsymbol{x})=V\Big(\,\boldsymbol{x}+\frac{1}{\hbar}\boldsymbol{\tau}\,\Big)=V\Big(\,\boldsymbol{x}+\frac{1}{\hbar}(\boldsymbol{\varphi}-\boldsymbol{\theta})\,\Big).$$

Hence, if we set $\boldsymbol{z}:=\boldsymbol{y}-\boldsymbol{x}$, we deduce

$$\mathcal{K}^\hbar(\boldsymbol{x},\boldsymbol{y})=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^m}V\left(\frac{1}{\hbar}\boldsymbol{z}+\frac{1}{\hbar}\boldsymbol{\nu}\right).$$

We set

$$V^\hbar(\boldsymbol{z}):=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^m}V\left(\boldsymbol{z}+\frac{1}{\hbar}\boldsymbol{\nu}\right). \tag{1.6}$$

The function $V^\hbar$ is $(\hbar^{-1}\mathbb{Z})^m$-periodic and we deduce that

$$\mathcal{K}^\hbar(\boldsymbol{x},\boldsymbol{y})=V^\hbar\left(\frac{1}{\hbar}\boldsymbol{z}\right),\;\;\boldsymbol{z}:=\boldsymbol{y}-\boldsymbol{x}. \tag{1.7}$$

The region $\widehat{B}_{1/\hbar}$ is a fundamental domain for the action of the lattice $(\hbar^{-1}\mathbb{Z})^m$ on $\mathbb{R}^m$. From the special form (1.6) of $V^\hbar$ and the fact that V is a Schwartz function we deduce that for any positive integers k, N we have

$$\|V^\hbar - V\|_{C^k(\widehat{B}_{1/\hbar})} = O(\hbar^N) \;\text{ as } \hbar \searrow 0. \tag{1.8}$$

From (1.3) we deduce that $Y^\hbar$ is a stationary Gaussian random function on $\mathbb{R}^m$ and its spectral measure is

$$\mu^\hbar(d\boldsymbol{\xi}) = \frac{1}{(2\pi)^m}\sum_{\boldsymbol{\nu}\in(2\pi\hbar\mathbb{Z})^m}(2\pi\hbar)^m w(\boldsymbol{\nu})\delta_{\boldsymbol{\nu}},$$

where $\delta_{\boldsymbol{\nu}}$ denotes the Dirac measure on $\mathbb{R}^m$ concentrated at $\boldsymbol{u}$. (Recall that the Fourier transform is normalized as in (1.4).)

Let us observe that, as $\hbar \to 0$, the measures $\mu^\hbar(|d\boldsymbol{\xi}|)$ converge to the measure

$$\mu^0(|d\boldsymbol{\xi}|) := \frac{1}{(2\pi)^m} w\big(\,|\boldsymbol{\xi}|\,\big)\,|d\boldsymbol{\xi}|$$

in the following sense: for any Schwartz function $u : \mathbb{R}^m \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^m} u(\boldsymbol{\xi})\mu^\hbar(|d\boldsymbol{\xi}|) = \int_{\mathbb{R}^m} u(\boldsymbol{\xi})\mu^0(|d\boldsymbol{\xi}|).$$

Denote by Y^0 the stationary, Gaussian random function on $\mathbb{R}^m$ with spectral measure $\mu^0(|d\boldsymbol{\xi}|)$. Its covariance kernel is

$$\mathcal{K}^0(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{-\boldsymbol{i}\langle\boldsymbol{\xi},\boldsymbol{y}-\boldsymbol{x}\rangle} w\big(\,|\boldsymbol{\xi}|\,\big)\,|d\boldsymbol{\xi}| = V(\boldsymbol{y}-\boldsymbol{x}).$$

From (1.8) we deduce that

$$V^\hbar \to V \;\text{ in } C^\infty \text{ as } \hbar \to 0$$

This suggests that the statistics of $Y^\hbar$ ought to be "close" to the statistics of Y^0.

For the goal we have in mind it is convenient to give a white noise description of these random functions. Recall (see [16, Chap.7]) that a Gaussian white-noise on $\mathbb{R}^m$ is a random measure $W(-)$ that associates to each Borel set $A \in \mathcal{B}(\mathbb{R}^m)$ a centered Gaussian random variable $W(A)$ with the property that

$$\mathbb{E}\big[\,W(A)W(B)\,\big] = |A \cap B|.$$

The fact that $Z(-)$ is a random *measure* is equivalent in this case to the condition

$$W(A \cup B) = W(A) + W(B),\;\; \forall A, B \in \mathcal{B}(\mathbb{R}^m),\;\; A \cap B = \emptyset.$$

Equivalently, a Gaussian white-noise on $\mathbb{R}^m$ is characterized by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an isometry

$$\boldsymbol{I} = \boldsymbol{I}_W : L^2(\mathbb{R}^m, d\boldsymbol{\xi}) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$$

onto a Gaussian subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$. More precisely, for $f \in L^2(\mathbb{R}^m, d\boldsymbol{\xi})$ the Gaussian random variable $\boldsymbol{I}_W[f]$ is the Ito integral

$$\boldsymbol{I}_W(f) = \int_{\mathbb{R}^m} f(\xi)W(d\boldsymbol{\xi}).$$

The isometry property of the Ito integral reads

$$\boldsymbol{E}\Big[\,\boldsymbol{I}_W(f)\boldsymbol{I}_W(g)\,\Big] = \int_{\mathbb{R}^m} f(\boldsymbol{\xi})g(\boldsymbol{\xi})\,d\boldsymbol{\xi}.$$

In particular

$$W(A) = \boldsymbol{I}_W[\mathbf{1}_A],\ \ \forall A \in \mathcal{B}(\mathbb{R}^m).$$

The existence of Gaussian white noises is a well settled fact, [12].

Fix two independent Gaussian white-noises W_1, W_2 on $\mathbb{R}^m$ defined on the the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\boldsymbol{I}_1$ and respectively $\boldsymbol{I}_2$ their associated Ito integrals,

$$\boldsymbol{I}_1, \boldsymbol{I}_2 : L^2(\mathbb{R}^m, d\boldsymbol{\xi}) \to L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

The independence of the white noises $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ is equivalent to the condition

$$\mathbb{E}\big[\,\boldsymbol{I}_1(f)\boldsymbol{I}_2(g)\,\big] = 0,\ \ \forall f, g \in L^2(\mathbb{R}^m, d\boldsymbol{\xi}).$$

This shows that we have a well defined isometry

$$\boldsymbol{I} : \underbrace{L^2(\mathbb{R}^m, d\boldsymbol{\xi}) \times L^2(\mathbb{R}^m, d\boldsymbol{\xi})}_{\mathfrak{H}} \to L^2(\Omega, \mathcal{F}, \mathbb{P}),\ \ \boldsymbol{I}(f_1 \oplus f_2) = \boldsymbol{I}_1(f_1) + \boldsymbol{I}_2(f_2). \tag{1.9}$$

whose image is a Gaussian Hilbert subspace $\mathscr{X} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$. The map $\boldsymbol{I}$ describes an isonormal Gaussian process parametrized by $\mathfrak{H}$. We will use $\boldsymbol{I}$ to give alternate descriptions to the functions $Y^\hbar$, $\hbar \geq 0$.

For each $\boldsymbol{\lambda}_0 \in \mathbb{R}^m$ and $r > 0$ we denote by $C_r\big(\boldsymbol{\lambda}^0\big)$ the cube[1] of size r centered at $\boldsymbol{\lambda}_0$ i.e.,

$$C_r\big(\boldsymbol{\lambda}_0\big) = \Big\{ \boldsymbol{\xi} \in \mathbb{R}^m;\ \ |\boldsymbol{\xi} - \boldsymbol{\lambda}_0|_\infty \leq \frac{r}{2} \Big\},$$

For $\boldsymbol{k} \in \mathbb{Z}^m$ we set $C_{\boldsymbol{k}} := C_1(\boldsymbol{k})$. For each $\boldsymbol{x} \in \mathbb{R}^m$ and $\hbar > 0$ we set

$$\begin{aligned}\tilde{Y}^\hbar(\boldsymbol{x}) &:= \sum_{\boldsymbol{k}\in\mathbb{Z}^m} \int_{\mathbb{R}^m} \sqrt{w(2\pi\hbar|\boldsymbol{k}|)} \cos 2\pi\hbar\langle \boldsymbol{k}, \boldsymbol{x}\rangle \mathbf{1}_{\hbar C_{\boldsymbol{k}}}(\boldsymbol{\xi}) W_1(d\boldsymbol{\xi})\\ &+ \sum_{\boldsymbol{k}\in\mathbb{Z}^m} \int_{\mathbb{R}^m} \sqrt{w(2\pi\hbar|\boldsymbol{k}|)} \sin 2\pi\hbar\langle \boldsymbol{k}, \boldsymbol{x}\rangle \mathbf{1}_{\hbar C_{\boldsymbol{k}}}(\boldsymbol{\xi}) W_2(d\boldsymbol{\xi}) \in \mathscr{X}.\end{aligned}$$

The isometry property of $\boldsymbol{I}$ shows that

$$\mathbb{E}\big[\,\tilde{Y}^\hbar(\boldsymbol{x})\tilde{Y}^\hbar(\boldsymbol{y})\,\big] = \mathcal{K}^\hbar(\boldsymbol{x}, \boldsymbol{y}).$$

Thus, the random function $\tilde{Y}^\hbar$ is stochastically equivalent to $Y^\hbar$. Next define

$$\tilde{Y}^0(\boldsymbol{x}) = \int_{\mathbb{R}^m} \sqrt{w(2\pi|\xi|)} \cos 2\pi\langle \boldsymbol{\xi}, \boldsymbol{x}\rangle W_1(d\boldsymbol{\xi}) + \int_{\mathbb{R}^m} \sqrt{w(2\pi|\xi|)} \sin 2\pi\langle \boldsymbol{\xi}, \boldsymbol{x}\rangle W_2(d\boldsymbol{\xi}) \in \mathscr{X}.$$

Then

$$\begin{aligned}\mathbb{E}\big[\,\tilde{Y}^0(\boldsymbol{x})\tilde{Y}^0(\boldsymbol{y})\,\big] &= \int_{\mathbb{R}^m} w\big(\,2\pi|\boldsymbol{\xi}|\,\big) \cos 2\pi\langle \xi, \boldsymbol{y} - \boldsymbol{x}\rangle d\boldsymbol{\xi}\\ &= \int_{\mathbb{R}^m} e^{-2\pi \boldsymbol{i}\langle \boldsymbol{\xi}, \boldsymbol{y}-\boldsymbol{x}\rangle} w\big(\,2\pi|\boldsymbol{\xi}|\,\big)\, |d\boldsymbol{\xi}| = \mathcal{K}^0(\boldsymbol{x}, \boldsymbol{y}).\end{aligned}$$

Thus, the random function $\tilde{Y}^0$ is stochastically equivalent to Y^0.

The above discussion shows that we can assume that the Gaussian random variables $Y^\hbar(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}$, $\hbar \geq 0$, live inside the *same* Gaussian Hilbert space $\mathscr{X}$.

We denote by $\widehat{\mathcal{F}}$ the σ-subalgebra of $\mathcal{F}$ generated by the random variables $\boldsymbol{I}(f_1 \oplus f_2)$, $f_1 \oplus f_2 \in \mathfrak{H}$ and we denote by $\widehat{\mathscr{X}}$ the *Wiener chaos*, [16, 19],

$$\widehat{\mathscr{X}} := L^2\big(\Omega, \widehat{\mathcal{F}}, \mathbb{P}\,\big). \tag{1.10}$$

[1]The astute reader may have observed that $C_r(\boldsymbol{\lambda}_0) = \widehat{B}_r(\boldsymbol{\lambda}^0)$ and may wonder why the new notation. The reason for this redundancy is that the cubes $\widehat{B}$ and C live in *different* vector spaces, dual to each other. The cube $\widehat{B}$ lives in the space with coordinates $\boldsymbol{x}$ and C lives in the dual frequency space with coordinates $\boldsymbol{\xi}$.

1.3. **Statements of the main results.** In the sequel we will use the notation $Q_\hbar = O(\hbar^\infty)$ to indicate that, for any $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that

$$|Q_\hbar| \leq C_N \hbar^N \;\text{ as } \hbar \searrow 0.$$

Theorem 1.1. *Fix a function $N : (0,\infty) \to \mathbb{N}$, $\hbar \mapsto N_\hbar$ such that*

$$2\hbar N_\hbar \leq 1, \;\; \forall \hbar > 0. \tag{†}$$

Then, for any box $B \subset \mathbb{R}^m$ we have

$$\mathbb{E}\big[\, \boldsymbol{Z}(Y^0, B)\,\big] = \bar{Z}_0\,|B|, \;\; \bar{Z}_0 = \frac{1}{\sqrt{\det(-2\pi\nabla^2 V(0))}}\,\mathbb{E}\big[\,|\det \nabla^2 Y^0(0)|\,\big], \tag{1.11a}$$

$$\mathbb{E}\big[\, \boldsymbol{Z}(Y^\hbar, \widehat{B}_{2N_\hbar})\,\big] = \mathbb{E}\big[\, \boldsymbol{Z}(Y^0, \widehat{B}_{N_\hbar})\,\big] + O(\hbar^\infty) \tag{1.11b}$$

□

For simplicity, for any Borel subset $B \subset \mathbb{R}^m$, and any $\hbar \in [0, \hbar_0]$ we set

$$\boldsymbol{Z}^\hbar(B) := \boldsymbol{Z}^\hbar(Y^\hbar, B), \;\; \zeta^\hbar(B) = |B|^{-1/2}\Big(\, \boldsymbol{Z}^\hbar(B) - \mathbb{E}\big[\, \boldsymbol{Z}^\hbar(B)\,\big]\,\Big).$$

For $R > 0$ we set

$$\boldsymbol{Z}^\hbar(R) := \boldsymbol{Z}(\widehat{B}_R), \;\; \zeta^\hbar(R) = \zeta^\hbar(\widehat{B}_R). \tag{1.12}$$

Theorem 1.2. *There exists a number $S^0 > 0$ such that, for any function*

$$N : (0,\infty) \to \mathbb{N}, \;\; \hbar \mapsto N_\hbar,$$

satisfying

$$2\hbar N_\hbar \leq \frac{1}{2}, \;\; \forall \hbar > 0, \tag{‡}$$

and

$$\lim_{\hbar\to 0} N_\hbar = \infty, \tag{$*$}$$

the following hold.

(i) *As $\hbar \to 0$*

$$\boldsymbol{var}\big[\, \boldsymbol{Z}^\hbar(2N_\hbar)\,\big] \sim S_0(2N_\hbar)^m, \;\; \boldsymbol{var}\Big[\boldsymbol{Z}^0(2N_\hbar)\Big].$$

(ii) *The families of random variables*

$$\Big\{\, \zeta^\hbar(2N_\hbar)\,\Big\}_{\hbar\in(0,\hbar_0]} \;\; \textit{and} \;\; \Big\{\, \zeta^0(2N_\hbar)\,\Big\}_{\hbar\in(0,\hbar_0]}$$

converge in distribution as $\hbar \to 0$ to normal random variables $\sim \mathcal{N}(0, S^0)$.

Fix $r \in (0, 1/2)$. Recall that $B_r \subset \mathbb{T}^m$ denotes the image of $\widehat{B}_r$ under the natural projection $\mathbb{R}^m \to \mathbb{T}^m$.

Theorem 1.3. *Let $\bar{Z}_0$ be as in (1.11a) and S_0 be as in Theorem 1.2. Then the following hold.*

(i) *As $\hbar \to 0$,*

$$\mathbb{E}\big[\, \boldsymbol{Z}(X^\hbar, B_r)\,\big] = \hbar^{-m}\big(\, \bar{Z}_0\,\mathrm{vol}(B_r) + o(\hbar^\infty)\,\big).$$

(ii) *As $\hbar \to 0$ we have*

$$\boldsymbol{var}\big[\, \boldsymbol{Z}(X^\hbar, B_r)\,\big] \sim S_0\hbar^{-m}\,\mathrm{vol}(B_r).$$

(iii) *As $\hbar\to 0$ the random variables*

$$\left(\frac{\hbar}{r}\right)^{\frac{m}{2}}\Big(\boldsymbol{Z}(X^\hbar,B_r)-\mathbb{E}\big[\,\boldsymbol{Z}(X^\hbar,B_r)\,\big]\Big) \tag{1.13}$$

converge in distribution to a random variable $\sim\mathcal{N}(0,S^0)$.

Corollary 1.4. *Let $r\in(0,1/2)$. Set*

$$\bar{\boldsymbol{Z}}^\hbar(X^\hbar,B_r):=\hbar^m\boldsymbol{Z}^\hbar(X^\hbar,B_r).$$

Let $(\hbar_n)$ be a sequence of postive numbers such that, for some $p\in(0,m)$ we have

$$\sum_{n\geq 1}\hbar_n^p<\infty.$$

Then

$$\bar{\boldsymbol{Z}}^{\hbar_n}(X^\hbar,B_r)\to\bar{Z}_0\operatorname{vol}(B_r)\ \ a.s..$$

Proof. Set $\alpha:=\frac{m-p}{2}$ so that $p=m-2\alpha$. Note that

$$\mathbb{E}\big[\,\bar{\boldsymbol{Z}}^{\hbar_n}(X^\hbar,B_r)\,\big]=\bar{Z}_0\operatorname{vol}(B_r)+O(\hbar^\infty)$$

$$\big[\,\bar{\boldsymbol{Z}}^{\hbar_n}(X^\hbar,B_r)\,\big]\sim S_0\operatorname{vol}(B_r)\hbar^m.$$

From Chebyshev's inequality we deduce

$$\mathbb{P}\big[\,\big|\bar{\boldsymbol{Z}}^{\hbar_n}(X^\hbar,B_r)-\bar{Z}_0\operatorname{vol}(B_r)\big|\geq h_n^\alpha\big]=O(\hbar_n^p).$$

Then

$$\sum_{n\geq 1}\mathbb{P}\big[\,\big|\bar{\boldsymbol{Z}}^{\hbar_n}(X^\hbar,B_r)-\bar{Z}_0\operatorname{vol}(B_r)\big|\geq h_n^\alpha\big]<\infty,$$

and the first Borel-Cantelli lemma implies the desired conclusion. □

Remark 1.5. The constant $\bar{Z}_0$ in (1.11a) depends only on m and w, $\bar{Z}_0=\bar{Z}_0(w,m)$ an represents the expected density of critical points per unit volume of the random function Y^0. We set

$$I_k(w):=\int_0^\infty w(r)r^k dr. \tag{1.14}$$

We have (see [22])

$$\begin{gathered}(2\pi)^{m/2}s_m=\frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}I_{m-1}(w),\ \ (2\pi)^{m/2}d_m=\frac{2\pi^{\frac{m}{2}}}{m\Gamma(\frac{m}{2})}I_{m+1}(w),\\ (2\pi)^{m/2}h_m=\frac{1}{3}\int_{\mathbb{R}^m}x_1^4w(|x|)dx=\frac{2\pi^{\frac{m}{2}}}{m(m+2)\Gamma(\frac{m}{2})}I_{m+3}(w).\end{gathered} \tag{1.15}$$

Denote by $\mathcal{S}_m$ the Gaussian Orthogonal Ensemble of real symmetric $m\times m$ matrices A with independent, normally distributed entries $(a_{ij})_{1\leq i,j\leq m}$ with variances

$$\boldsymbol{E}[a_{ii}^2]=2,\ \ \boldsymbol{E}[a_{ij}^2]=1,\ \ \forall 1\leq i\neq j\leq m$$

As explained in [24], we have

$$\bar{Z}_0(w,m)=\left(\frac{h_m}{2\pi d_m}\right)^{\frac{m}{2}}\mathbb{E}_{\mathcal{S}_m}\big[\,|\det A|\,\big]=\left(\frac{I_{m+1}(w)}{2\pi(m+2)I_{m+3}(w)}\right)^{\frac{m}{2}}\mathbb{E}_{\mathcal{S}_m}\big[\,|\det A|\,\big]. \tag{1.16}$$

In [22, Cor.1.7] we have shown that, as $m \to \infty$, we have

$$\bar{Z}_0(w,m) \sim \frac{8}{\sqrt{\pi m}}\Gamma\left(\frac{m+3}{2}\right)\left(\frac{2I_{m+3}(w)}{\pi(m+2)I_{m+1}(w)}\right)^{\frac{m}{2}}. \tag{1.17}$$

The asymptotic behavior of $\bar{Z}_0(w,m)$ as $m \to \infty$ depends rather dramatically on the size of the tail of the Schwartz function w: the heavier the tail, the faster the growth. For example, in [22, Sec.3] we have shown the following.

- If $w(t) \sim \exp(-\log t)\log(\log t)\,)$ as $t \to \infty$, then
$$\log \bar{Z}_0(w,m) \sim \frac{m}{2}e^{m+2}(e^2-1) \text{ as } m \to \infty.$$
- If $w(t) \sim \exp\bigl(-(\log t)^{\frac{p}{p-1}}\,\bigr)$ as $t \to \infty$, $p > 1$, then, for some explicit constant $C_p > 0$, we have
$$\log \bar{Z}_0(w,m) \sim C_p m^p, \text{ as } m \to \infty$$
- If $w(t) \sim e^{-t^2}$ as $t \to \infty$, then
$$\log \bar{Z}_0(w,m) \sim\sim \frac{m}{2}\log m, \text{ as } m \to \infty.$$

(b) The constant S_0 in Theorem 1.2 seems very difficult to estimate. As the proof of Theorem 1.2 will show, the constant S_0 is a sum of a series with nonnegative terms

$$S_0 = \sum_{q\geq 1}\bar{S}_q^0,$$

where the terms $\bar{S}_q^0$ are defined explicitly in (2.22). In [24] we have proved that $S_0 > 0$ by showing that

$$\bar{S}_2^0 = \int_{\mathbb{R}^m}\bigl|\,P(\xi_1,\dots,\xi_m)w(|\boldsymbol{\xi}|)\,\bigr|^2 d\boldsymbol{\xi},$$

where $P(\xi_1,\dots,\xi_m)$ is a certain nonzero polynomial. The constant $\bar{S}_2^0$ depends on w and m. In [24, Appendix A] we described methods of producing asymptotic estimates for $\bar{S}_2^0(w,m)$ as $m \to \infty$, but the results are not too pretty. □

1.4. **Outline of proofs.** The strategy of proof is inspired from [5, 11]. As explained earlier, the Gaussian random variables $Y^\hbar(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^m$, $\hbar \geq 0$, are defined on the same probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ and inside the same Gaussian Hilbert space $\mathscr{X}$.

Using the Kac-Rice formula and the asymptotic estimates in [23] we show in Subsection 2.1 that, for any $\hbar \geq 0$ sufficiently small, and any box B, the random variables $\boldsymbol{Z}^\hbar(B)$ belongs to the Wiener chaos $\widehat{\mathscr{X}}$ and we describe its Wiener chaos decomposition. The key result behind this fact is Proposition 2.1 whose rather involved technical proof is deferred to Appendix A. The Wiener chaos decomposition of $\boldsymbol{Z}^\hbar(B)$ leads immediately to (1.11a) and (1.11b).

To prove that the random variables $\zeta^\hbar(2N_\hbar)$ and $\zeta^0(2N_\hbar)$ converge in law to normal random variable $\bar{\zeta}^0(\infty)$ and respectively $\zeta^0(\infty)$ we imitate the strategy in [11, 24] based on the very general Breuer-Major type central limit theorem [26, Thm. 6.3.1], [25, 27, 28, 29].

The case of the variables $\zeta^0(N_\hbar)$ is covered in [24] where we have shown that there exists $S^0 > 0$ and a normal random bariable $\zeta^0(\infty) \sim \mathcal{N}(0, S^0)$ such that, as $N \to \infty$, the random variable $\zeta^0(N)$ converges in law to $\zeta^0(\infty)$.

The case $\zeta^\hbar(2N_\hbar)$ is conceptually similar, but the extra dependence on $\hbar$ adds an extra layer of difficulty. Here are the details.

Denote by $\zeta_q^\hbar$ the q-th chaos component of $\zeta_q^\hbar(2N_\hbar) \in \widehat{\mathscr{X}}$. According to [26, Thm.6.3.1], to prove that $\zeta^\hbar(2N_\hbar)$ converges in law to a normal random variable $\bar{\zeta}^0(\infty)$ it suffices to prove the following.

(i) For every $q \in \mathbb{N}$ there exists $\bar{S}_q^0 \geq 0$ such that

$$\lim_{\hbar\to 0} \boldsymbol{var}\big[\,\zeta_q^\hbar\,\big] = \bar{S}_q^0.$$

(ii) Exists $\hbar_0 > 0$ such that

$$\lim_{Q\to\infty} \sup_{0\leq\hbar\leq\hbar_0} \sum_{q\geq Q} \boldsymbol{var}\big[\,\zeta_q^\hbar\,\big] = 0.$$

(iii) For each $q \in \mathbb{N}$, the random variables $\zeta_q^\hbar(2N_\hbar)$ converge in law to a normal random variable, necessarily of variance $\bar{S}_q^0$.

We prove (i) and (ii) in Subsection 2.4; see (2.23) and respectively Lemma 2.6.

To prove (iii) we rely on the fourth-moment theorem [26, Thm. 5.2.7], [28]. The details are identical to the ones employed in the proof of [11, Prop. 2.4]. The variance of the limiting normal random variable $\bar{\zeta}^0(\infty)$ is

$$\boldsymbol{var}\big[\,\bar{\zeta}_0(\infty)\,\big] = \sum_{q\geq 1} \bar{S}_q^0 < \infty.$$

The explicit description of the components $\bar{S}_q^0$ will then show that $S^0 = \bar{S}^0$.

The proof of Theorem 1.3 is, up to a suitable rescaling, identical to the proof of Theorem 1.2. We explain this in more detail in Subsection 2.6

1.5. **Related results.** Central limit theorems concerning crossing counts of random functions go back a while, e.g. T. Malevich [20] (1969) and J. Cuzik [9] (1976).

The usage of Wiener chaos decompositions and of Breuer-Major type results in proving such central limit theorems is more recent, late 80s early 90s. We want to mention here the pioneering contributions of Chambers and Slud [8], Slud [30, 31], Kratz and León [17], Sodin and Tsirelson [32].

This topic was further elaborated by Kratz and León in [18] where they also proved a central limit theorem concerning the length of the zero set of a random function of two variables. We refer to [6] for particularly nice discussion of these developments.

Azaïs and León [5] used the technique of Wiener chaos decomposition to give a shorter and more conceptual proof to a central limit theorem due to Granville and Wigman [13] concerning the number of zeros of random trigonometric polynomials of large degree. This technique was then succesfully used by Azaïs, Dalmao and León [4] to prove a CLT concerning the number of zeros of Gaussian even trigonometric polynomials and by Dalmao in [10] to prove a CLT concerning the number of zeros of one-variable polynomials in the Kostlan-Shub-Smale probabilistic ensemble. Recently, Adler and Naitzat [1] used Hermite decompositions to prove a CLT concerning Euler integrals of random functions.

2. Proofs of the main results

2.1. **Hermite decomposition of the number of critical points.** For every $\hbar \geq 0$, $\boldsymbol{v} \in \mathbb{R}^m$ and $B \in \mathcal{B}(\mathbb{R}^m)$ we denote by $\boldsymbol{Z}^\hbar(\boldsymbol{v}, B)$ the number of solutions $\boldsymbol{x}$ of the equation

$$\nabla Y^\hbar(\boldsymbol{x}) = \boldsymbol{v}, \;\; \boldsymbol{x} \in B.$$

For $\varepsilon > 0$ we define

$$\delta_\varepsilon : \mathbb{R}^m \to \mathbb{R}, \;\; \delta_\varepsilon(\boldsymbol{v}) = \varepsilon^{-m}\boldsymbol{1}_{\widehat{B}_{\varepsilon/2}(0)}(\boldsymbol{v}).$$

Note that δ_ε is supported on the cube of size ε centered at the origin and its total integral is 1. As $\varepsilon \searrow 0$, the function δ_ε converges in the sense of distributions to the Dirac δ_0. We set

$$\boldsymbol{Z}^\hbar_\varepsilon(\boldsymbol{v}, B) = \int_B \big|\det \nabla^2 Y^\hbar(\boldsymbol{x})\big|\, \delta_\varepsilon\big(\nabla Y^\hbar(\boldsymbol{x}) - \boldsymbol{v}\big) d\boldsymbol{x}, \;\; \boldsymbol{Z}^\hbar(B) := \boldsymbol{Z}^\hbar(\boldsymbol{v}, B)\big|_{\boldsymbol{v}=0}.$$

We define a *box* in $\mathbb{R}^m$ to be a set $B \subset \mathbb{R}^m$ of the form

$$B = [a_1, b_1] \times \cdots \times [a_m, b_m], \;\; a_1 < b_1, \dots, a_m < b_m.$$

If $B \subset \mathbb{R}^m$ is a box [2, Thm.11.3.1], we deduce that X is a.s. a Morse function on T and in particular, for any $\boldsymbol{v} \in \mathbb{R}^m$, the equation $\nabla X^\hbar(\boldsymbol{x}) = \boldsymbol{v}$ almost surely has no solutions $\boldsymbol{x} \in \partial B$.

The proof of the Kac-Rice formula [2, Thm. 11.2.3] shows that $\boldsymbol{Z}^\hbar(\boldsymbol{v}, B) \in L^1(\Omega)$ and

$$\boldsymbol{Z}^\hbar_\varepsilon(\boldsymbol{v}, B) \to \boldsymbol{Z}^\hbar(\boldsymbol{v}, B) \;\text{ a.s. as } \varepsilon \to 0.$$

Proposition 2.1. *There exists $\hbar_0 > 0$, sufficiently small, and $C_0 > 0$ such that, for any $\hbar \in [0, \hbar_0)$ and any box $B \subset \widehat{B}_2(0)$ the following hold.*

(i) *For any $\boldsymbol{v} \in \mathbb{R}^m$, $\boldsymbol{Z}^\hbar(\boldsymbol{v}, B) \in L^2(\Omega, \widehat{\mathfrak{F}}, \mathbb{P})$.*

(ii) *The function*

$$\mathbb{R}^m \ni \boldsymbol{v} \mapsto \mathbb{E}\big[\, \boldsymbol{Z}^\hbar(\boldsymbol{v}, B)^2 \,\big] \in \mathbb{R}$$

is continuous.

(iii) *For any $\boldsymbol{v} \in \mathbb{R}^m$*

$$\lim_{\varepsilon\to 0} \boldsymbol{Z}^\hbar_\varepsilon(\boldsymbol{v}, B) = \boldsymbol{Z}^\hbar(\boldsymbol{v}, B) \;\text{ in } L^2(\Omega).$$

(iv) *The function*

$$[0, \hbar_0] \ni \hbar \mapsto \boldsymbol{Z}^\hbar(B) \in L^2(\Omega, \widehat{\mathfrak{F}}, \mathbb{P})$$

is continuous.

We defer the proof of Proposition 2.1 to the Appendix A. The case $\hbar = 0$ of this proposition is discussed in [11, Prop.1.1]. That proof uses in an essential fashion the isotropy of the random function Y^0. The random functions $Y^\hbar$, $\hbar \neq 0$, are not isotropic, but they are "nearly" so for $\hbar$ small.

Since for any Borel set $B \subset \mathbb{R}^m$, and any $\varepsilon > 0$ the random variables $\boldsymbol{Z}^\hbar_\varepsilon(\boldsymbol{v}, B)$ belong to the Wiener chaos $\widehat{\mathscr{X}}$ defined in (1.10), we deduce from Proposition 2.1(iii) that, for any $\hbar \leq \hbar_0$, and any box $B \subset \widehat{B}_{1/\hbar}$, the number of critical points $\boldsymbol{Z}^\hbar(B)$ belongs to the Wiener chaos $\widehat{\mathscr{X}}$.

Fix $\hbar_0$ as in Proposition 2.1. Consider the random field

$$\widehat{\boldsymbol{Y}}^\hbar(\boldsymbol{x}) := \nabla Y^\hbar(\boldsymbol{x}) \oplus \nabla^2 Y^\hbar(\boldsymbol{x}), \;\; \boldsymbol{x} \in \mathbb{R}^m, \;\; \hbar \in [h, \hbar_0].$$

of dimension

$$D = m + \nu(m), \;\; \nu(m) = \frac{m(m+2)}{2}$$

Note that

$$\mathbb{E}\big[\, Y^\hbar_i(\boldsymbol{x}) Y^\hbar_{j,k}(\boldsymbol{x}) \,\big] = -V^\hbar_{i,j,k}(0) = 0,$$

since $V^\hbar(\boldsymbol{x})$ is an even function. Hence, the two components of $\widehat{\boldsymbol{Y}}^\hbar$ are independent. We can find invertible matrices $\Lambda^\hbar_1$ and $\Lambda^\hbar_2$ of dimensions $m \times m$ and respectively $\nu(m) \times \nu(m)$, that depend continuously on $\hbar \in [0, \hbar_0]$ such that the probability distributions of the random vectors

$$U^\hbar(\boldsymbol{x}) = (\Lambda_1^\hbar)^{-1}\nabla Y^\hbar(\boldsymbol{x}) \in \mathbb{R}^m,\;\; A^\hbar(\boldsymbol{x}) := (\Lambda_2^\hbar)^{-1}\nabla^2 Y^\hbar(\boldsymbol{x}) \in \mathbb{R}^{\nu(m)}$$

are the canonical Gaussian measures on the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^{\nu(m)}$ respectively. More precisely, we can choose as $\Lambda_i^\hbar$, $i = 1, 1$, the square roots of the covariance matrices of $\nabla^i Y^\hbar(\boldsymbol{x})$.

Consider the functions

$$f^\hbar : \mathbb{R}^{\nu(m)} \to \mathbb{R},\;\; f^\hbar(A) = \bigl|\det \Lambda_2^\hbar A\,\bigr|,$$

$$G_\varepsilon^\hbar : \mathbb{R}^m \times \mathbb{R}^{\nu(m)} \to \mathbb{R},\;\; G_\varepsilon(U, A) = \delta_\varepsilon(\Lambda_1^\hbar U) f_\hbar(A).$$

Fix a box B, independent of $\hbar$. Proposition 2.1 shows that, for $\hbar$ sufficiently small, we have

$$\boldsymbol{Z}^\hbar(B) = \lim_{\varepsilon\to 0}\int_B G_\varepsilon^\hbar(U(\boldsymbol{x}), A(\boldsymbol{x})).$$

Recall that an orthogonal basis of $L^2\bigl(\mathbb{R}, \boldsymbol{\gamma}(dx)\bigr)$ is given by the Hermite polynomials, [16, Ex. 3.18], [21, V.1.3],

$$H_n(x) := (-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}\Bigl(e^{-\frac{x^2}{2}}\Bigr) = n!\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^r}{2^r r!(n-2r)!}x^{n-2r}. \tag{2.1}$$

In particular

$$H_n(0) = \begin{cases} 0, & n \equiv 1 \bmod 2,\\ (-1)^r\frac{(2r)!}{2^r r!}, & n = 2r. \end{cases} \tag{2.2}$$

For every multi-index $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{N}_0^{\mathbb{N}}$ such that all but finitely many α_k-s are nonzero, and any

$$\underline{x} = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}}$$

we set

$$|\alpha| := \sum_k \alpha_k,\;\; \alpha! := \prod_k \alpha_k!,\;\; H_\alpha(\underline{x}) := \prod_k H_{\alpha_k}(x_k).$$

Following [11, Eq.(5)] we define for every $\alpha \in \mathbb{N}_0^m$ the quantity

$$d_\alpha := \frac{1}{\alpha!}(2\pi)^{-\frac{m}{2}}H_\alpha(0). \tag{2.3}$$

The function $f^\hbar : \mathbb{R}^{\boldsymbol{\nu}(m)} \to \mathbb{R}$ has a $L^2(\mathbb{R}^{\nu(m)}, \boldsymbol{\Gamma})$-orthogonal decomposition

$$f^\hbar(A) = \sum_{n\geq 0} f_n^\hbar(A),$$

where

$$f_n^\hbar(A) = \sum_{\beta\in\mathbb{N}_0^{\nu(m)},\,|\beta|=n} f_\beta^\hbar H_\beta(A),\;\; f_\beta^\hbar = \frac{1}{\beta!}\int_{\mathbb{R}^{\nu(m)}} f^\hbar(A) H_\beta(A)\boldsymbol{\Gamma}(dA). \tag{2.4}$$

Note that

$$f_0^\hbar = \mathbb{E}\bigl[\,|\det\nabla^2 Y^\hbar(0)|\,\bigr]. \tag{2.5}$$

The function $\delta_\varepsilon(U)$ has a $L^2(\mathbb{R}^m, \boldsymbol{\Gamma})$-orthogonal decomposition

$$\delta_\varepsilon(U) = \sum_{\alpha\in\mathbb{N}_0^m} d_{\alpha,\varepsilon}^\hbar H_\alpha(U),$$

where

$$d^{\hbar}_{\alpha,\varepsilon} = \frac{1}{\alpha!}\int_{\mathbb{R}} \delta_\varepsilon(\Lambda_1^{\hbar} U) H_\alpha(U)\boldsymbol{\Gamma}(dU).$$

Note that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \delta_\varepsilon(\Lambda_1^{\hbar} U) H_\alpha(U)\boldsymbol{\Gamma}(dU) = \frac{1}{\det\Lambda_1^{\hbar}} H_\alpha(0),$$

so that

$$\lim_{\varepsilon\to 0} d^{\hbar}_{\alpha,\varepsilon} = \frac{1}{\det\Lambda_1^{\hbar}} d_\alpha, \tag{2.6}$$

uniformly for $\hbar\in[0,\hbar_0]$. We set

$$\omega_\hbar := \frac{1}{\det\Lambda_1^{\hbar}}.$$

Remark 2.2. The matrix $\Lambda_1^{\hbar}$ is the square root of the covariance matrix of the random vector $\nabla Y^{\hbar}(0)$, i.e.,

$$\Lambda_1^{\hbar} = \sqrt{-\nabla^2 V^{\hbar}(0)}.$$

The function $V = V^{\hbar=0}$ is radially symmetric and thus

$$\nabla^2 V(0) = -\lambda^2 \mathbb{1}_m,$$

for some $\lambda>0$. Hence

$$\Lambda_1^0 = \lambda\mathbb{1}_m,\;\; \omega_0 = \lim_{\hbar\to 0}\omega_\hbar = \lambda^{-m} = \frac{1}{\sqrt{\det(-\nabla^2 V(0))}}. \tag{2.7}$$

□

If we set

$$\mathfrak{I}_m := \mathbb{N}_0^m \times \mathbb{N}_0^{\nu(m)}$$

Then

$$\boldsymbol{Z}^{\hbar}_{\varepsilon}(B) = \sum_{q=0}^{\infty}\int_B \rho^{\hbar}_{q,\varepsilon}(\boldsymbol{x})d\boldsymbol{x}, \tag{2.8}$$

where

$$\rho^{\hbar}_{q,\varepsilon}(\boldsymbol{x}) = \sum_{\substack{(\alpha,\beta)\in\mathfrak{I}_m,\\ |\alpha|+|\beta|=q}} d^{\hbar}_{\alpha,\varepsilon} f^{\hbar}_{\beta} H_\alpha(U(\boldsymbol{x}))H_\beta(A(\boldsymbol{x})).$$

If we let $\varepsilon\to 0$ in (2.8) and use Proposition 2.1(iii) and (2.6) we deduce

$$Z^{\hbar}(B) = \sum_{q\geq 0} Z^{\hbar}_q(B),\;\; Z^{\hbar}_q(B) = \int_B \rho^{\hbar}_q(\boldsymbol{x})dbx, \tag{2.9a}$$

$$\rho^{\hbar}_q(\boldsymbol{x}) = \sum_{\substack{(\alpha,\beta)\in\mathfrak{I}_m,\\ |\alpha|+|\beta|=q}} \omega_\hbar d_\alpha f^{\hbar}_{\beta} H_\alpha(U(\boldsymbol{x}))H_\beta(A(\boldsymbol{x})). \tag{2.9b}$$

To proceed further we need to use some basic Gaussian estimates.

2.2. **A technical interlude.** Let $\boldsymbol{V}$ be a real Euclidean space of dimension N. We denote by $\mathcal{A}(\boldsymbol{V})$ the space of symmetric positive semidefinite operators $A:\boldsymbol{V}\to\boldsymbol{V}$. For $A\in\mathcal{A}(\boldsymbol{V})$ we denote by $\boldsymbol{\gamma}_A$ the centered Gaussian measure on $\boldsymbol{V}$ with covariance form A. Thus

$$\boldsymbol{\gamma}_{\mathbb{1}}(d\boldsymbol{v})=\frac{1}{(2\pi)^{\frac{N}{2}}}e^{-\frac{1}{2}|\boldsymbol{v}|^2}d\boldsymbol{v},$$

and $\boldsymbol{\gamma}_A$ is the push forward of $\boldsymbol{\gamma}_{\mathbb{1}}$ via the linear map $\sqrt{A}$,

$$\boldsymbol{\gamma}_A=(\sqrt{A})_*\boldsymbol{\gamma}_{\mathbb{1}}. \tag{2.10}$$

For any measurable $f:\boldsymbol{V}\to\mathbb{R}$ with at most polynomial growth we set

$$\boldsymbol{E}_A(f)=\int_{\boldsymbol{V}}f(\boldsymbol{v})\boldsymbol{\gamma}_A(d\boldsymbol{v}).$$

Proposition 2.3. *Let $f:\boldsymbol{V}\to\mathbb{R}$ be a locally Lipschitz function which is positively homogeneous of degree $\alpha\geq 1$. Denote by L_f the Lipschitz constant of the restriction of f to the unit ball of $\boldsymbol{V}$. There exists a constant $C>0$ which depends only on N and α such that, for any $\Lambda>0$ and any $A,B\in\mathcal{A}(\boldsymbol{V})$ such that $\|A\|,\|B\|\leq\Lambda$ we have*

$$\big|\,\boldsymbol{E}_A(f)-\boldsymbol{E}_B(f)\,\big|\leq CL_f\Lambda^{\frac{\alpha-1}{2}}\|A-B\|^{\frac{1}{2}}. \tag{2.11}$$

Proof. We present the very elegant argument we learned from George Lowther on MathOverflow. In the sequel we will use the same letter C to denote various constant that depend only on α and N.

First of all let us observe that (2.10) implies that

$$\boldsymbol{E}_A(f)=\int_{\boldsymbol{V}}f(\sqrt{A}\boldsymbol{v})\boldsymbol{\gamma}_{\mathbb{1}}(d\boldsymbol{v}).$$

We deduce that for any $t>0$ we have

$$\boldsymbol{E}_{tA}(f)=\int_{\boldsymbol{V}}f(\sqrt{tA}\boldsymbol{v})\boldsymbol{\gamma}_{\mathbb{1}}(d\boldsymbol{v})=t^{\frac{\alpha}{2}}\int_{\boldsymbol{V}}f(\sqrt{A}\boldsymbol{v})\boldsymbol{\gamma}_{\mathbb{1}}(d\boldsymbol{v})=t^{\frac{\alpha}{2}}\boldsymbol{E}_A(f),$$

and thus it suffices to prove (2.11) in the special case $\Lambda=1$, i.e. $\|A\|,\|B\|\leq 1$. We have

$$\begin{aligned}
\big|\,\boldsymbol{E}_A(f)-\boldsymbol{E}_B(f)\,\big| &\leq\int_{\boldsymbol{V}}\big|\,f(\sqrt{A}\boldsymbol{v})-f(\sqrt{B}\boldsymbol{v})\,\big|\boldsymbol{\gamma}_{\mathbb{1}}(d\boldsymbol{v})\\
&=\int_{\boldsymbol{V}}|\boldsymbol{v}|^\alpha\Big|\,f\Big(\sqrt{A}\frac{1}{|\boldsymbol{v}|}\boldsymbol{v}\Big)-f\Big(\sqrt{B}\frac{1}{|\boldsymbol{v}|}\boldsymbol{v}\Big)\,\Big|\boldsymbol{\gamma}_{\mathbb{1}}(d\boldsymbol{v})\\
&\leq L_f\int_{\boldsymbol{V}}|\boldsymbol{v}|^\alpha\Big|\,\sqrt{A}\frac{1}{|\boldsymbol{v}|}\boldsymbol{v}-\sqrt{B}\frac{1}{|\boldsymbol{v}|}\boldsymbol{v}\,\Big|\boldsymbol{\gamma}_{\mathbb{1}}(d\boldsymbol{v})\\
&\leq L_f\|\sqrt{A}-\sqrt{B}\|\int_{\boldsymbol{V}}|\boldsymbol{v}|^\alpha\boldsymbol{\gamma}_{\mathbb{1}}(d\boldsymbol{v})\leq CL_f\|A-B\|^{\frac{1}{2}}.
\end{aligned}$$

□

2.3. **Proof of Theorem 1.1.** Note that

$$\mathbb{E}\big[\,\boldsymbol{Z}^{\hbar}(B)\,\big]=\mathbb{E}\big[\,Z_0^{\hbar}(B)\,\big]=|B|\omega_{\hbar}f_0^{\hbar}d(0)$$

(use (2.3) and (2.5))

$$=(2\pi)^{-m/2}|B|\omega_{\hbar}\mathbb{E}\big[\,|\det\nabla^2Y^{\hbar}(0)|\,\big].$$

Using (1.7) and Remark 2.2 we deduce that that

$$\omega_{\hbar}-\omega_0=O(\hbar^{\infty}).$$

From (1.7) we also deduce that

$$\|\nabla^2V^{\hbar}(0)-\nabla^2V(0)\|=O(\hbar^{\infty}.$$

Invoking Proposition 2.3 we deduce that

$$\mathbb{E}\big[\,|\det\nabla^2Y^{\hbar}(0)|\,\big]=\mathbb{E}\big[\,|\det\nabla^2Y^{0}(0)|\,\big]+O(\hbar^{\infty}).$$

Hence

$$\begin{aligned}\mathbb{E}\big[\,\boldsymbol{Z}^{\hbar}(B)\,\big]&=\mathbb{E}\big[\,\boldsymbol{Z}^{0}(B)\,\big]+O(\hbar^{\infty}),\\ \mathbb{E}\big[\,\boldsymbol{Z}^{0}(B)\,\big]&=(2\pi)^{-m/2}\omega_0\,|B|\,\mathbb{E}\big[\,|\det\nabla^2Y^{0}(0)|\,\big].\end{aligned}\tag{2.12}$$

Using (2.7) in the above equality we obtain (1.11a).

Let $N_{\hbar}$ satisfy (†). Recall that $\mathbb{A}^m$ denotes the affine lattice

$$\mathbb{A}^m=\left(\frac{1}{2}+\mathbb{Z}\right)^m.\tag{2.13}$$

We have

$$\widehat{B}_{2N_{\hbar}}=\bigcup_{\boldsymbol{a}\in\mathbb{A}^m,\ |\boldsymbol{a}|_{\infty}\leq N_{\hbar}}\widehat{B}(\boldsymbol{a}),\ \ \widehat{B}(a):=\widehat{B}_1(\boldsymbol{a}).\tag{2.14}$$

The cubes in the above union have disjoint interiors. According to [2, Thm.11.3.1], for $\hbar\leq\hbar_0$ the function $Y^{\hbar}$ is a.s. Morse. Give a box $B\subset\mathbb{R}^m$, the function $Y^{\hbar}$ will a.s. have no critical points on the boundary of B Thus

$$\boldsymbol{Z}^{\hbar}\big(\,\widehat{B}_{2N_{\hbar}}\,\big)=\sum_{\boldsymbol{a}\in\mathbb{A}^m\cap\widehat{B}_{2N_{\hbar}}}\boldsymbol{Z}^{\hbar}\big(\,\widehat{B}(\boldsymbol{a})\,\big).$$

From (†) we deduce that $\widehat{B}(\boldsymbol{a})\subset\widehat{B}_{1/\hbar}(0)$ so (1.8) holds on $\widehat{B}(\boldsymbol{a})$. We deduce

$$\boldsymbol{Z}^{\hbar}\big(\,\widehat{B}_{2N_{\hbar}}\,\big)\stackrel{(2.12)}{=}\sum_{\boldsymbol{a}\in\mathbb{A}^m\cap\widehat{B}_{2N_{\hbar}}}\Big(\,\boldsymbol{Z}^{0}\big(\,\widehat{B}(\boldsymbol{a})\,\big)+O(\hbar^{\infty})\,\Big)$$

From (†) we deduce that $2N_{\hbar}\leq\frac{1}{\hbar}$ so that

$$\#\big(\boldsymbol{a}\in\mathbb{A}^m\cap\widehat{B}_{2N_{\hbar}}\,\big)=O(\hbar^{-m}).$$

Hence

$$\boldsymbol{Z}^{\hbar}\big(\,\widehat{B}_{2N_{\hbar}}\,\big)=\left(\sum_{\boldsymbol{a}\in\mathbb{A}^m\cap\widehat{B}_{2N_{\hbar}}}\boldsymbol{Z}^{0}\big(\,\widehat{B}(\boldsymbol{a})\,\big)\right)+O(\hbar^{\infty})=\boldsymbol{Z}^{\hbar}\big(\,\widehat{B}_{2N_{\hbar}}\,\big)+O(\hbar^{\infty}).$$

2.4. **Variance estimates.** For $\hbar \in [0, \hbar_0]$ we define

$$\psi^\hbar : \mathbb{R}^m \to \mathbb{R}, \;\; \psi^\hbar(\boldsymbol{x}) = \begin{cases} \max_{|\alpha|\leq 4} \bigl| \, \partial_{\boldsymbol{x}}^\alpha V^\hbar(\boldsymbol{x}) \, \bigr|, & |\boldsymbol{x}|_\infty \leq \frac{1}{2\hbar}, \\ \\ 0, & |\boldsymbol{x}|_\infty > \frac{1}{2\hbar}. \end{cases} \tag{2.15}$$

Lemma 2.4. *For any $p \in [0, \infty]$ we have*

$$\bigl\| \, \psi^\hbar - \psi^0 \, \bigr\|_{L^p(\mathbb{R}^m)} = O(\hbar^\infty). \tag{2.16}$$

Proof. We distinguish two cases.

1. $p = \infty$. Note that that (1.8) implies

$$\sup_{|\boldsymbol{x}|_\infty \leq 1/(2\hbar)} |\psi^\hbar(\boldsymbol{x}) - \psi^0(\boldsymbol{x})| = O(\hbar^\infty).$$

Since V is a Schwartz function we deduce that

$$\sup_{|\boldsymbol{x}|_\infty > 1/(2\hbar)} |\psi^\hbar(\boldsymbol{x}) - \psi^0(\boldsymbol{x})| = \sup_{|\boldsymbol{x}|_\infty > 1/(2\hbar)} |\psi^0(\boldsymbol{x})| = O(\hbar^\infty).$$

2. $p \in [1, \infty)$. We have

$$\int_{\mathbb{R}^m} |\psi^\hbar(\boldsymbol{x}) - \psi^0(\boldsymbol{x})|^p d\boldsymbol{x} = \int_{|\boldsymbol{x}|_\infty \leq 1/(2\hbar)} |\psi^\hbar(\boldsymbol{x}) - \psi^0(\boldsymbol{x})|^p d\boldsymbol{x} + \int_{|\boldsymbol{x}|_\infty > 1/(2\hbar)} |\psi^0(\boldsymbol{x})|^p d\boldsymbol{x}.$$

The integrand in the first integral in the right-hand side is $O(\hbar^\infty)$ and the volume of the region is integration is $O(\hbar^{-m})$ so the first integral is $O(\hbar^\infty)$. Since V is a Schwartz function we deduce that

$$|\psi^0(\boldsymbol{x})| = O\bigl(\, |\boldsymbol{x}|^{-N} \, \bigr), \;\; \forall N \in \mathbb{N}.$$

This shows that the second integral is also $O(\hbar^\infty)$. □

Proposition 2.5. *There exists $S^0 \in (0, \infty)$ such that*

$$\lim_{\hbar \searrow 0} \boldsymbol{var}\bigl[\zeta^\hbar(2N_\hbar) \, \bigr] = S^0 = \lim_{\hbar \searrow 0} \boldsymbol{var}\bigl[\zeta^0(2N_\hbar) \, \bigr].$$

Proof. The [24] we proved that the limit

$$\lim_{N \to \infty} \boldsymbol{var}\bigl[\zeta^0(2N) \, \bigr] \tag{2.17}$$

exists, it is finite and nonzero. We denote by S^0 this limit. It remains to prove two facts.

($\mathbf{F}_1$) The limit $\bar{S}^0 := \lim_{\hbar \searrow 0} \boldsymbol{var}\bigl[\zeta^\hbar(2N_\hbar) \, \bigr]$ exists and it is finite.

($\mathbf{F}_2$) $S^0 = \bar{S}^0$.

To prove these facts, we will employ a refinement of the strategy used in the proof of [24, Prop.3.3].

Proof of $\mathbf{F}_1$. Using (2.9a) we deduce

$$\zeta^\hbar(2N_\hbar) = (2N_\hbar)^{-m/2} \Bigl(\boldsymbol{Z}^\hbar(\widehat{B}_{2N_\hbar}) - \mathbb{E}\bigl[\, \boldsymbol{Z}^\hbar(\widehat{B}_{2N_\hbar}) \, \bigr] \Bigr) = (2N_\hbar)^{-m/2} \sum_{q > 0} Z_q^\hbar(\widehat{B}_{2N_\hbar}).$$

We set

$$S^\hbar = \boldsymbol{var}\bigl[\, \zeta^\hbar(N_\hbar) \, \bigr] = \mathbb{E}\bigl[\, \zeta^\hbar(2N_\hbar)^2 \, \bigr] = \sum_{q > 0} \underbrace{(2N_\hbar)^{-m} \mathbb{E}\bigl[\, Z^\hbar(\widehat{B}_{2N_\hbar})^2 \, \bigr]}_{=: S_q^\hbar}$$

To estimate $S_q^\hbar$ we write

$$Z_q^\hbar(\widehat{B}_{2N_\hbar}) = \int_{\widehat{B}_{2N_\hbar}} \rho_q^\hbar(\boldsymbol{x})d\boldsymbol{x},$$

where $\rho_q^\hbar(\boldsymbol{x})$ is described in (2.9b). Then

$$S_q^\hbar = (2N_\hbar)^{-m} \int_{\widehat{B}_{2N_\hbar}\times\widehat{B}_{2N_\hbar}} \mathbb{E}\big[\,\rho_q^\hbar(\boldsymbol{x})\rho_q^\hbar(\boldsymbol{y})\,\big]d\boldsymbol{x}d\boldsymbol{y}$$

(use the stationarity of $Y^\hbar(\boldsymbol{x})$)

$$= (2N_\hbar)^{-m} \int_{\widehat{B}_{2N_\hbar}\times\widehat{B}_{2N_\hbar}} \mathbb{E}\big[\,\rho_q^\hbar(0)\rho_q(\boldsymbol{y}-\boldsymbol{x})\,\big]d\boldsymbol{x}d\boldsymbol{y}$$

$$= \int_{\widehat{B}_{4N_\hbar}} \boldsymbol{E}\big[\,\rho_q^\hbar(0)\rho_q^\hbar(\boldsymbol{u})\,\big]\prod_{k=1}^m\left(1-\frac{|u_k|}{2N_\hbar}\right)d\boldsymbol{u}.$$

The last equality is obtained by integrating along the fibers of the map

$$\widehat{B}_{2N_\hbar}\times\widehat{B}_{2N_\hbar}\ni(\boldsymbol{x},\boldsymbol{y})\mapsto\boldsymbol{y}-\boldsymbol{x}\in\widehat{B}_{4N_\hbar}.$$

At this point we need to invoke (2.9b) to the effect that

$$\rho_q^\hbar(\boldsymbol{x}) = \sum_{\substack{(\alpha,\beta)\in\mathfrak{I}_m,\\ |\alpha|+|\beta|=q}} \omega_\hbar d_\alpha f_\beta^\hbar H_\alpha(U(\boldsymbol{x}))H_\beta(A(\boldsymbol{x})).$$

We can rewrite this in a more compact form. Set

$$\Xi^\hbar(\boldsymbol{x}) := \big(U(\boldsymbol{x}),\ A(\boldsymbol{x})\,\big).$$

For $\gamma=(\alpha,\beta)\in\mathfrak{I}_m$ we set

$$\boldsymbol{a}^\hbar(\gamma) := \omega_\hbar d_\alpha f_\beta^\hbar,\ \ H_\gamma(\Xi^\hbar(\boldsymbol{x})) := H_\alpha(U(\boldsymbol{x}))H_\beta(A(\boldsymbol{x})).$$

Then

$$\rho_q^\hbar(\boldsymbol{x}) = \sum_{\gamma\in\mathfrak{I}_m,\ |\gamma|=q} \boldsymbol{a}^\hbar(\gamma)H_\gamma\big(\,\Xi^\hbar(\boldsymbol{x})\,\big), \tag{2.18}$$

$$\mathbb{E}\big[\,\rho_q^\hbar(0)\rho_q^\hbar(u)\,\big] = \sum_{\substack{\gamma,\gamma'\in\mathfrak{I}_m\\ |\gamma|=|\gamma'|=q}} \boldsymbol{a}^\hbar(\gamma)\boldsymbol{a}^\hbar(\gamma')\mathbb{E}\big[\,H_\gamma(\,\Xi^\hbar(0)\,)H_{\gamma'}(\,\Xi^\hbar(\boldsymbol{u})\,)\,\big].$$

We set $\omega(m):=m+\nu(m)$, and we denote by $\Xi_i(\boldsymbol{x})$, $1\leq i\leq\omega(m)$, the components of $\Xi(\boldsymbol{x})$ labelled so that $\Xi_i(\boldsymbol{x})=U_i(\boldsymbol{x})$, $\forall 1\leq i\leq m$. For $\boldsymbol{u}\in\mathbb{R}^m$, $\hbar\in[0,\hbar_0]$ and $1\leq i,j\leq\omega(m)$ we define the covariances

$$\Gamma_{ij}^\hbar(\boldsymbol{u}) := \mathbb{E}\big[\,\Xi_i^\hbar(0)\Xi_j^\hbar(\boldsymbol{u})\,\big].$$

Observe that there exists a positive constant K, independent of $\hbar\in[0,\hbar_0]$ such that

$$\big|\,\Gamma_{i,j}^\hbar(\boldsymbol{u})\,\big|\leq K\psi^\hbar(\boldsymbol{u}),\ \ \forall i,j=1,\dots,\omega(m),\ \ \boldsymbol{u}\in\widehat{B}_{4N_\hbar}, \tag{2.19}$$

where $\psi^\hbar$ is the function defined in (2.15). From (‡) we deduce that,

$$\widehat{B}_{4N_\hbar}\subset\widehat{B}_{1/\hbar} \tag{2.20}$$

Using the Diagram Formula (see e.g.[19, Cor. 5.5] or [16, Thm. 7.33]) we deduce that for any $\gamma,\gamma'\in\mathfrak{I}_m$ such that $|\gamma|=|\gamma'|=q$, there exists a *universal* homogeneous polynomial of degree q, $P_{\gamma,\gamma'}$ in the variables $\Gamma_{ij}(\boldsymbol{u})$ such that

$$\mathbb{E}\big[\,H_\gamma(\,\Xi^\hbar(0)\,)H_{\gamma'}(\,\Xi^\hbar(\boldsymbol{u})\,)\,\big] = P_{\gamma,\gamma'}\big(\,\Gamma_{ij}^\hbar(\boldsymbol{u})\,\big).$$

Hence

$$S_q^\hbar = (2N_\hbar)^{-m} \sum_{\substack{\gamma,\gamma'\in\mathfrak{I}_m\\ |\gamma|=|\gamma'|=q}} \boldsymbol{a}^\hbar(\gamma)\boldsymbol{a}^\hbar(\gamma') \underbrace{\int_{\widehat{B}_{4N_\hbar}} P_{\gamma,\gamma'}\big(\,\Gamma^\hbar_{ij}(\boldsymbol{u})\,\big)\prod_{k=1}^m\left(1-\frac{|u_k|}{2N_\hbar}\right)d\boldsymbol{u}}_{=:R^\hbar(\boldsymbol{\gamma},\boldsymbol{\gamma}')}. \tag{2.21}$$

From (2.19) we deduce that for any $\gamma,\gamma'\in\mathfrak{I}_m$ such that $|\gamma|=|\gamma'|=q$ there exists a constant $C_{\gamma,\gamma'}>0$ such that

$$\big|\,P_{\gamma,\gamma'}\big(\,\Gamma^\hbar_{ij}(\boldsymbol{u})\,\big)\,\big| \leq C_{\gamma,\gamma'}\psi^\hbar(\boldsymbol{u})^q,\;\;\forall \boldsymbol{u}\in\widehat{B}_{4N_\hbar}.$$

We know from ($*$) that $N_\hbar\to\infty$ as $\hbar\to 0$. Arguing exactly as in the proof of Lemma 2.4 we deduce that

$$\lim_{\hbar\to 0} R^\hbar(\boldsymbol{\gamma},\boldsymbol{\gamma}') = R^0(\gamma,\gamma') := \int_{\mathbb{R}^m} P_{\gamma,\gamma'}\big(\,\Gamma^0_{ij}(\boldsymbol{u})\,\big)d\boldsymbol{u}, \tag{2.22}$$

and thus

$$\lim_{\hbar\to 0} S_q^\hbar = \bar{S}_q^0 := \sum_{\substack{\gamma,\gamma'\in\mathfrak{I}_m\\ |\gamma|=|\gamma'|=q}} \boldsymbol{a}^0(\gamma)\boldsymbol{a}^0(\gamma')R^0(\gamma,\gamma') = \int_{\mathbb{R}^m}\mathbb{E}\big[\,\rho^0_q(0)\rho^0_q(\boldsymbol{u})\,\big]d\boldsymbol{u}. \tag{2.23}$$

Since $S_q^\hbar\geq 0$, $\forall q,\hbar$, we have

$$S_q^\hbar \geq 0,\;\;\forall q.$$

Lemma 2.6. *For any positive integer Q we set*

$$S^\hbar_{>Q} := \sum_{q>Q} S^\hbar.$$

Then

$$\lim_{Q\to\infty}\big(\,\sup_\hbar S_{>Q}\,\big) = 0, \tag{2.24}$$

the series

$$\sum_{q\geq 1}\bar{S}^0_q$$

is convergent and, if $\bar{S}^0$ is its sum, then

$$\bar{S}^0 = \lim_{\hbar\to 0} S^\hbar = \lim_{\hbar\to 0}\sum_{q\geq 1} S^\hbar_q. \tag{2.25}$$

Proof. For $\boldsymbol{x}\in\mathbb{R}^m$ we denote by $\theta_{\boldsymbol{x}}$ the shift operator associated with the stationary fields $\boldsymbol{Y}^\hbar$, i.e.,

$$\theta_{\boldsymbol{x}} Y^\hbar(\bullet) = Y^\hbar(\bullet+\boldsymbol{x}).$$

This extends to a unitary map $L^2(\Omega)\to L^2(\Omega)$ that commutes with the chaos decomposition of $L^2(\Omega)$. Moreover, for any box B and any $\hbar\in[0,\hbar_0]$ we have

$$\boldsymbol{Z}^\hbar(B+\boldsymbol{x}) = \theta_{\boldsymbol{x}}\boldsymbol{Z}^\hbar(B).$$

If we denote by $\mathcal{L}_\hbar$ the set

$$\mathcal{L}_\hbar := \mathbb{A}^m\cap\widehat{B}_{4N_\hbar}, \tag{2.26}$$

then we deduce

$$\zeta^\hbar(2N_\hbar) = (2N)^{-m/2}\sum_{\boldsymbol{p}\in\mathcal{L}_\hbar}\theta_{\boldsymbol{p}}\zeta^\hbar(B),\;\; B=\widehat{B}_1. \tag{2.27}$$

We denote by $\mathcal{P}_{>Q}$ the projection

$$\mathcal{P}_{>Q} = \sum_{q>Q} \mathcal{P}_q,$$

where $\mathcal{P}_q$ denotes the projection on the q-th chaos component of $\widehat{\mathscr{X}}$. We have

$$\mathcal{P}_{>Q}\zeta^{\hbar}(N_{\hbar}) = (2N_{\hbar})^{-m/2} \sum_{\boldsymbol{p}\in\mathcal{L}_{\hbar}} \theta_{\boldsymbol{p}}\mathcal{P}_{>Q}\zeta^{\hbar}(B).$$

Using the stationarity of $Y^{\hbar}$ we deduce

$$S^{\hbar}_{>Q,} = \mathbb{E}\Big[\,\big|\mathcal{P}_{>Q}\zeta^{\hbar}(2N_{\hbar})\big|^2\,\Big] = (2N_{\hbar})^{-m} \sum_{\boldsymbol{p}\in\mathcal{L}_{\hbar}} \nu(\boldsymbol{p},N_{\hbar})\boldsymbol{E}\big[\,\mathcal{P}_{>Q}\zeta\big(B\big)\cdot\theta_{\boldsymbol{p}}\mathcal{P}_{>Q}\zeta^{\hbar}\big(B\big)\,\big], \tag{2.28}$$

where $\nu(\boldsymbol{p},N_{\hbar})$ denotes the number of points $\boldsymbol{x}\in\mathcal{L}_{\hbar}$ such that $\boldsymbol{x}-\boldsymbol{p}\in\widehat{B}_{2N_{\hbar}}$. Clearly

$$\nu(\boldsymbol{p},N_{\hbar}) \le (2N_{\hbar})^m. \tag{2.29}$$

With K denoting the positive constant in (2.19) we deduce from Lemma 2.4 that we can choose positive numbers a,ρ such that

$$\psi^{\hbar}(\boldsymbol{x}) \le \rho < \frac{1}{K},\;\; \forall |\boldsymbol{x}|_{\infty} > a,\;\; \forall \hbar\in[0,\hbar_0].$$

We split $S^{\hbar}_{>Q}$ into two parts,

$$S^{\hbar}_{>Q} = S^{\hbar}_{>Q,0} + S^{\hbar}_{>Q,\infty},$$

where $S^{\hbar}_{>Q,0}$ is made up of the terms in (2.28) corresponding to points $\boldsymbol{p}\in\mathcal{L}_{\hbar}$ such that $|\boldsymbol{p}|_{\infty} < a+1$, while $S^{\hbar}_{>Q,\infty}$ corresponds to points $\boldsymbol{p}\in\mathcal{L}_{\hbar}$ such that $|\boldsymbol{p}|_{\infty}\ge a+1$.

We deduce from (2.29) that for $2M > a+1$ we have

$$\Big|\,S^{\hbar}_{>Q,0}\,\Big| \le (2N_{\hbar})^{-m}(2a+2)^m(2N_{\hbar})^m\mathbb{E}\Big[\,\big|\mathcal{P}_{>Q}\zeta^{\hbar}(B)\big|^2\,\Big].$$

Proposition 2.1(iv) implies that, as $Q\to\infty$, the right-hand side of the above inequality goes to 0 uniformly with respect to $\hbar$.

To estimate $S^{\hbar}_{>Q,\infty}$ observe that for $\boldsymbol{p}\in\mathcal{L}_{\hbar}$ such that $|\boldsymbol{p}|_{\infty} > a+1$ we have

$$\boldsymbol{E}\big[\,\mathcal{P}_{>Q}\zeta^{\hbar}(B)\cdot\theta_{\boldsymbol{p}}\mathcal{P}_{>Q}\zeta^{\hbar}(B)\,\big] = \sum_{q>Q}\int_B\int_B \mathbb{E}\big[\,\rho^{\hbar}_q(\boldsymbol{x})]\rho^{\hbar}_q(\boldsymbol{y}+\boldsymbol{p})\,\big]d\boldsymbol{x}d\boldsymbol{y}, \tag{2.30}$$

where we recall from (2.18) that

$$\rho^{\hbar}_q(\boldsymbol{x}) = \sum_{\gamma\in\mathcal{I}_m,\,|\gamma|=q} \boldsymbol{a}^{\hbar}(\gamma)H_{\gamma}\big(\,\Xi^{\hbar}(\boldsymbol{x})\,\big),\;\; \mathcal{I}_m := \mathbb{N}_0^m\times\mathbb{N}_0^{\nu(m)},\;\; \nu(m) = \frac{m(m+1)}{2}.$$

Thus

$$\mathbb{E}\big[\,\rho_q(\boldsymbol{x})\rho_q(\boldsymbol{y}+\boldsymbol{p})\,\big] = \mathbb{E}\Big[\,\Big(\sum_{\gamma\in\mathcal{I}_m,\,|\gamma|=q}\boldsymbol{a}^{\hbar}(\gamma)H_{\gamma}\big(\,\Xi^{\hbar}(\boldsymbol{t})\,\big)\Big)\Big(\sum_{\gamma\in\mathcal{I}_m,\,|\gamma|=q}\boldsymbol{a}^{\hbar}(\gamma)H_{\gamma}\big(\,\Xi^{\hbar}(\boldsymbol{y}+\boldsymbol{p})\,\big)\Big)\Big]$$

Arcones' inequality [3, Lemma 1] implies that

$$\mathbb{E}\big[\,\rho_q(\boldsymbol{x})\rho_q(\boldsymbol{y}+\boldsymbol{p})\,\big] \le K^q\psi^{\hbar}(\boldsymbol{p}+\boldsymbol{y}-\boldsymbol{x})^q \sum_{\gamma\in\mathcal{I}_m,\,|\gamma|=q} |\boldsymbol{a}^{\hbar}(\gamma)|^2\gamma!. \tag{2.31}$$

We're not out of the woods yet since the series $\sum_{\gamma\in\mathcal{I}_m}|\boldsymbol{a}^\hbar(\gamma)|^2\gamma!$ is divergent. On the other hand, for $\gamma=(\alpha,\beta)\in\mathcal{I}_m$ we have

$$\boldsymbol{a}^\hbar(\gamma)=\omega_\hbar d_\alpha f^\hbar_\beta$$

where, according to (2.3) we have $d_\alpha=\frac{1}{\alpha!}(2\pi)^{-\frac{m}{2}}H_\alpha(0)$. Recalling that

$$H_{2r}(0)=(-1)^r\frac{(2r)!}{2^r r!},\;\;H_{2r+1}(0)=0.$$

we deduce that

$$(2r)!\Big|\frac{1}{(2r)!}H_{2r}(0)\Big|^2=\frac{1}{2^{2r}}\binom{2r}{r}\leq 1,$$

and

$$d_\alpha^2\alpha!\leq C=\frac{1}{(2\pi)^{m/2}}.$$

Using (2.7) and (2.4) we conclude that

$$\sum_{\gamma\in\mathcal{I}_m,\,|\gamma|=q}|\boldsymbol{a}^\hbar(\gamma)|^2\gamma!\leq\omega_\hbar(2\pi)^{-m/2}q^m\sum_{\beta\in\mathbb{N}_0^{\nu(m)},|\beta|\leq q}(f^\hbar_\beta)^2\beta!\leq Cq^m\mathbb{E}\big[\,|\det\nabla^2Y^\hbar(0)|^2\,\big].$$

Using this in (2.30) and (2.31) we deduce

$$\mathbb{E}\big[\,\mathcal{P}_{>Q}\zeta^\hbar(B)\cdot\theta_{\boldsymbol{p}}\mathcal{P}_{>Q}\zeta^\hbar(B)\,\big]$$

$$\leq\underbrace{C\mathbb{E}\big[\,|\det\nabla Y^\hbar(0)|^2\,\big]}_{=:C'}\sum_{q>Q}q^mK^q\int_B\int_B\psi^\hbar(\boldsymbol{s}+\boldsymbol{u}-\boldsymbol{t})^qd\boldsymbol{u}d\boldsymbol{t}$$

Hence

$$\big|\,S^\hbar_{>Q,\infty}\big|\leq C'\Big(\sum_{q>Q}q^mK^q\rho^{q-1}\Big)\Big(\sum_{\boldsymbol{p}\in\mathcal{L}_\hbar;\,|\boldsymbol{s}|_\infty>a+1}\int_B\int_B\psi^\hbar(\boldsymbol{p}+\boldsymbol{y}-\boldsymbol{x})d\boldsymbol{y}d\boldsymbol{x}\Big),$$

where we have used the fact that for $|p|_\infty\geq a+1$, $|\boldsymbol{y}|,|\boldsymbol{x}|\leq 1$ we have $\psi^\hbar(\boldsymbol{p}+\boldsymbol{y}-\boldsymbol{x})<\rho$. Since $\rho<\frac{1}{K}$, the sum

$$\sum_{q>Q}q^mK^q\rho^{q-1}$$

is the tail of a convergent power series. On the other hand,

$$\sum_{\substack{\boldsymbol{p}\in\mathcal{L}_\hbar,\\|\boldsymbol{p}|_\infty>a+1}}\int_B\int_B\psi^\hbar(\boldsymbol{p}+\boldsymbol{y}-\boldsymbol{x})d\boldsymbol{y}d\boldsymbol{x}\leq\sum_{\boldsymbol{p}\in\mathcal{L}_\hbar}\int_{[-1,1]^m}\psi^\hbar(\boldsymbol{p}+\boldsymbol{y})\leq 2\int_{\mathbb{R}^m}\psi^\hbar(\boldsymbol{y})d\boldsymbol{y}\stackrel{(2.16)}{=}O(1).$$

This proves that $\sup_\hbar|S^\hbar_{>Q,\infty}|$ goes to zero as $Q\to\infty$ and completes the proof of (2.24). The claim (2.25) follows immediately from (2.24). This concludes the proof of Lemma 2.6 and of fact $\mathbf{F}_1$. □

Proof of $\mathbf{F}_2$. In [24] we have shown that the limit S^0 in (2.17)4 is the sum of a series

$$S^0=\sum_{q\geq 1}S^0_q,\;\;S^0_q=\int_{\mathbb{R}^m}\mathbb{E}\big[\,\rho^0_q(0)\rho^0_q(\boldsymbol{u})\,\big]d\boldsymbol{u}.$$

The equality (2.23) shows that $\bar{S}^0=S^0>0$. This concludes the proof of Proposition 2.5. □

2.5. **Proof of Theorem 1.2.** In [24] we have shown that, as $\hbar \to 0$, the random variables converge in law to a random variable $\sim \mathcal{N}(0, S^0)$.

As explained in Subsection 1.4, to conclude the proof of Theorem 1.2 it suffices to establish the asymptotic normality as $\hbar \to 0$ of the family

$$\zeta_q^\hbar = \frac{1}{(2N)^{m/2}} \int_{\widehat{B}_{2N_\hbar}} \rho_q(\boldsymbol{x}) d\boldsymbol{x}, \;\; \forall q \geq 1.$$

This follows from the fourth-moment theorem [26, Thm. 5.2.7], [28]. Here are the details.

Recall from [16, IV.1] that we have a surjective isometry $\Theta_q : \mathscr{X}^{\odot q} \to \mathscr{X}^{:q:}$, where $\mathscr{X}^{\odot q}$ is the q-th symmetric power and $\mathscr{X}^{:q:}$ is the q-th chaos component of $\widehat{\mathscr{X}}$. The multiple Ito integral $\boldsymbol{I}_q$ is then the map

$$\boldsymbol{I}_q = \frac{1}{\sqrt{q!}} \Theta_q.$$

We can write $\zeta_q^\hbar$ as a multiple Ito integral

$$\zeta_q^\hbar = \boldsymbol{I}_q\big[\, g_q^\hbar \,\big], \;\; g_q^\hbar \in \mathscr{X}^{\odot n}.$$

According to [26, Thm.5.2.7(v)], to prove that $\zeta_q^\hbar$ converge in law to a normal variable it suffices to show that

$$\lim_{\hbar \to 0} \| g_q^\hbar \otimes_r g_q^\hbar \|_{\mathscr{X}^{\otimes(2q-2r)}} = 0, \;\; \forall r = 1, \dots, q-1. \tag{2.32}$$

In our context, using the isometry $\boldsymbol{I}$ in (1.9) we can view $g_q^\hbar$ as a function

$$g_q^\hbar \in L^2\big(\, (\mathbb{R}^m \times \mathbb{R}^m)^q \,\big), \;\; g_q^\hbar = g_q^\hbar(\boldsymbol{z}_1, \dots, \boldsymbol{z}_q), \;\; \boldsymbol{z}_j \in \mathbb{R}^m \times \mathbb{R}^m,$$

and then

$$g_q^\hbar \otimes_r g_q^\hbar \in L^2\big(\, (\mathbb{R}^m \times \mathbb{R}^m)^{2(q-r)} \,\big), \;\; g_q^\hbar \otimes_r g_q^\hbar(\boldsymbol{z}_{q-r+1}, \boldsymbol{z}'_{q-r+1}, \dots, \boldsymbol{z}_q, \boldsymbol{z}'_q)$$

$$= \int_{(\mathbb{R}^m \times \mathbb{R}^m)^r} g_q^\hbar(\boldsymbol{z}_1, \dots, \boldsymbol{z}_q, \boldsymbol{z}_{q-r+1}, \dots, \boldsymbol{z}_q) g_q^\hbar(\boldsymbol{z}_1, \dots, \boldsymbol{z}_q, \boldsymbol{z}'_{q-r+1}, \dots, \boldsymbol{z}'_q) d\boldsymbol{z}_1 \cdots d\boldsymbol{z}_q.$$

To show (2.32) we invoke the arguments following the inequality (18) in the second step of the proof of [11, Prop. 2.4] which extend to the setup in this paper.

2.6. **Proof of Theorem 1.3.** We set

$$N_\hbar := \left\lceil \frac{r}{2\hbar} \right\rceil, \;\; s_\hbar := \frac{r}{2\hbar N_\hbar}.$$

Then $N_\hbar \in \mathbb{N}$,

$$\lim_{\hbar \to 0} N_\hbar = \infty, \;\; \widehat{B}_{2N_\hbar s_\hbar} = \widehat{B}_{\hbar^{-1} r} \subset \widehat{B}_{1/(2\hbar)}, \;\; \lim_{\hbar \to 0} s_\hbar = 1.$$

Thus, $\widehat{B}_{\hbar^{-1} r}$ is a cube, centered at 0 with vertices in the lattice $(s_\hbar \mathbb{Z})^m$ and $s(\hbar) \approx 1$ for $\hbar$ small.

To reach the conclusion (i) run the arguments in the the proof of Theorem 1.1 with the following modified notations: the box $\widehat{B}_{2N_\hbar}$ should be replaced with the box $\widehat{B}_{2N_\hbar s_\hbar} = s_\hbar \widehat{B}_{2N_\hbar}$, the lattice $\mathbb{A}^m$ in (2.13) replaced by $s_\hbar \mathbb{A}^m$, and $\widehat{B}(\boldsymbol{a})$ redefined as $s_\hbar \widehat{B}_1(\boldsymbol{a}) = \widehat{B}_{s_\hbar}(\boldsymbol{a})$.

To reach the conclusions (ii) and (iii) of Theorem 1.3 run the arguments in the subsections 2.4 and 2.5 with the following modified notations: the box $\widehat{B}_{2N_\hbar}$ should be replaced with the box $\widehat{B}_{2N_\hbar s_\hbar} = s_\hbar \widehat{B}_{2N_\hbar}$, in (2.26) the set $\mathcal{L}_\hbar$ should be redefined to be

$$\mathcal{L}_\hbar := s_\hbar\big(\mathbb{A}^m \cap \widehat{B}_{4N_\hbar}\,\big),$$

and the box B in (2.27) should be redefined to be $\widehat{B}_{s_\hbar} = s_\hbar \widehat{B}_1$.

Appendix A. Proof of Proposition 2.1

We will follow the strategy in the proof of [11, Prop. 1.1]. Some modifications are required since the random functions $Y^\hbar$ are not isotropic for $\hbar > 0$.

Denote by $p^\hbar_{\boldsymbol{x}}(-)$ and $p^\hbar_{\boldsymbol{x},\boldsymbol{y}}(-,-)$ the probability densities of the Gaussian vectors $\nabla Y^\hbar(\boldsymbol{x})$ and respectively $(\nabla Y^\hbar(\boldsymbol{x}), \nabla Y^\hbar(\boldsymbol{y}))$. For simplicity we denote by $|S|$ the Lebesgue volume of a Borel subset $S \subset \mathbb{R}^m$.

Due to the stationarity of $Y^\hbar$ it suffices to assume that the box B is centered at 0. The Gaussian random function $Y^\hbar$ is stationary. Using the Kac-Rice formula [2, Ch.11] or [6, Ch.6] we deduce that, $\forall \boldsymbol{v} \in \mathbb{R}^m$ we have

$$\mathbb{E}\big[\, \boldsymbol{Z}^\hbar(\boldsymbol{v}, B)\,\big] = \mathbb{E}\big[\, |\det \nabla^2 Y^\hbar(0)\,|\,\big] p_0(\boldsymbol{v})\, |B|, \tag{A.1}$$

$$\begin{aligned}\mathbb{E}\big[\, &\boldsymbol{Z}^\hbar(\boldsymbol{v}, B)\big(\boldsymbol{Z}^\hbar(\boldsymbol{v}, B) - 1\big)\,\big] \\ &= \int_{2B} \big| B \cap (B - \boldsymbol{y}) \big| \; \underbrace{\boldsymbol{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, |\det \nabla^2 Y^\hbar(0) \det \nabla^2 Y^\hbar(\boldsymbol{y})|\,\Big]}_{g^\hbar(\boldsymbol{v},\boldsymbol{y})} \; p_{0,\boldsymbol{y}}(\boldsymbol{v}, \boldsymbol{v}) d\boldsymbol{y},\end{aligned} \tag{A.2}$$

where, for typographical reasons, we denoted by $\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}$ the conditional expectation

$$\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}[-] = \mathbb{E}\big[\, - | C_{\boldsymbol{y}}(\boldsymbol{v})\,\big], \;\; C_{\boldsymbol{y}}(\boldsymbol{v}) := \{\nabla Y^\hbar(0) = \nabla Y^\hbar(\boldsymbol{y}) = \boldsymbol{v}\}.$$

The two sides of the equality (A.2) are simultaneously finite or infinite. Let us point out that the integrand on the right-hand side of this equality could blow-up at $\boldsymbol{y} = 0$ because the Gaussian vector $(\nabla Y^\hbar(0), \nabla Y^\hbar(0))$ is degenerate and therefore

$$\lim_{\boldsymbol{y} \to 0} p_{0,\boldsymbol{y}}(\boldsymbol{v}, \boldsymbol{v}) = \infty.$$

The most demanding part in the proof of Proposition 2.1 is showing that the right-hand side of (A.2) is finite. This boils down to understanding the singularity at the origin of the integrand in (A.2). In [23] we proved this fact in the case $\boldsymbol{v} = 0$. To deal with the general case we will use a blend of the ideas in [11] and [23].

Step 1. We will show that there exist $\hbar_1 > 0$, $r_1 > 0$ and $C_1 > 0$ such that for any $\hbar \leq h_1$ we have

$$p^\hbar_{0,\boldsymbol{y}}(0,0) < \infty, \;\; \forall \boldsymbol{y} \neq 0, \tag{A.3a}$$

$$0 < p_{0,\boldsymbol{y}}(\boldsymbol{v}, \boldsymbol{v}) \leq C_1 |\boldsymbol{y}|^{-m}, \;\; \forall 0 < |\boldsymbol{y}| < r_1, \;\; \forall \boldsymbol{v} \in \mathbb{R}^m. \tag{A.3b}$$

These two facts follow from [23, Lemma 3.5] and the obvious inequality

$$p_{0,\boldsymbol{y}}(\boldsymbol{v}, \boldsymbol{v}) \leq p_{0,\boldsymbol{y}}(0, 0).$$

Step 2. We will show that there exist $\hbar_2 > 0$, $r_2 > 0$ and $C_2 > 0$ such that for any $\hbar \leq h_1$ we have

$$|g^\hbar(\boldsymbol{v}, \boldsymbol{y})| \leq C_2 \|\boldsymbol{y}\|^2, \;\; \forall |\boldsymbol{y}| \leq r_2, \;\; \boldsymbol{v} \in \mathbb{R}^m. \tag{A.4}$$

We set

$$f^\hbar(\boldsymbol{y}, \boldsymbol{v}) := \boldsymbol{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, |\det \nabla^2 Y^\hbar(0)|^2\,\Big].$$

From the Cauchy inequality and the stationarity of $Y^\hbar$ we deduce

$$g^\hbar(\boldsymbol{v}, \boldsymbol{y})^2 \leq \boldsymbol{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, |\det \nabla^2 Y^\hbar(0)|^2\,\Big] \cdot \boldsymbol{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, |\det \nabla^2 Y^\hbar(\boldsymbol{y})|^2\,\Big]$$

$$= f^\hbar(\boldsymbol{y},\boldsymbol{v})f^\hbar(-\boldsymbol{y},\boldsymbol{v}) = f^\hbar(\boldsymbol{y},\boldsymbol{v})^2.$$

We now invoke the Hadamard's inequality [15, Thm. 7.8.1]: *if $A:\mathbb{R}^m\to\mathbb{R}^m$ is an $m\times m$ symmetric positive operator and $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_m\}$ is an orthonormal basis of $\mathbb{R}^m$, then*

$$\det A \leq \prod_{j=1}^m (A\boldsymbol{e}_j,\boldsymbol{e}_j).$$

Applying this inequality to $A=\nabla^2 Y^\hbar(0))^2$ and a fixed orthonormal basis $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_m\}$ such that

$$\boldsymbol{e}_1 := |\boldsymbol{y}|^{-1}\boldsymbol{y},$$

we deduce

$$|\det\nabla^2 Y^\hbar(0)|^2 \leq |\boldsymbol{y}|^{-2}\|\nabla^2 Y^y(0)\boldsymbol{y}\|^2\|\nabla^2 Y^\hbar(0)\|^{2(m-1)}.$$

Hence

$$\begin{aligned}|g^\hbar(\boldsymbol{v},\boldsymbol{y})| &\leq f^\hbar(\boldsymbol{v},\boldsymbol{y})\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\,|\det\nabla^2 Y^\hbar(0)|^2\,\Big]^2\\ &\leq |\boldsymbol{y}|^{-2}\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\,\|\nabla^2 Y^y(0)\boldsymbol{y}\|^4\,\Big]^{\frac{1}{2}}\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\,\|\nabla^2 Y^y(0)\|^{4(m-1)}\,\Big]^{\frac{1}{2}} \qquad (A.5)\end{aligned}$$

Now observe that

$$\|\nabla^2 Y^y(0)\boldsymbol{y}\|^2 = |\boldsymbol{y}|^2\sum_{j=1}^m Y^\hbar_{1j}(0)^2,$$

where, for any smooth function $F:\mathbb{R}^m\to\mathbb{R}$, we set

$$F_{i,j,\dots k} := \partial_{\boldsymbol{e}_i}\partial_{\boldsymbol{e}_j}\cdots\partial_{\boldsymbol{e}_k}F.$$

Thus

$$|\nabla^2 Y^y(0)\boldsymbol{y}|^4 \leq m|\boldsymbol{y}|^4\sum_{j=1}^m Y^\hbar_{1j}(0)^4,$$

$$\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\,|\nabla^2 Y^y(0)\boldsymbol{y}|^4\,\Big] \leq m|\boldsymbol{y}|^4\sum_{j=1}^m \mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\,Y^\hbar_{1j}(0)^4\,\Big],$$

$$g^\hbar(\boldsymbol{v},\boldsymbol{y}) \leq \sqrt{m}\left(\sum_{j=1}^m \mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\,Y^\hbar_{1j}(0)^4\,\Big]\right)^{\frac{1}{2}}\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\,\|\nabla^2 Y^\hbar(0)\|^{4(m-1)}\,\Big]^{\frac{1}{2}}. \qquad (A.6)$$

For each $j=1,\dots,m$ define the random function

$$F_j:[0,1]\to\mathbb{R},\;\; F_j(t)=Y^\hbar_j(t\boldsymbol{y}).$$

Then

$$F'_j(t) = |\boldsymbol{y}|Y^\hbar_{1,j}(t\boldsymbol{y}),\;\; F''_j(t) = |\boldsymbol{y}|^2 Y^\hbar_{1,1,j}(t\boldsymbol{y})$$

Using the Taylor formula with integral remainder we deduce

$$F_j(1)-F_j(0) = F'_j(0)+\int_0^1 F''_j(t)(1-t)dt,$$

i.e.,

$$Y^\hbar_j(\boldsymbol{y})-Y^\hbar_j(0) = Y^\hbar_{1j}(0)+|\boldsymbol{y}|\int_0^1 Y^\hbar_{11j}(t\boldsymbol{y})(t\boldsymbol{y})dt.$$

Hence

$$Y^{\hbar}_{1,j}(0) = Y^{\hbar}_j(\boldsymbol{y}) - Y^{\hbar}_j(0) - |\boldsymbol{y}| \int_0^1 Y^{\hbar}_{1,1,j}(t\boldsymbol{y})(t\boldsymbol{y})dt.$$

Setting $v_j := (\boldsymbol{v}, \boldsymbol{e}_j)$ and observing that under the condition $C_{\boldsymbol{y}}(\boldsymbol{v})$ we have

$$Y^{\hbar}_j(0) = Y^{\hbar}_j(\boldsymbol{y}) = v_j, \;\; \forall j = 1, \dots, m,$$

we deduce

$$\begin{aligned}\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, Y^{\hbar}_{1,j}(0)^4 \,\Big] &= \mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\left[\left(Y^{\hbar}_j(\boldsymbol{y}) - Y^{\hbar}_j(0) - |\boldsymbol{y}| \int_0^1 Y^{\hbar}_{1,1,j}(t\boldsymbol{y})(1-t)dt \right)^4 \,\Big|\, C_{\boldsymbol{y}}(\boldsymbol{v}) \right] \\ &= |\boldsymbol{y}|^4 \mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\left[\left(\int_0^1 Y^{\hbar}_{1,1,j}(t\boldsymbol{y})(1-t)dt \right)^4 \right] \leq |\boldsymbol{y}|^4 \mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\left[\int_0^1 \big| Y^{\hbar}_{1,1,j}(t\boldsymbol{y}) \big|^4 dt \right] \\ &= |\boldsymbol{y}|^4 \int_0^1 \mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, \big| Y^{\hbar}_{1,1,j}(t\boldsymbol{y}) \big|^4 \,\Big] dt.\end{aligned}$$

We conclude that

$$g^{\hbar}(\boldsymbol{v}, \boldsymbol{y}) \;\;\leq\;\; \sqrt{m}|\boldsymbol{y}|^2 \left(\int_0^1 \sum_{j=1}^m \mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, \big| Y^{\hbar}_{1,1,j}(t\boldsymbol{y}) \big|^4 \,\Big] dt \right)^{\frac{1}{2}} \mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, \|\nabla^2 Y^{\hbar}(0)\|^{4(m-1)} \,\Big]^{\frac{1}{2}}. \tag{A.7}$$

Step 2 will be completed once we prove the following result.

Lemma A.1. *There exist $\hbar_2 > 0$, $r_2 > 0$ and $C_2 > 0$ such that for any $\hbar \leq h_1$ and any $\boldsymbol{v} \in \mathbb{R}^m$ we have*

$$\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, \|\nabla^2 Y^{\hbar}(0)\|^{4(m-1)} \,\Big] \leq C_2(1 + |\boldsymbol{v}|)^{4(m-1)}, \;\; \forall \hbar < \hbar_2, \;\; |\boldsymbol{y}| < r_2, \tag{A.8a}$$

$$\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, \big| Y^{\hbar}_{1,1,j}(t\boldsymbol{y}) \big|^4 \,\Big] \leq C_2(1 + |\boldsymbol{v}|)^4, \;\; \forall j, \;\; \hbar < \hbar_2, \;\; |\boldsymbol{y}| < r_2, \;\; t \in [0,1]. \tag{A.8b}$$

Proof. The random matrix $\nabla^2 Y^{\hbar}$ conditioned by $C_{\boldsymbol{y}}(\boldsymbol{v})$ is *Gaussian*. The same is true of $Y^{\hbar}_{1,1,j}(t\boldsymbol{y})$ so it suffices to show that there exist $\hbar_2 > 0$, $r_2 > 0$ and $C_2 > 0$ such that for any $\hbar \leq h_1$ and any $\boldsymbol{v} \in \mathbb{R}^m$ we have

$$\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, \big| Y^{\hbar}_{i,j}(0) \big|^2 \,\Big] \leq C_2(1 + |\boldsymbol{v}|)^2, \;\; \forall i, j, \; \forall \hbar < \hbar_2, \;\; |\boldsymbol{y}| < r_2, \tag{A.9a}$$

$$\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\Big[\, \big| Y^{\hbar}_{1,1,j}(t\boldsymbol{y}) \big|^2 \,\Big] \leq C_2(1 + |\boldsymbol{v}|)^2, \;\; \forall j, \; \forall \hbar < \hbar_2, \; |\boldsymbol{y}| < r_2, \;\; t \in [0,1]. \tag{A.9b}$$

As in [23] we introduce the index sets

$$\boldsymbol{J} = \{\pm 1, \pm 2, \dots, m\}, \;\; J_{\pm} = \big\{ j \in \boldsymbol{J}; \;\; \pm j > 0 \,\big\}.$$

We consider the $\mathbb{R}^m \oplus \mathbb{R}^m$ valued Random Gaussian vector $G^{\hbar}(t) = G^{\hbar}_- \oplus G^{\hbar}_+$, $t \in [0,1]$, where

$$G^{\hbar}_- := \sum_{i=1}^m G^{\hbar}_{-i} \boldsymbol{e}_i = \nabla Y^{\hbar}(0), \;\; G^{\hbar}_+ := \sum_{j=1}^m G^{\hbar}_j \boldsymbol{e}_j = \nabla Y^{\hbar}(t\boldsymbol{y}).$$

The covariance form of this vector is the $2m \times 2m$ symmetric matrix

$$S_{\hbar} = S_{\hbar}(t) = \left[\begin{array}{cc} S^{-,-}_{\hbar} & S^{-,+}_{\hbar} \\ & \\ S^{+,-}_{\hbar} & S^{+,+}_{\hbar} \end{array} \right] = \left[\begin{array}{cc} A_{\hbar} & B_{\hbar} \\ B_{\hbar} & A_{\hbar} \end{array} \right],$$

$$A_{\hbar} = -\nabla^2 V^{\hbar}(0), \;\; B_{\hbar} = B_{\hbar}(t, \boldsymbol{y}) = -\nabla^2 V^{\hbar}(t\boldsymbol{y}).$$

From (A.3b) we deduce that $S_\hbar$ is invertible if $\hbar < \hbar_1$ and $|\boldsymbol{y}| \leq r_1$. Its inverse has the block form

$$\left[\begin{array}{cc} C_\hbar(t) & -D_\hbar(t) \\ -D_\hbar(t) & C_\hbar(t) \end{array}\right] = \left[\begin{array}{cc} C_\hbar(t) & -A_\hbar^{-1}B_\hbar(t)C_\hbar(t) \\ -A_\hbar^{-1}B_\hbar(t)C_\hbar(t) & C_\hbar(t) \end{array}\right],$$

where,

$$C_\hbar(t) = C_\hbar(t,\boldsymbol{y}) := \big(\, A_\hbar - B_\hbar(t,\boldsymbol{y})A_\hbar^{-1}B_\hbar(t,\boldsymbol{y})\,\big)^{-1}.$$

In [23, Lemma 3.6] we have shown that there exists $\hbar_2 \in (0,\hbar_1)$ such that the $m \times m$ matrix

$$K^\hbar := (K^\hbar_{ij})_{1\leq i,j\leq m},\;\; K^\hbar_{ij} := V^\hbar_{1,1,i,j}(0)$$

is invertible for $\hbar \in [0,\hbar_2]$ and

$$\lim_{t\to 0} t^2 C_\hbar(t,\boldsymbol{y}) = (K^\hbar)^{-1},\;\; \textit{uniformly} \text{ in } \hbar\in[0,\hbar_2] \text{ and } |\boldsymbol{y}|\leq r_1\;. \tag{A.10}$$

Next, observe that

$$\begin{aligned} C_\hbar(t,\boldsymbol{y}) - D_\hbar(t,\boldsymbol{y}) &= A_\hbar^{-1}\big(\, A_\hbar - B_\hbar(t,\boldsymbol{y})\,\big)C_\hbar(t,\boldsymbol{y}) \\ &= A_\hbar^{-1}\frac{1}{t^2}\big(\, A_\hbar - B_\hbar(t,\boldsymbol{y})\,\big)t^2 C_\hbar(t,\boldsymbol{y}), \end{aligned}$$

and

$$\frac{1}{t^2}\big(\, A_\hbar - B_\hbar(t,\boldsymbol{y})\,\big) = \frac{1}{t^2}\Big(\, \nabla^2 V^\hbar(t\boldsymbol{y}) - \nabla^2 V^\hbar(0)\,\Big)$$

Since the function

$$\boldsymbol{x} \mapsto \nabla^2 V^\hbar(\boldsymbol{x})$$

is even and $V^\hbar \to V^0$ in C^∞ as $\hbar \to 0$ we deduce that the limit

$$\lim_{t\to 0}\frac{1}{t^2}\Big(\, \nabla^2 V^\hbar(t\boldsymbol{y}) - \nabla^2 V^\hbar(0)\,\Big)$$

exists, it is finite and it is *uniform* in $\hbar\in[0,\hbar_2]$ and $|\boldsymbol{y}|\leq r_1$. Using (A.10) we conclude that there exists a constant $c_1 > 0$ such that

$$\|C_\hbar(t,\boldsymbol{y}) - D_\hbar(t,\boldsymbol{y})\| \leq c_1,\;\; \forall t\in[0,1],\;\; \hbar\in[0,\hbar_2],\;\; |\boldsymbol{y}|\leq r_1. \tag{A.11}$$

We can now prove (A.9a) and (A.9b).

Proof of (A.9a). Fix $i_0, j_0 \in \{1,\dots,m\}$. The random variable $Y^\hbar_{i_0,j_0}(0)$, conditioned by $C_{\boldsymbol{y}}(\boldsymbol{v})$, is a normal random variable $\bar{Y}^\hbar_{i_0,j_0}$, and its mean and variance are determined by the regression formula, [6, Prop. 1.2]. To apply this formula we need to compute the correlations between $Y^\hbar_{i_0,j_0}(0)$ and $G^\hbar$. These are given by the expectations

$$\boldsymbol{\Xi}^\hbar_j = \boldsymbol{\Xi}^\hbar_j(\boldsymbol{y}) = \mathbb{E}\big[\, Y^\hbar_{i_0,j_0}(0)G_j(1)\,\big],\;\; j\in\boldsymbol{J}.$$

We have

$$\boldsymbol{\Xi}^\hbar_j(\boldsymbol{y}) = \begin{cases} V^\hbar_{i_0,j_0,|j|}(0), & j\in\boldsymbol{J}_- \\ V^\hbar_{i_0,j_0,j}(\boldsymbol{y}), & j\in\boldsymbol{J}_+. \end{cases}$$

We regard the collection $(\boldsymbol{\Xi}^\hbar_j(\boldsymbol{y}))_{j\in\boldsymbol{J}}$ as a linear map

$$\boldsymbol{\Xi}^\hbar(\boldsymbol{y}) : \mathbb{R}^m \oplus \mathbb{R}^m \to \mathbb{R},\;\; \boldsymbol{\Xi}^\hbar\big(\,(z_j)_{j\in\boldsymbol{J}}\,\big) = \sum_{j\in\boldsymbol{J}} \boldsymbol{\Xi}^\hbar_j(\boldsymbol{y})z_j.$$

In particular, we think of $\boldsymbol{\Xi}^\hbar$ as a *row* vector so its transpose $(\boldsymbol{\Xi}^\hbar)^\top$ is a *column* vector.

Observe that since the function $V^\hbar$ is even, the third order derivative $V^\hbar_{ijk}$ are odd functions. Thus

$$V^\hbar_{i_0,j_0,j}(0) = 0$$

and there exists $r_2 \in (0, r_1)$ and $c_2 > 0$ such that

$$|V^{\hbar}_{i,j,k}(\boldsymbol{y})| \leq c_2|\boldsymbol{y}|, \;\; \forall i,j,k, \; \forall \hbar \in [0,\hbar_2], \; |\boldsymbol{y}| \leq r_2.$$

Hence

$$\|\boldsymbol{\Xi}^{\hbar}(\boldsymbol{y})\| \leq c_2|\boldsymbol{y}|, \; \forall \hbar \in [0,\hbar_2], \; |\boldsymbol{y}| \leq r_2. \tag{A.12}$$

Denote by $\hat{\boldsymbol{v}} \in \mathbb{R}^m \oplus \mathbb{R}^m$ the vector $\boldsymbol{v} \oplus \boldsymbol{v}$.

According to the regression formula, the mean of the conditioned random variable $\bar{Y}^{\hbar}_{i_0,j_0}$ is

$$\mathbb{E}\big[\,\bar{Y}^{\hbar}_{i_0,j_0}\,\big] = -\boldsymbol{\Xi}^{\hbar}(\boldsymbol{y})\big(\,S_{\hbar}^{-1}\hat{\boldsymbol{v}}\,\big) = \boldsymbol{\Xi}^{\hbar}(\boldsymbol{y})\big(\,(C_\hbar - D_\hbar)\boldsymbol{v} \oplus (C_\hbar - D_\hbar)\boldsymbol{v}\,\big).$$

Using (A.11) and (A.12) we deduce that there exists $c_3 > 0$ such that

$$\Big|\mathbb{E}_{\boldsymbol{y},\boldsymbol{v}}\big[\,Y^{\hbar}_{i_0,j_0}\,\big]\Big| \leq c_3|\boldsymbol{y}||\boldsymbol{v}|, \;\; \forall \hbar \in [0,\hbar_2], \;\; |\boldsymbol{y}| \leq r_2, \; \boldsymbol{v} \in \mathbb{R}^m. \tag{A.13}$$

According to the regression formula, the variance of the conditioned random variable $\bar{Y}^{\hbar}_{i_0,j_0}$ is

$$\boldsymbol{var}\big[\,\bar{Y}^{\hbar}_{i_0,j_0}\,\big] = \boldsymbol{var}\big[\,Y^{\hbar}_{i_0,j_0}\,\big] - \boldsymbol{\Xi}^{\hbar}(\boldsymbol{y})S_{\hbar}^{-1}(\boldsymbol{\Xi}^{\hbar}(\boldsymbol{y}))^{\top} = V^{\hbar}_{i_0,j_0,i_0,j_0}(0) - \boldsymbol{\Xi}^{\hbar}(\boldsymbol{y})S_{\hbar}^{-1}(\boldsymbol{\Xi}^{\hbar}(\boldsymbol{y}))^{\top}$$

Using (1.8), (A.11) and (A.12) we deduce that there exists $c_4 > 0$ such that

$$\boldsymbol{var}_{\boldsymbol{y},\boldsymbol{v}}\big[\,Y^{\hbar}_{i_0,j_0}\,\big] \leq c_4|\boldsymbol{v}|, \;\; \forall \hbar \in [0,\hbar_2], \;\; |\boldsymbol{y}| \leq r_2, \; \boldsymbol{v} \in \mathbb{R}^m. \tag{A.14}$$

The inequality (A.9a) now follows from (A.13) and (A.14).

Proof of (A.9b). Fix $j_0 \in \{1,2,\dots,m\}$ The random variable $Y^{\hbar}_{1,1,j_0}(t\boldsymbol{y})$, conditioned by $C_{\boldsymbol{y}}(\boldsymbol{v})$ is a normal random variable $\bar{Y}^{\hbar}_{1,1,j_0}$. To describe its mean and its variance we need to compute the correlations

$$\boldsymbol{\Omega}^{\hbar}_j(t,\boldsymbol{y}) := \mathbb{E}\big[\,Y^{\hbar}_{1,1,j_0}(t\boldsymbol{y})G^{\hbar}_j\,\big] = \begin{cases} -V^{\hbar}_{1,1,j_0,|j|}(t\boldsymbol{y}), & j \in \boldsymbol{J}_- \\ -V^{\hbar}_{1,1,j_0,j}\big((1-t)\boldsymbol{y}\big), & j \in \boldsymbol{J}_+. \end{cases}.$$

Again, we think of the collection $(\boldsymbol{\Omega}^{\hbar}_j(t,\boldsymbol{y}))_{j\in\boldsymbol{J}}$ as defining a linear map

$$\boldsymbol{\Omega}^{\hbar} : \mathbb{R}^m \oplus \mathbb{R}^m \to \mathbb{R}.$$

The row vector $\boldsymbol{\Omega}^{\hbar}$ splits as a direct sum of row vectors

$$\boldsymbol{\Omega}^{\hbar} = \boldsymbol{\Omega}^{\hbar}_- \oplus \boldsymbol{\Omega}^{\hbar}_+, \;\; \boldsymbol{\Omega}^{\hbar}_{\pm} = \big(\,\boldsymbol{\Omega}^{\hbar}_j\,\big)_{j\in\boldsymbol{J}_{\pm}}.$$

The mean of the random variable $\bar{Y}^{\hbar}_{1,1,j_0}$ is

$$\mathbb{E}\big[\,\bar{Y}^{\hbar}_{1,1,j_0}\,\big] = -\boldsymbol{\Omega}^{\hbar}\big(\,(C_\hbar - D_\hbar)\boldsymbol{v} \oplus (C_\hbar - D_\hbar)\boldsymbol{v}\,\big).$$

We conclude as befgore that

$$\Big|\,\mathbb{E}\big[\,Y^{\hbar}_{1,1,j_0}(t\boldsymbol{y})\,\big]\,\Big| = O\big(|\boldsymbol{v}|\cdot|\boldsymbol{y}|\big), \;\; |\boldsymbol{y}| \leq r_2, \;\; \boldsymbol{v} \in \mathbb{R}^m, \;\; t \in [0,1], \tag{A.15}$$

where *the constant implied by the O-symbol is independent of $\hbar$ and t. This convention will stay in place for the remainder of this proof.*

Next, we have

$$\begin{aligned} \boldsymbol{var}\big[\,\bar{Y}^{\hbar}_{1,1,j_0}\,\big] &= \boldsymbol{var}\big[\,Y^{\hbar}_{1,1,j_0}(t\boldsymbol{y})\,\big] - \boldsymbol{\Omega}^{\hbar}S_{\hbar}^{-1}(\boldsymbol{\Omega}^{\hbar})^{\top} \\ &= V^{\hbar}_{1,1,j_0,1,1,j_0}(t\boldsymbol{y}) - \big[\,\boldsymbol{\Omega}^{\hbar}_- \;\; \boldsymbol{\Omega}^{\hbar}_+\,\big]\cdot\begin{bmatrix} C_\hbar & -D_\hbar \\ -D_\hbar & C_\hbar \end{bmatrix}\cdot\begin{bmatrix} (\boldsymbol{\Omega}^{\hbar}_-)^{\top} \\ (\boldsymbol{\Omega}^{\hbar}_+)^{\top} \end{bmatrix} \\ &= V^{\hbar}_{1,1,1,1,j_0,j_0}(t\boldsymbol{y}) - \big[\,\boldsymbol{\Omega}^{\hbar}_- \;\; \boldsymbol{\Omega}^{\hbar}_+\,\big]\cdot\begin{bmatrix} C_\hbar(\boldsymbol{\Omega}^{\hbar}_-)^{\top} - D_\hbar(\boldsymbol{\Omega}^{\hbar}_+)^{\top} \\ -D_\hbar(\boldsymbol{\Omega}^{\hbar}_-)^{\top} + C_\hbar(\boldsymbol{\Omega}^{\hbar}_+)^{\top} \end{bmatrix} \end{aligned}$$

$$= V^{\hbar}_{1,1,1,1,j_0,j_0}(t\boldsymbol{y}) - \Big(\boldsymbol{\Omega}^{\hbar}_{-} C_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{-})^{\top} - \boldsymbol{\Omega}^{\hbar}_{-} D_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{+})^{\top} - \boldsymbol{\Omega}^{\hbar}_{+} D_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{-})^{\top} + \boldsymbol{\Omega}^{\hbar}_{+} C_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{+})^{\top}\Big)$$

$(D_{\hbar} = C_{\hbar} + O(1))$

$$= V^{\hbar}_{1,1,1,1,j_0,j_0}(t\boldsymbol{y}) - \Big(\boldsymbol{\Omega}^{\hbar}_{-} C_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{-})^{\top} - \boldsymbol{\Omega}^{\hbar}_{-} C_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{+})^{\top} - \boldsymbol{\Omega}^{\hbar}_{+} C_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{-})^{\top} + \boldsymbol{\Omega}^{\hbar}_{+} C_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{+})^{\top}\Big) + O(1)$$

$$= V^{\hbar}_{1,1,1,1,j_0,j_0}(t\boldsymbol{y}) - \Big(\big(\boldsymbol{\Omega}^{\hbar}_{-} - \boldsymbol{\Omega}^{\hbar}_{+}\big) C_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{-})^{\top} - \big(\boldsymbol{\Omega}^{\hbar}_{-} - \boldsymbol{\Omega}^{\hbar}_{+}\big) C_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{+})^{\top}\Big) + O(1)$$

$$= V^{\hbar}_{1,1,1,1,j_0,j_0}(t\boldsymbol{y}) - (\boldsymbol{\Omega}^{\hbar}_{-} - \boldsymbol{\Omega}^{\hbar}_{+}) C_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{-} - \boldsymbol{\Omega}^{\hbar}_{+})^{\top} + O(1)$$

Now observe that $\Phi^{\hbar} = \boldsymbol{\Omega}^{\hbar}_{-} - \boldsymbol{\Omega}^{\hbar}_{+}$ is the vector in $\mathbb{R}^m$ with components

$$\Phi^{\hbar}_j(t,\boldsymbol{y}) = V^{\hbar}_{1,1,j_0,j}\big((1-t)\boldsymbol{y}\big) - V^{\hbar}_{1,1,j_0,j}\big((t\boldsymbol{y}\big),$$

that satisfy

$$\big|\,\Phi^{\hbar}_j(t,\boldsymbol{y})\,\big| = O(|\boldsymbol{y}|), \ \ \forall j.$$

From (A.10) we deduce

$$\|C_{\hbar}(t,\boldsymbol{y})\| = O\big(|\boldsymbol{y}|^{-2}\big)$$

so that

$$\big|\,(\boldsymbol{\Omega}^{\hbar}_{-} - \boldsymbol{\Omega}^{\hbar}_{+}) C_{\hbar}(\boldsymbol{\Omega}^{\hbar}_{-} - \boldsymbol{\Omega}^{\hbar}_{+})^{\top}\,\big| = O(1).$$

This completes the proof of (A.9b) and thus of Lemma A.1 and of statement (i) in Proposition 2.1. □

Step 3. The map

$$\boldsymbol{v} \mapsto \mathbb{E}\big[\,\boldsymbol{Z}^{\hbar}(\boldsymbol{v},B)\big(\boldsymbol{Z}^{\hbar}(\boldsymbol{v},B) - 1\big)\,\big]$$

is continuous. This follows by using the argument in *Point 2* in the proof of [11, Prop. 1.1]. Combined with (A.1) will prove the statement (ii) in Proposition 2.1.

Step 4. Prove the statement (iii) in Proposition 2.1. This follows by using the argument in *Point 3* in the proof of [11, Prop. 1.1].

Step 5. Using the results in **Step 1** and **Step 2** and the dominated convergence theorem we obtain the statement (iv). □

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Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-4618.
E-mail address: nicolaescu.1@nd.edu
URL: http://www.nd.edu/~lnicolae/