THE CROFTON FORMULA

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ABSTRACT. I discuss the classical Crofton formula for curves in the plane.

1. AFFINE LINES IN THE PLANE

Denote by $\mathbf{Graff}_1(\mathbb{R}^2)$ the set of affine lines in the plane. For any $L \in \mathbf{Graff}_1$ denote by $[L]^\perp$ the line through the origin perpendicular to L. The resulting map

$$\mathbf{Graff}_1(\mathbb{R}^2) \ni L \mapsto [L]^\perp \in \mathbb{RP}^1$$

defines a structure of real line bundle $\mathbf{Graff}_1(\mathbb{R}^2) \to \mathbb{RP}^1$ canonically isomorphic to the tautological line bundle $\mathcal{U}_1 \to \mathbb{RP}^1$. More precisely, the isomorphism associates to a line $\ell \in \mathbb{R}^1$ and a point $\boldsymbol{p} \in \ell$ the affine line L through $\boldsymbol{p}$ and perpendicular to ℓ.

We can identify $\mathbf{Graff}_1(\mathbb{R}^2)$ with the Möbius band $\mathbb{R}^2/\sim$, where

$$(\varphi, t) \sim (\varphi + n\pi, (-1)^n t), \;\; n \in \mathbb{Z}.$$

More precisely, to the equivalence class $[\varphi, t]$ we associate the line

$$L_{[\varphi,t]} = \big\{ (x,y) \in \mathbb{R}^2; \;\; t = x\cos\varphi + y\sin\varphi \;\big\}.$$

We have a metric $g = d\varphi^2 + dt^2$ on $\mathbf{Graff}_1(\mathbb{R}^)$ with volume density $|dV_g| = |d\varphi \wedge dt|$. This density is invariant with respect to the group of affine isometries of $\mathbb{R}^2$ which acts in the obvious way on $\mathbf{Graff}_1(\mathbb{R})$.

2. THE CROFTON FORMULA

Suppose that C is simple closed C^2-curve in $\mathbb{R}^2$ parametrized by arclength

$$[0,S] \ni s \mapsto \big(\, x(s), y(s) \,\big) \in \mathbb{R}^2, \;\; |x'(s)|^2 + |y'(s)|^2 = 1, \;\; S := \mathrm{length}\,(C).$$

For any affine line $L \in \mathbf{Graff}_1(\mathbb{R}^2)$ we denote by $|C \cap L| \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ the cardinality of the intersection $L \cap C$

Theorem 2.1 (Crofton)**.** *The function*

$$\mathbf{Graff}_1(\mathbb{R}^2) \ni L \mapsto |L \cap C| \in \mathbb{R}$$

is measurable and

$$\int_{\mathbf{Graff}_1(\mathbb{R}^2)} |L \cap C| \; |dV_g(L)| = 2\mathrm{length}\,(C).$$

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Proof. Consider the incidence set

$\mathcal{I}(C) = \big\{ (\boldsymbol{p}, L) \in C \times \mathbf{Graff}_1(\mathbb{R}^2);\;\; \boldsymbol{p} \in L \;\big\} = \big\{ (s,\varphi,t) \in [0,S] \times [0,\pi) \times \mathbb{R};\;\; t = x(s)\cos\varphi + y(s)\sin\varphi \;\big\}.$

We can regard $\mathcal{I}(C)$ as the graph of the map

$$[0,S] \times [0,\pi) \to \mathbb{R},\;\; (s,\varphi) \mapsto t(s,\varphi) = x(s)\cos\varphi + y(s)\sin\varphi.$$

Viewed as a submanifold of $\mathbb{R}^3$ is equipped with the induced metric

$$\hat{g} = dt^2 + ds^2 + d\varphi^2 = \big(\, dt(s,\varphi)\,\big)^2 + ds^2 + d\varphi^2.$$

We use (s,φ) as coordinates on $\mathcal{I}(C)$ so that the vector fields $\partial_s, \partial_\varphi$ can be identified with the 3-dimensional vector fields

$$\partial_s = (1,0,\partial_s t),\;\; \partial_\varphi = (0,1,\partial_\varphi t).$$

Consider the maps

$$\begin{array}{ccc} & \mathcal{I}(C) & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ C & & \mathbf{Graff}_1(\mathbb{R}^2), \end{array} \qquad \alpha(\boldsymbol{p},L) = \boldsymbol{p},\;\; \beta(\boldsymbol{p},L) = L.$$

In coordinates we have

$$\alpha(s,\varphi) = s,\;\; \beta(s,\varphi) = (\varphi, t(s,\varphi)). \tag{2.1}$$

Note that the fiber $\beta^{-1}(L)$ can be identified with the set $L \cap C$. The fiber of β over $\boldsymbol{p} \in C$ can be identified with $\mathbb{RP}^1(\boldsymbol{p}) \subset \mathbf{Graff}_1(\mathbb{R}^2)$, the space of affine lines through $\boldsymbol{p}$.

Using the coarea formula [1] we deduce

$$\int_{\mathbf{Graff}_1(\mathbb{R}^2)} |L \cap C\,|dV_g| = \int_{\mathcal{I}(C)} J_\beta |dV_{\hat{g}}| = \int_C \left(\int_{\mathbb{RP}^1(\boldsymbol{p})} \frac{J_\beta}{J_\alpha} d\sigma_{\boldsymbol{p}} \right) ds, \tag{2.2}$$

where J_α, J_β are the Jacobians of α and respectively β, and $d\sigma_{\boldsymbol{p}}$ denotes the arctlength along $\mathbb{RP}^1(\boldsymbol{p})$ with respect to the metric induced by the metric $\hat{g}$ on $\mathcal{I}(C)$.

Using [1, Lemma 1.2] we deduce

$$J_\beta^2 = \frac{\mathbb{G}_g(\beta_*\partial_s, \beta_*\partial_\varphi)}{\mathbb{G}_{\hat{g}}(\partial_s, \partial_\varphi)},\;\; J_\alpha^2 = \frac{|\alpha_*\partial_s|^2 \cdot |\partial_\varphi|^2_{\hat{g}}}{\mathbb{G}_{\hat{g}}(\partial_s, \partial_\varphi)}.$$

Hence

$$\frac{J_\beta^2}{J_\alpha^2} = \frac{\mathbb{G}_g(\beta_*\partial_s, \beta_*\partial_\varphi)}{|\alpha_*\partial_s|^2 \cdot |\partial_\varphi|^2_{\hat{g}}} = \frac{\mathbb{G}_g(\beta_*\partial_s, \beta_*\partial_\varphi)}{|\partial_\varphi|^2_{\hat{g}}}.$$

Using (2.1) we deduce

$$|\partial_\varphi|^2_{\hat{g}} = 1 + |\partial_\varphi t|^2,\;\; \mathbb{G}_g(\beta_*\partial_s, \beta_*\partial_\varphi) = \det \begin{bmatrix} |\beta_*\partial_s|^2_g & g(\beta_*\partial_s, \beta_*\partial_\varphi) \\ g(\beta_*\partial_s, \beta_*\partial_\varphi) & |\beta_*\partial_\varphi|^2_g \end{bmatrix}$$

$$= \det \begin{bmatrix} |\partial_s t|^2 & \partial_s t \cdot \partial_\varphi t \\ \partial_s t \cdot \partial_\varphi t & 1 + |\partial_\varphi t|^2 \end{bmatrix} = |\partial_s t|^2.$$

If $\boldsymbol{p} = (x(s_0), y(s_0))$ then the fiber $\mathbb{RP}^1(\boldsymbol{p})$ admits the parametrization

$$[0,\pi] \ni \varphi \mapsto (s_0, \varphi, t(s_0,\varphi)) \in \mathcal{I}(C)$$

so that

$$|d\sigma_{\boldsymbol{p}}|^2 = (1 + |\partial_\varphi t|^2)|d\varphi|^2.$$

Hence

$$\frac{J_\beta}{J_\alpha}|d\sigma_{\boldsymbol{p}}| = |\partial_s t||d\varphi|.$$

Hence

$$\int_{\mathbb{RP}^1(\boldsymbol{p})} \frac{J_\beta}{J_\alpha}|d\sigma_{\boldsymbol{p}}| = \int_0^\pi |x'(s_0)\cos\varphi + y'(s_0)\sin\varphi|d\varphi = \frac{1}{2}\int_0^{2\pi} |x'(s_0)\cos\varphi + y'(s_0)|d\varphi.$$

At this point we want to invoke the following technical result whose proof we postpone.

Lemma 2.2. *Suppose that $\boldsymbol{v}_0 \in \mathbb{R}^2$ is a nonzero vector. Then*

$$\int_{|\boldsymbol{r}|=1} |\boldsymbol{v}_0\cdot\boldsymbol{r}|\;|d\boldsymbol{r}| = 4|\boldsymbol{v}_0|.$$

□

If we use Lemma 2.2 with $\boldsymbol{v}_0 = (x'(s_0), y'(s_0))$ we deduce

$$\int_0^{2\pi} |x'(s_0)\cos\varphi + y'(s_0)\sin\varphi|d\varphi = 4,\quad \int_{\mathbb{RP}^1(\boldsymbol{p})} \frac{J_\beta}{J_\alpha}|d\sigma_{\boldsymbol{p}}| = 2.$$

Using the last equality in (2.2) we conclude

$$\int_{\mathbf{Graff}_1(\mathbb{R}^2)} |L\cap C\;|dV_g| = 2\int_C ds = 2\mathrm{length}\,(C).$$

□

Proof of Lemma 2.2 Choose a new orthonormal basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ of $\mathbb{R}^2$ such that $\boldsymbol{v}_0 = |\boldsymbol{v}_0|\boldsymbol{e}_1$. We denote by (x_1, x_2) the resulting orthonormal coordinates. In these coordinates

$$\boldsymbol{v}_0 = (|\boldsymbol{v}_0|, 0)$$

and if $\boldsymbol{r} = (\cos\theta, \sin\theta)$ then

$$|\boldsymbol{v}_0\cdot\boldsymbol{r}| = |\boldsymbol{v}_0|\cdot|\cos\theta|$$

and thus

$$\int_{|\boldsymbol{r}|=1} |\boldsymbol{v}_0\cdot\boldsymbol{r}|\;|d\boldsymbol{r}| = |\boldsymbol{v}_0|\int_0^{2\pi} |\cos\theta|d\theta = 4|\boldsymbol{v}_0|.$$

□

References

[1] L.I. Nicolaescu: The coarea formula, notes for a seminar.

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