REGULARIZATION OF DIVERGENT SERIES AND TAUBERIAN THEOREMS

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Introduction

The concepts of convergence and divergence, while not defined explicitly until early 19th century, have been studied since the third century BC. Throughout this time, mathematicians have largely focused upon convergent sequences and series, having seemingly little analytical reason to study series that diverged. However, as the area of mathematical analysis developed over the past few centuries, studying properties of divergent series and methods that sum divergent series has produced results that are very useful in the areas of harmonic analysis, number theory, and combinatorics/probability theory. This paper is essentially an investigation of such methods–called summation methods–and includes classical examples, as well as proofs of general properties about these methods.

The first section of the paper contains definitions and basic terminology used throughout the paper, as well as a brief commentary on summation methods and the divergent series to which they are applied. Since all the summation methods within this paper will all essentially involve the process of averaging, a definition of weighted averaging is also provided.

Within section two of the paper, we present classical examples of summation methods, starting with the identity summation method. Next, the summation method involving standard averaging by arithmetic means, called the Cesàro method, is discussed. After this point, more involved summation methods are discussed at length, including Hölder methods, higher order Cesàro means, and the Euler method. For each of these summation methods, its corresponding collection of averaging weights can be organized as an infinite matrix, and is listed explicitly as such when possible. These matrices are afterwards referred to as "averaging" matrices.

In the latter part of this section, we present three theorems pertaining to averaging matrices–the first states that the product of any two averaging matrices will also be an averaging matrix. The second theorem gives a necessary and sufficient condition that ensures a matrix will be "regular"–that is, it will not affect the limit of series that is convergent in the normal sense–a property that is typically desirable in summation methods. Finally we offer a theorem that proves the product of two regular averaging matrices is also a regular averaging matrix.

The section is concluded with a discussion of a summation method unlike those mentioned before. This method, called the Abel method, is a less obvious averaging method whose collection weights cannot be organized within an infinite matrix. Rather, to each real number in the interval $(0, 1)$ there corresponds a sequence of averaging weights–thus, we have a continuous, uncountable collection of weights for the Abel method.

The third section of the paper includes two Abelian theorems. An Abelian theorem relates the relative "strengths" of two specific summation methods–we say that a summation method X is "stronger" than another method Y, if it sums all Y-summable series and possibly more. The first Abelian theorem included in the section states that the Abel method is stronger than the Cesàro method, and the second that the Cesàro method is stronger than the identity method (convergence in the normal sense).

The final section of the paper contains two Tauberian theorems, which are partial converses to the Abelian theorems. That is, given summation methods X and Y with X stronger than Y, a Tauberian theorem specifies what additional properties a Y-summable series must possess to be X-summable as well. The two Tauberian theorems included in this section are exactly the partial converses to the two Abelian theorems listed in section three. The proofs of the Tauberian theorems require the most sophisticated mathematics seen within this dissertation, and are quite involved. For further information regarding this topic, the reader is encouraged to turn to the text that was central in the writing of this paper, G.H. Hardy's "Divergent Series".

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1. The Basics

We begin by briefly covering basic concepts involving sequences and series, as well as defining general terminology. Given any series, $\sum_{k\geq 0} a_k$, we say that such a series converges if the sequence of partial sums $s_n = \sum_{k\geq 0}^{n} a_k$ converges to some limit as n approaches infinity.

For every set A we denote by $\mathbb{R}^A$ the set of functions $A \to \mathbb{R}$. We define

$$\boldsymbol{Seq} := \mathbb{R}^{\mathbb{N}},$$

where $\mathbb{N}$ denotes the set of non-negative integers. Denote by $\boldsymbol{Seq}_c$ the subspace of $\boldsymbol{Seq}$ consisting of converging sequences. We regard the limit of a convergent sequence as a linear map $\lim : \boldsymbol{Seq}_c \to \mathbb{R}$.

Definition 1.1. A *summation method* is a quadruple $(\mathcal{T}, \mathcal{X}, U, u_0)$ where

- $\mathcal{X}$ is a subspace of $\boldsymbol{Seq}$,
- U is a topological space, $u_0 \in U$,

- $\mathcal{T}$ is a linear operator $\mathcal{T} : \mathcal{X} \to \mathbb{R}^{U_*}$, $U_* = U \setminus \{u_0\}$

such that the following hold:

(a) $\boldsymbol{Seq}_c \subset \mathcal{X}$
(b) For any sequence $\boldsymbol{s} \in \mathcal{X}$, the functions $\mathcal{T}\boldsymbol{s} : U_* \to \mathbb{R}$ has a finite limit at u_0,

Typically, we will refer to the $(\mathcal{T}, \mathcal{X}, U, u_0)$ method as the $\mathcal{T}$-method. The vector space $\mathcal{X}$ is called the space of $\mathcal{T}$-*convergent sequences*. The method $\mathcal{T}$ is said to be *regular* if $\forall \boldsymbol{s} \in \boldsymbol{Seq}_c$ we have $\lim_{n\to\infty} \boldsymbol{s} = \lim_{u\to u_0} \mathcal{T}\boldsymbol{s}(u)$ □

For any *summation method* $\mathcal{T}$ discussed within this paper, consisting of the quadruple $(\mathcal{T}, \mathcal{X}, U, u_0)$ as stated above, the topological space U will be one of the following two types.

In the first case, we take U to be the compactification

$$U = \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}.$$

the transformation $\mathcal{T}$ takes a sequence, say $\boldsymbol{s}$, to another sequence, call it $\boldsymbol{s}'$. In this case, we are interested in investigating the limit of the transformed sequence at infinity. Thus, for any *summation method* with $U = \overline{\mathbb{N}}$, $u_0 = \infty$, (i.e. $\lim_{u\to u_0} \mathcal{T}\boldsymbol{s}(u) = \lim_{u\to\infty} \mathcal{T}\boldsymbol{s}(u)$).

In the other case, where we have a *summation method* $\mathcal{T}$ with topological space $U = [0, 1]$, the transformation $\mathcal{T}$ takes a sequence $\boldsymbol{s}$ to a function $\mathcal{T}\boldsymbol{s} : [0, 1) \to \mathbb{R}$. In this case, $u_0 = 1$, and we investigate the $\lim_{u\to 1^-} \mathcal{T}\boldsymbol{s}(u)$.

As indicated above, since $\mathcal{T}$ is a *summation method*, if $\boldsymbol{s}$ converges to a finite limit at infinity, $\mathcal{T}\boldsymbol{s}$ also converges to a finite limit at u_0. Moreover, $\boldsymbol{s}$ may be divergent, yet $\mathcal{T}\boldsymbol{s}$ may converge to a finite limit at u_0. Naturally, we tend to think of sequences which converge to a finite limit as "well-behaved" or "tame", while we think of divergent sequences as disordered or wild. Since a *summation method* has the ability to transform a divergent sequence into a function that possesses a finite limit at infinity, while not disrupting the finite limit of a sequence that is already convergent, it is natural to think of $\mathcal{T}$ as a "taming"-transformation.

Remark 1.2. Note that there exists a canonical bijection between any sequence, say a_n, and its series $\sum_{k\geq 0} a_k$. Namely to the series $\sum_{k\geq 0} a_k$ we associate the sequence of partial sums

$$s_n = \sum_{k=0}^{n} a_k, \quad \forall n \geq 0$$

Conversely, to the sequence $s(n)$ we associate the series

$$\sum_{k=0}^{n} a_k = s_n, \quad a_0 = s_0, \quad a_n = s_n - s_{n-1}, \quad \forall n \geq 1.$$

And so, while each method actually operates upon a *sequence*, we will think of that sequence as the sequence of partial sums of a series. Thus we see the reason for calling such processes *summation methods.* □

All of the summation-transformations within this paper will be obtained by averaging. For every fixed $u \in U_*$ we will consider a sequence of weights:

$$w(u) = w_0(u), w_1(u), ..., w_n(u), ...$$

As a true weighted-average, the sequence satisfies the following two properties:

$$w_n(u) \geq 0, \quad \forall n, \tag{$*$}$$

$$\sum_{n\geq 0} w_n(u) = 1, \quad \forall u \tag{$**$}$$

Given any sequence $\boldsymbol{s} = s_0, s_1, \ldots, s_n, \ldots$, we form its averages

$$\mathcal{T}\boldsymbol{s}(u) = w_0(u)s_0 + w_1(u)s_1 + \ldots + w_n(u)s_n + \ldots = \sum_{n\geq 0} w_n(u)s_n$$

If $\lim_{u\to u_0} \mathcal{T}\boldsymbol{s}(u) = c$, where c is finite, then we say that $\boldsymbol{s}$ is $\mathcal{T}$-*convergent* and $\sum a_k$ is $\mathcal{T}$-summable, with $\mathcal{T}$-limit c.

☞ While within this paper we will apply *summation methods* directly to given sequences, it is important that the reader keep in mind the relationship between sequences and series, and the series that the sequence originates from.

✍ **Notation** Let f,g be real functions.

- $f \sim g$ if and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

- $f(n) = O(g(n))$ as $n \to \infty$ if and only if there exists $M > 0$ and $N \in \mathbb{R}$ such that

$$|f(n)| \leq M|g(n)| \quad \forall n > N$$

- $f(n) = o(g(n))$ if and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

2. Examples of summation techniques

When $U_* = \mathbb{N}$, the collection of weights $\{w(u)\}_{u\in U_*}$ can be organized as an infinite matrix. The collection $\{w(u)\}$ occupies the u^{th} row of the infinite matrix.

Definition 2.1 (Averaging Matrix)**.** Let $\mathcal{M}$ be an infinite square matrix, with entries $\mathcal{M}_{(ij)} = m_{ij}$ where i corresponds to row, and j to column. We call $\mathcal{M}$ an *averaging* matrix if it satisfies the following properties.

$$m_{ij} \geq 0, \quad \forall i, j \in \mathbb{N}. \tag{2.1a}$$

$$\sum_{j\geq 0} m_{ij} = 1, \quad \forall i \in \mathbb{N}. \tag{2.1b}$$

□

Note: row/column indices of matrices will start with 0, and increase as usual. We will now give several examples of standard *summation methods* that can be represented by averaging matrices. We refer [1] for a wealth of many other examples.

Example 2.2 (The Identity summation method)**.** Let $\mathcal{I}$ be the identity map $\boldsymbol{Seq} \to \boldsymbol{Seq}$. Here we take U to be the compactification

$$U = \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}.$$

Then $u_0 = \infty$, $U_* = \mathbb{N}$. We define the family of weights used in this trivial method as follows:

$$w_n(u) = \begin{cases} 0, & \text{if } n \neq u; \\ 1, & \text{if } n = u. \end{cases}$$

Therefore, $\mathcal{I}$ is represented by the infinite identity matrix:

$$\mathcal{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

To calculate $\mathcal{I}\boldsymbol{s}$, we write $\boldsymbol{s}$ as an infinite column vector and multiply:

$$\mathcal{I}\boldsymbol{s} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \end{bmatrix}$$

Clearly $\mathcal{I}\boldsymbol{s} = \boldsymbol{s}$, and thus, $\lim_{u\to\infty} \mathcal{I}\boldsymbol{s}(u) = \lim_{u\to\infty} s_u$, so the $\mathcal{I}$-convergent sequences are sequences that converge in the normal sense, just as we would expect. □

Example 2.3 (Cesàro method)**.** The Cesàro method uses standard averaging, and is the most simple non-trivial *summation method*. The weights are evenly distributed, and are defined as follows:

$$w_n(u) = \begin{cases} \frac{1}{u+1}, & \text{if } n \leq u; \\ 0, & \text{if } n > u. \end{cases}$$

which can be represented by the following infinite averaging matrix,

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Applying the Cesàro method to any sequence $\boldsymbol{s}$ yields the following transformed sequence.

$$\mathcal{C}\boldsymbol{s} = s_0, \quad \frac{s_0 + s_1}{2}, \quad \frac{s_0 + s_1 + s_2}{3}, \quad \cdots$$

So in general:

$$\mathcal{C}\boldsymbol{s}(u) = \frac{1}{u+1} \sum_{i=0}^{u} \boldsymbol{s}_i$$

If the sequence $\mathcal{C}\boldsymbol{s}$ is convergent, then the sequence $\boldsymbol{s}$ is called $(C, 1)$-convergent and the limit is called the $(C, 1)$-value of the sequence. Additionally, the series $\sum_{k=0} a_k$ that bijects with the sequence $\boldsymbol{s}$ is called $(C, 1)$-summable. □

Example 2.4 (Hölder means)**.** Hölder methods are a generalization of the Cesàro method. The k^{th} Hölder method, referred to as the $\mathcal{H}^k$ method, is described by applying the Cesàro method k successive times to a given sequence. Thus, the infinite matrix that describes the weights for the k^{th} Hölder method is given by $\mathcal{C}^k$. The general expression for the weights of the k^{th} Hölder method will not be listed explicitly, as this expression is rather involved. However, the following theorem ensures that all the rows of the matrix $\mathcal{C}^i$ will satisfy the two necessary weighting properties, (2.1a) and (2.1b).

□

Theorem 2.5. *If $\mathcal{A}$ and $\mathcal{B}$ are both lower-triangular* averaging *matrices, then $\mathcal{AB}$ will be a lower-triangular* averaging *matrix as well.*

Proof. Since $\mathcal{A}_{(ij)} = a_{ij} \geq 0$ and $\mathcal{B}_{(ij)} = b_{ij} \geq 0$ $\forall i, j \in \mathbb{N}$, it is obvious that $\mathcal{AB}_{(ij)} \geq 0$ $\forall i, j \in \mathbb{N}$. It is also clear that lower-triangularity is preserved in matrix multiplication. Thus, we need only show that $\mathcal{AB}$ satisfies (2.1b).

Since $\mathcal{A}$ and $\mathcal{B}$ satisfy property (2.1b) individually, and are lower triangular we deduce

$$\sum_{j\geq 0} a_{ij} = \sum_{j=0}^{i} a_{ij} = \sum_{j\geq 0} b_{ij} = \sum_{j=0}^{i} b_{ij} = 1 \quad \forall i.$$

By matrix multiplication:

$$\mathcal{AB}_{(ij)} = \sum_{k=0}^{i} a_{ik} b_{kj}$$

Since $\mathcal{AB}$ is lower-triangular, the sum of the elements on the i^{th} row of $\mathcal{AB}$, for any i, is expressed as:

$$\sum_{j\geq 0} \mathcal{AB}_{(ij)} = \sum_{j=0}^{i} \mathcal{AB}_{(ij)} = \sum_{j=0}^{i}\sum_{k=0}^{i} a_{ik} b_{kj} = \Big(\sum_{k=0}^{i} a_{ik}\Big)\Big(\sum_{j=0}^{i} b_{kj}\Big) = 1, \quad \forall i.$$

This shows that $\mathcal{AB}$ is also an averaging matrix. □

Thus, according to the previous theorem, the space of lower-triangular averaging matrices is closed under multiplication. There are other interesting properties of averaging matrices that are preserved under multiplication, and we will discuss these later in the paper.

Example 2.6 (Cesàro means)**.** Multiplication by the matrix $\mathcal{C}$ above is equivalent to taking a standard average, which essentially involves two steps: a summation, and then a division. In this way, the Hölder method can be thought of process of repetitive averaging–that is, a summation, followed by a division, followed by a summation, followed by a division, etc. While the Hölder method involves alternating between the process of summation and division, the Cesàro method involves a finite amount of summations performed successively, followed by only a single division. Although perhaps not as apparent as in the case of the Hölder method, the Cesàro method is indeed an averaging method–a fact that can be verified through investigation of its weighting sequences. To introduce the general Cesàro method, we will first define a few terms. Given any sequence $\boldsymbol{s} = s_0, s_1, \ldots, s_n, \ldots$, we define

$$S_n^1 = s_0 + s_1 + s_2 + \ldots + s_n$$

and define S_n^k by the follow recursion relation:

$$S_n^k = S_0^{k-1} + S_1^{k-1} + \ldots + S_n^{k-1}$$

We now go through the following steps to describe the weighting sequence of the Cesàro method and explain the need to define the above terms. Consider the generating functions:

$$\sum_{n\geq 0} S_n^1 x^n \;=\; (1 + x + x^2 + \ldots) \sum_{n\geq 0} s_n x^n \;=\; (1-x)^{-1} \sum_{n\geq 0} s_n x^n$$

$$\Rightarrow \;\; (1-x) \sum_{n\geq 0} S_n^1 x^n = \sum_{n\geq 0} s_n x^n$$

Similarly,

$$(1-x) \sum_{n\geq 0} S_n^2 x^n = \sum_{n\geq 0} S_n^1 x^n,$$

so that,

$$(1-x)^2 \sum_{n\geq 0} S_n^2 x^n = \sum_{n\geq 0} s_n x^n$$

Iterating the above procedure we obtain the following identity.

$$(1-x)^k \sum_{n\geq 0} S_n^k x^n = \sum_{n\geq 0} s_n x^n \iff \sum_{n\geq 0} S_n^k x^n = (1-x)^{-k} \sum_{n\geq 0} s_n x^n \tag{2.2}$$

Taking the $(k-1)^{st}$ derivative of the equality

$$(1-x)^{-1} = \sum_{n\geq 0} x^n \qquad |x| < 1$$

we deduce

$$\frac{(k-1)!}{(1-x)^k} = \sum_{n\geq 0} \frac{n!}{(n-k+1)!} x^{n-k+1},$$

i.e.,

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{n}{k-1} x^{n-k+1}$$

Since the terms in the above sum are 0 for $n < (k-1)$, we begin the the summation at $n = (k-1)$, which yields

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n. \tag{2.3}$$

Thus, by (2.2) and (2.3):

$$\sum_{n\geq 0} S_n^k x^n = (1-x)^{-k} \sum_{n\geq 0} s_n x^n = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n \sum_{n\geq 0} s_n x^n.$$

Hence

$$S_n^k = \sum_{m=0}^{n} \binom{n-m+k-1}{k-1} s_m.$$

Thus, we can see how to partially construct a matrix to model the weights of the k^{th} Cesàro method:

$$\begin{bmatrix} \delta_0\binom{k-1}{k-1} & 0 & 0 & 0 & \cdots & \cdots \\ \delta_1\binom{k}{k-1} & \delta_1\binom{k-1}{k-1} & 0 & 0 & \cdots & \cdots \\ \delta_2\binom{k+1}{k-1} & \delta_2\binom{k}{k-1} & \delta_2\binom{k-1}{k-1} & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta_n\binom{n+k-1}{k-1} & \delta_n\binom{n+k-2}{k-1} & \cdots & \delta_n\binom{k-1}{k-1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Now we just need to define the coefficient δ_n, for each row, to ensure that each row sums to 1. The sum of the n^{th} row is:

$$\sum_{i=0}^{n} \delta_n \binom{i+k-1}{k-1} = \delta_n \left\{ \binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{n+k-1}{k-1} \right\}.$$

From the classical identity

$$\sum_{i=0}^{n} \binom{i+k-1}{k-1} = \binom{n+k}{k}$$

we conclude that if the sum of the nth is 1 if and only if

$$\delta_n := \frac{1}{\binom{n+k}{k}}.$$

It is worth noting that the denominator of each coefficient δ_n is exactly the number of terms of our original sequence that have been summed in each particular row. For example, when $n = 2$, we have summed $\binom{k+1}{k-1}$ copies of $s(0)$, $\binom{k}{k-1}$ copies of $s(1)$, and $\binom{k-1}{k-1}$ copies of $s(2)$, for a total of $\binom{k+2}{k}$ terms–thus, we divide by this number since this is an *averaging* method.

That being said, we now we have a legitimate sequence of weights for the Cesàro method that we can write explicitly. The weighting sequence for the k^{th} Cesàro methodis,

$$w_n(u) = \begin{cases} \delta_u\binom{u-n+k-1}{k-1}, & \text{if } n \le u; \\ 0, & \text{if } n > u. \end{cases}$$

where δ_u is defined as above. Note that when $k = 1$, we obtain the sequence of weights represented by the matrix $\mathcal{C}$ given before–thus the 1^{st}Cesàro method is equivalent to the 1^{st} Hölder method, as we expected. □

Example 2.7 (The Euler method)**.** The weights of Euler method also involve binomial coefficients, yet the distribution of the weights is concentrated toward the middle, rather than the beginning (as was the case for Cesàro method). For the Euler method, we define the following weighting system:

$$w_n(u) = \varepsilon_u \binom{u}{n}, \quad \text{where } \varepsilon_u := \frac{1}{2^u}.$$

Thus, the transformation matrix of the Euler Method is the following:

$$\mathcal{E} = \begin{bmatrix} \varepsilon_0\binom{0}{0} & 0 & 0 & 0 & \cdots & \cdots \\ \varepsilon_1\binom{1}{0} & \varepsilon_1\binom{1}{1} & 0 & 0 & \cdots & \cdots \\ \varepsilon_2\binom{2}{0} & \varepsilon_2\binom{2}{1} & \varepsilon_2\binom{2}{2} & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon_u\binom{u}{0} & \varepsilon_u\binom{u}{1} & \cdots & \varepsilon_u\binom{u}{u} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

It is clear that the sum of any row of $\mathcal{E}$ is 1, since

$$\sum_{n\geq 0} \varepsilon_u \binom{u}{n} = \sum_{n=0}^{u} \frac{1}{2^u}\binom{u}{n} = \frac{1}{2^u}\sum_{n=0}^{u}\binom{u}{n} = 1 \quad \forall u.$$

Given any sequence $\boldsymbol{s}$, if the $\lim_{u\to\infty} \mathcal{E}\boldsymbol{s}(u)$ exists and is finite, then we say that $\boldsymbol{s}$ is $\mathcal{E}-$convergent and its Euler Sum is $(\lim_{u\to\infty} \mathcal{E}\boldsymbol{s}(u))$.

□

At this point in the paper, we have seen many examples of averaging methods that were able to be represented by infinite matrices, and we claimed that these were all legitimate *summation methods*. That is, we claimed that they were able to induce convergence in some divergent sequences, while not affecting the limit of already convergent sequences. We will now prove this claim, and in fact, elucidate the key properties that determine whether or not a matrix will be a summation method.

Theorem 2.8 (Characterization of regular averaging matrices)**.** *Suppose $\mathcal{T}$ is a summation method given by an averaging matrix $(c_{mn})_{m,n\geq 0}$. Then the summation method is regular if and only if*

$$\lim_{m\to\infty} c_{mn} = 0, \quad \forall n. \tag{2.4}$$

Proof. First we will prove sufficiency of (2.4). Suppose $\boldsymbol{s}$ is a convergent sequence with limit s. Define

$$t_m := \sum_{n=0}^{\infty} c_{mn}s_n.$$

We want to show that $\lim_{m\to\infty} t_m = s$. Thus, we need to show that

$$\forall \varepsilon > 0, \;\; \exists M > 0 : |t_m - c_m s| < \varepsilon, \;\; \forall m > M.$$

Observe that

$$|t_m - s| \overset{(2.1b)}{=} |\sum_{n=0}^{\infty} c_{mn}s_n - \sum_{n=0}^{\infty} c_{mn}s| = |\sum_{n=0}^{\infty} c_{mn}(s_n - s)| \leq \sum_{n=0}^{\infty} |c_{mn}||s_n - s|. \tag{2.5}$$

Since $s_n \to s$ as $n \to \infty$, $\exists N > 0$ such that

$$|s_n - s| < \frac{\varepsilon}{2K}, \;\; \forall n > N. \tag{2.6}$$

We now express the infinite sum on the right hand side of the inequality (2.5) as follows:

$$\sum_{n=0}^{\infty}|c_{mn}||s_n-s| = \sum_{n=0}^{N}|c_{mn}||s_n-s| + \sum_{n>N}^{\infty}|c_{mn}||s_n-s|$$
$$\overset{(2.6)}{\leq} \sum_{n=0}^{N}|c_{mn}||s_n-s| + \frac{\varepsilon}{2K}\sum_{n>N}^{\infty}|c_{mn}| \overset{(i)}{\leq} \sum_{n=0}^{N}|c_{mn}||s_n-s| + \frac{\varepsilon}{2}.$$

Now consider:

$$\sum_{n=0}^{N}|c_{mn}||s_n-s| = |c_{m0}||s_0-s| + |c_{m1}||s_1-s| + \cdots + |c_{mN}||s_N-s|$$

Set

$$A_N := \max\{|s_0-s|, |s_1-s|, ..., |s_N-s|\}.$$

Thus,

$$\sum_{n=0}^{N}|c_{mn}||s_n-s| \leq (|c_{m0}|+|c_{m1}|+\cdots+|c_{mN}|)A_N$$

But by (2.4), $\lim_{m\to\infty}|c_{mi}| \to 0\ \forall i$. Thus for any $i = 0, 1, \ldots, N$, there exists $M_i > 0$ such that

$$\forall m > M_i: \quad |c_{mi}| < \frac{\varepsilon}{2(N+1)A_N}.$$

Let

$$M := \max\{M_0, M_1, ..., M_N\}.$$

Thus, if $m > M$ then

$$\sum_{n=0}^{N}|c_{mn}||s_n-s| < \sum_{n=0}^{N}\frac{\varepsilon}{2(N+1)A_N}A_N = \frac{\varepsilon}{2}$$

Therefore

$$|t_m - c_m s| \leq \sum_{n=0}^{\infty}|c_{mn}||s_n-s| = \sum_{n=0}^{N}|c_{mn}||s_n-s| + \sum_{n>N}^{\infty}|c_{mn}||s_n-s| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\lim_{m\to\infty} t_m = s$. We have thus shown the sufficiency of (2.4).

To prove the necessity we argue by contradiction. Thus we assume that $\mathcal{T}$ is regular yet (2.4) is not satisfied. Then there exists N such that the sequence of terms on the N-th column does not converge to zero. Consider the sequence $\boldsymbol{s} = s_0, s_1, \ldots, s_n, \ldots$, where

$$s_n = \begin{cases} 0, & n \neq N; \\ 1, & n = N. \end{cases}$$

Clearly $\lim_{n\to\infty} s_n = 0$. Thus, since $\mathcal{T}$ is regular, $\lim_{m\to\infty}\mathcal{T}\boldsymbol{s} = 0$. On the other hand, we have

$$\mathcal{T}\boldsymbol{s}(m) = \sum_{n=0}^{\infty} c_{mn}s_n = c_{mN}s_N = c_{mN}.$$

Thus.

$$\lim_{m\to\infty}\mathcal{T}\boldsymbol{s} = \lim_{m\to\infty} c_{m,N} \neq 0 = \lim_{n\to\infty} s_n \quad \text{(Contradiction!)}$$

This contradiction implies that (2.4) is necessary

$\square$

The reader is now encouraged to revisit the aforementioned averaging method matrices, and verify that each one is indeed regular–that is, each one satisfies the conditions of Theorem 2.8.

Now that we specific conditions for regularity, we can give a more general proof of Theorem 2.5.

Theorem 2.9. *If $\mathcal{A}$ and $\mathcal{B}$ are both regular* averaging *matrices, then $\mathcal{AB}$ will be a regular* averaging *matrix as well.*

Proof. $\mathcal{AB}$ satisfies (2.1a) and (2.1b) automatically, by Theorem 2.5. We need only show that it satisfies (2.4) listed above.

After matrix multiplication, the entry $\mathcal{AB}_{(ij)}$ is given by:

$$\mathcal{AB}_{(ij)} = \sum_{k=0}^{i} a_{i,k} b_{k,j}$$

We want to show that:

$$\lim_{i \to \infty} \mathcal{AB}_{(ij)} = 0 \quad \forall j \in \mathbb{N}$$

Let j be some fixed column, and let $\varepsilon > 0$ be arbitrarily small. Since $\mathcal{B}$ satisfies (iii), we know that $\exists\ N > 0$ such that $b_{i,j} < \frac{\varepsilon}{2}$ for every $i > N$.

Since N is finite, we know that $\sum_{k=0}^{N} b_{k,j}$ is bounded, so $\exists K$ such that:

$$\sum_{k=0}^{N} b_{k,j} < K$$

Also, $\mathcal{A}$ satisfies (2.4), so

$$\begin{aligned}
&\exists\ M_0\ :\ a_{i,0} < \frac{\varepsilon}{2K} \quad \forall i > M_0 \\
&\exists\ M_1\ :\ a_{i,1} < \frac{\varepsilon}{2K} \quad \forall i > M_1 \\
&\qquad\vdots \\
&\exists\ M_N\ :\ a_{i,N} < \frac{\varepsilon}{2K} \quad \forall i > M_N
\end{aligned}$$

Set

$$M_* := \max\{M_0, M_1, \ldots, M_N\}.$$

Now consider:

$$\mathcal{AB}_{(ij)} = \sum_{k=0}^{i} a_{i,k} b_{k,j} = \sum_{k=0}^{N} a_{i,k} b_{k,j} + \sum_{k>N}^{i} a_{i,k} b_{k,j}$$

For all $i > M_*$:

$$\sum_{k=0}^{N} a_{i,k} b_{k,j} \leq \frac{\varepsilon}{2K} \sum_{k=0}^{N} b_{k,j} \leq \frac{\varepsilon}{2K} * K \leq \frac{\varepsilon}{2}$$

Employing how we've defined N, as well as using the fact that $\sum_{j \geq 0} a_{i,j} = 1$:

$$\sum_{k>N}^{i} a_{i,k} b_{k,j} \leq \frac{\varepsilon}{2} \sum_{k>N}^{i} a_{i,k} \leq \frac{\varepsilon}{2}$$

Thus,

$$\mathcal{AB}_{(ij)} = \sum_{k=0}^{i} a_{i,k}b_{k,j} = \sum_{k=0}^{N} a_{i,k}b_{k,j} + \sum_{k>N}^{i} a_{i,k}b_{k,j} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \lim_{i\to\infty} \mathcal{AB}_{(ij)} = 0.$$

□

Example 2.10 (Abel Summation)**.** In this case, $U = [0,1], u_0 = 1$. For any $u \in [0,1)$, we let

$$w_n(u) = (1-u)u^n, \ \ \forall n \geq 0.$$

In this case, the $u - average$ of a sequence $\boldsymbol{s}$

$$(1-u)\sum_{n\geq 0} \boldsymbol{s}_n u^n$$

The sequence is called Abel convergent or $A - convergent$ if the limit

$$\lim_{n\to 1^-} (1-u)\sum_{n\geq 0} \boldsymbol{s}_n u^n$$

exists and is finite.

When $\boldsymbol{s}_n$ are the partial sums of the series $\sum_{n\geq 0} a_n$, then the notion $A - convergence$ can be rephrased in a very intuitive way. Denote by $a(t)$ a generating series of the sequence $(a_n)_{n\geq 0}$.

$$a(t) = \sum_{n\geq 0} a_n t^n$$

Then the generating series of the sequence $\boldsymbol{s}_n$ is

$$\sum_{n\geq 0} \boldsymbol{s}_n t^n = (1+t+t^2+\cdots)\sum_{n\geq 0} a_n t^n = \frac{1}{1-t}\sum_{n\geq 0} a_n t^n.$$

Hence,

$$(1-u)\sum_{n\geq 0} \boldsymbol{s}_n u^n = \sum_{n\geq 0} a_n u^n$$

We see that the series $\sum_{n\geq 0} a_n$ is A-convergent if the following two things happens:

- The series $\sum_{n\geq 0} a_n u^n$ is convergent for any $u \in [0,1)$.
- The $\lim_{u\to 1^-} \sum_{n\geq 0} a_n u^n$ exists and it is finite. This limit is called the Abel sum of the series.

For example, the alternating series $\sum_{n\geq 0}(-1)^n$ is A-convergent and its Abel sum is $\frac{1}{2}$. Indeed,

$$\sum_{n\geq 0}(-1)^n u^n = \sum_{n\geq 0}(-u)^n = \frac{1}{1+u}, \ \ \forall u \in [0,1).$$

□

3. Abelian theorems

We now prove a theorem that demonstrates how the Abelian Method is indeed stronger than any Cesàro Method, in the sense that any series that is Cesàro summable is immediately Abel summable, but not vice-versa.

Theorem 3.1. *If the series $\sum_{n\geq 0} a_n$ is Cesàro summable to S then it is also Abel summable, with the same sum, S. In other words, if $\sum a_n = S\ (C,1)$, then $\sum a_n = S\ (A)$.*

Proof. The proof of the theorem will follow immediately form the following result.

Lemma 3.2. *If $d_n > 0$, $\sum d_n = \infty$, $\sum d_n x^n$ is convergent when $0 \leq x < 1$ and $c_n \sim S d_n$, where $S \neq 0$, then*

$$C(x) = \sum c_n x^n \sim SD(x) = S \sum d_n x^n, \quad \textit{as } x \nearrow 1. \qquad \square$$

Let us first explain how we can deduce Theorem 3.1 from Lemma 3.2. Without loss of generality, we may assume that $S \neq 0$. Set

$$s_n := a_0 + \cdots + a_n, \quad S_n^1 := s_0 + \cdots + s_n.$$

Because of the Cesàro convergence we have

$$S_n^1 \sim Sn \text{ as } n \to \infty.$$

In particular, this implies that the series $\sum_{n\geq 0} S_n^1 x^n$ is absolutely convergent for $|x| < 1$. We conclude that the series

$$f(x) = \sum_{n\geq 0} a_n x^n = (1-x)^2 \sum_{n\geq 0} S_n^1 x^n$$

is absolutely convergent for $|x| < 1$. Using the identity

$$(1-x)^{-2} = \frac{d}{dx}(1-x)^{-1} = \sum n x^{n-1} = \sum (n+1)x^n$$

we deduce

$$\sum_{n\geq 0} a_n x^n = \frac{1}{(1-x)^{-2}} \sum_{n\geq 0} S_n^1 x^n = \frac{\sum S_n^1 x^n}{\sum (n+1)x^n}$$

But recall $\sum a_n = S(C,1)$, which implies $S_n^1 \sim nS$ (so of course $S_n^1 \sim (n+1)S$). Thus, by Lemma 3.2 $f(x) \to S$ as $x \to 1$, so $\sum a_n = S(\mathrm{A})$.

$\square$

Proof of Lemma 3.2. Without loss of generality, we may suppose $S = 1$. Because $c_n \sim d_n$, for any $\varepsilon > 0$, there exists N such that $n > N$ ensures that

$$1 - \varepsilon < \frac{c_n}{d_n} < 1 + \varepsilon$$

We obtain

$$C(x) = \sum_{n=0}^{N} c_n x^n + \sum_{n>N}^{\infty} c_n x^n \leq (1+\varepsilon)D(x) + \sum_{n=0}^{N} |c_n| x^n$$

and similarly,

$$C(x) \geq (1-\varepsilon)D(x) - \sum_{n=0}^{N} d_n x^n - \sum_{n=0}^{N} |c_n| x^n$$

Since $d_n > 0$ we see that,

$$D(x) \geq \sum_{n=0}^{N} d_n x^n \quad \forall N$$

Thus,

$$\lim_{x\to 1} D(x) \geq \lim_{x\to 1} \sum_{n=0}^{N} d_n x^n \quad \forall N$$

Since $D(x) \to \infty$ as $x \to 1$, when we divide both bounding inequalities of $C(x)$ by $D(x)$ we see that

$$\limsup_{x\to 1} \frac{C(x)}{D(x)} \leq 1+\varepsilon, \quad \liminf_{x\to 1} \frac{C(x)}{D(x)} \geq 1-\varepsilon,$$

and so $C(x) \sim D(x)$. □

We now know that any Cesàro summable series is automatically Abel-summable. Thus, to show that the Abel method is "stronger" than Cesàro, we need only produce a series that is Abel-summable, but not Cesàro-summable.

Example 3.3. Consider $a_n = (-1)^{n+1}n$. As usual, we define $\sum_{k=0}^{n} a_k = s_n$. We can see $\sum_{n\geq 0} a_n$ clearly by looking at the sequence of partial sums,

$$\boldsymbol{s} = s_0, s_1, s_2, \ldots, s_n, \ldots = 0, 1, -1, 2, -2, 3, -3, 4, \ldots$$

Thus, we can write s_n explicitly as,

$$s_n = \begin{cases} -\frac{n}{2}, & \text{n even;} \\ \frac{n+1}{2} & \text{n odd.} \end{cases}$$

which obviously diverges as $n \to \infty$. Additionally, $\sum a_n$ is not Cesàro-summable $(C, 1)$, i.e. $\mathcal{C}\boldsymbol{s}$ diverges. Recall that

$$\mathcal{C}\boldsymbol{s}(u) = \frac{1}{u+1} \sum_{i=0}^{u} \boldsymbol{s}_i$$

Now we investigate the sequence partial sums of $\mathcal{C}\boldsymbol{s}$:

$$\mathcal{C}\boldsymbol{s} = \mathcal{C}s(0), \mathcal{C}s(1), \ldots \mathcal{C}s(n) \ldots = 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \ldots \Rightarrow$$

$$\mathcal{C}s(n) = \begin{cases} 0, & \text{n even;} \\ 1/2 & \text{n odd.} \end{cases}$$

Thus, $\mathcal{C}\boldsymbol{s}$ diverges as claimed, and $\sum a_n$ is not Cesàro-summable. Now apply the Abel method to the same series.

$$\sum_{n\geq 0} a_n u^n = \sum_{n\geq 0} (-1)^{n+1} n u^n = -\sum_{n\geq 0} n(-u)^n$$

But we can quickly derive the following identity for the expression on the right-hand side:

$$\sum_{n\geq 0} (-u)^n = \frac{1}{1+u} \quad \forall u \in [0, 1)$$

Differentiating both sides, and multiplying by $(-u)$ we obtain,

$$-\sum_{n\geq 0} n(-u)^n = \frac{u}{(1+u)^2} \quad \forall u \in [0, 1)$$

Therefore, we have

$$\sum_{n\geq 0} a_n u^n = \frac{u}{(1+u)^2} \quad \forall u \in [0,1)$$

and

$$\lim_{u\to 1^-} \sum_{n\geq 0} a_n u^n = \lim_{u\to 1^-} \frac{u}{(1+u)^2} = \frac{1}{4}$$

so $\sum_{n\geq 0} a_n$ is Abel-summable. Thus, we have found a series that is Abel-summable, but not Cesàro-summable. Using this fact, along with the statement of Theorem 3.1, we see how the Abel method is "stronger" than the Cesàro method (i.e. it can sum all of the Cesàro-summable series, and more).

4. Tauberian theorems

We have so far proved the following sequence of implications

usual convergence $\Rightarrow$ Cesàro summability $\Rightarrow$ Abel summability

In this section we want to describe conditions when the opposite implications hold. Such conditions are known in the literature as *Tauberian conditions*. Before starting, we briefly explain notation that will be used.

$$x_n = O_L(y_n) \iff \exists K: \;\; x_n > -Ky_n \;\; \forall n.$$

Our first result The following theorem lists conditions that guarantee that an Abel-summable series will be Cesàro-summable as well.

Theorem 4.1. *If $\sum a_n = s(A)$, a_n is real, and $s_n = O_L(1)$, then $\sum a_n = s(C,1)$.*

To prove 4.1 we follow the strategy in [1]. We first observe that Theorem 4.1 is a consequence of the following more general result.

Theorem 4.2. *If*

$$f(x) = \sum a_n x^n \sim \frac{C}{1-x} \tag{4.1}$$

when $x \to 1$, and a_n is real, and $a_n = O_L(1)$, then $(a_0 + a_1 + \ldots + a_n) = s_n \sim Cn$.

In fact, we can easily show that Theorem 4.1 is a corollary of Theorem 4.2. If the conditions of Theorem 4.1 are satisfied, then

$$\sum_{n\geq 0} s_n x^n = \frac{1}{1-x} \sum_{n\geq 0} a_n x^n \sim \frac{s}{1-x},$$

and $s_n = O_L(1)$. Thus, assuming Theorem 4.2 is true, we can apply it to s_n (rather than a_n, as used in Theorem 4.2) to obtain

$$(s_0 + s_1 + \ldots + s_n) \sim sn \iff \frac{s_0 + \ldots + s_n}{n} \sim s$$

and so $\sum a_n = s(C,1)$. Thus Theorem 4.2 $\Rightarrow$ 4.1. Now we progress to the substantial proof of Theorem 4.2 (and consequently 4.1). We will prove this as a special case of the theorem that involves Stieltjes integrals.

Theorem 4.3. *If $\alpha(t)$ increases with t,*

$$I(y) = \int e^{-yt} d\alpha(t) \tag{4.2}$$

is convergent for $y > 0$, and $I(y) \sim Cy^{-1}$ where $C \geq 0$, when $y \to 0$, then $\alpha(t) \sim Ct$.

Proof. We will need two lemmas to complete this proof.

Lemma 4.4. *If $g(x)$ is a real function, and Riemann integrable in $(0,1)$, then there are polynomials $p(x)$ and $P(x)$ such that $p(x) < g(x) < P(x)$ and*

$$\int_0^1 (P(x) - p(x))dx = \int_0^\infty e^{-t}(P(e^{-t}) - p(e^{-t})dt < \varepsilon. \tag{4.3}$$

$$g(x) = \begin{cases} 1, & a \leq x \leq b; \\ 0, & \text{otherwise.} \end{cases}$$

where $0 \leq a < b \leq 1$. Let $\varepsilon > 0$ be arbitrary. Although g is discontinuous, we can certainly find a continuous function $h(x)$ such that

$$g \leq h, \qquad \int_0^1 (h - g)dx < \varepsilon/6$$

By Stone-Weierstrass theorem, there exists polynomial Q(x) such that $|h - Q| < \varepsilon/6$ for all $x \in [0,1]$. If P(x) = Q + $\varepsilon/6$, then $g \leq h < P$ and since

$$|P - g| \leq |P - Q| + |Q - h| + |h - g| \Rightarrow$$

$$\int_0^1 |P - g|dx \leq \int_0^1 |P - Q|dx + \int_0^1 |Q - h|dx + \int_0^1 |h - g|dx < \varepsilon/2$$

We can go through an entirely symmetric process for p such that $p < g$ to ensure that $\int_0^1 |g - p|dx < \varepsilon/2$. This being true, along with the fact that $p < g < P$ implies

$$\int_0^1 |P - g|dx + \int_0^1 |g - p|dx = \int_0^1 (P - g)dx + \int_0^1 (g - p)|dx = \int_0^1 (P - p)dx < \varepsilon$$

It follows from the previous argument that the Lemma applies to any step function containing a finite number of jumps. We know that if g is a Riemann integrable function, then there are finite step-functions g_1 and g_2 such that

$$g_1 \leq g \leq g_2, \qquad \int_0^1 (g_2 - g_1)dx < \varepsilon/3$$

We now associate polynomials p_1, P_1 with g_1, and p_2, P_2 with g_2, in the way that the Lemma suggests. Then

$$\int_0^1 (P_2 - g_2)dx < \varepsilon/3 \qquad \int_0^1 (g_1 - p_1)dx < \varepsilon/3$$

So

$$\int_0^1 (P_2 - p_1)dx = \int_0^1 [(P_2 - g_2) + (g_2 - g_1) + (g_1 - p_1)]dx < 3\varepsilon$$

We can carry out a simple substution in the above integral, with $x = e^{-t}$,

$$\int_0^1 (P_2(x) - p_1(x))dx = \int_0^\infty e^{-t}(P_2(e^{-t}) - p_1(e^{-t}))dt$$

□

Lemma 4.5. *Suppose that $\alpha(t)$ increases with t, that I(y) is convergent for y¿0, that $I(y) \sim Cy^{-1}$, and that g(x) is of bounded variation in (0,1). Then*

$$\chi(y) = \int e^{-yt} g(e^{-yt}) d\alpha(t)$$

exists for all positive values of y except values τ/ω, where ω is a discontinuity of α, and τ a discontinuity of $g(e^{-t})$; and

$$\chi(y) \sim \frac{C}{y} \int e^{-t} g(e^{-t}) dt \tag{4.4}$$

when $y \to 0$ through any sequence of positive values which excludes these exceptional values.

Proof. Since τ and ω are at most countable individually, their intersection is also countable (i.e. the points that we throw away), so we exclude at most a countable set of values, call them y_k, of y, which will not affect the value of the integral. Because a function of bounded variation is Riemann integrable, by Lemma 4.4 we can choose polynomials p and P such that

$$p < g < P, \qquad \int e^{-t}[P(e^{-t}) - p(e^{-t})] dt < \varepsilon$$

So

$$\int e^{-t} p(e^{-t}) dt < \int e^{-t} g(e^{-t}) dt < \int e^{-t} P(e^{-t}) dt;$$

and since α increases with t,

$$\int e^{-yt} p(e^{-yt}) d\alpha(t) \leq \int e^{-yt} g(e^{-yt}) d\alpha(t) \leq \int e^{-yt} P(e^{-yt}) d\alpha(t),$$

when $y \neq y_k$.

$$\int e^{-yt} e^{-nyt} d\alpha(t) = \int e^{-(n+1)yt} d\alpha(t) \sim \frac{C}{(n+1)y} = \frac{C}{y} \int e^{-t} e^{-nt} dt,$$

and thus

$$\int e^{-yt} P(e^{-yt}) d\alpha(t) \sim \frac{C}{y} \int e^{-t} P(e^{-t}) dt.$$

Hence, if $y \to 0$ through any sequence of not including values y_k, we have

$$\limsup_{y \to 0} y \int e^{-yt} g(e^{-yt}) d\alpha(t) \leq \limsup_{y \to 0} y \int e^{-yt} P(e^{-yt}) d\alpha(t)$$

$$= C \int e^{-t} P(e^{-t}) dt < \int e^{-t} g(e^{-t}) dt + C\varepsilon$$

In a completely similar process involving p, we arrive at the following relationship,

$$\liminf_{y \to 0} y \int e^{-yt} g(e^{-yt}) d\alpha(t) > C \int e^{-t} g(e^{-t}) dt - C\varepsilon;$$

Which essentially proves (4.4). ∎

Now we are able to prove Theorem 4.3. Suppose, without loss of generality, that $\alpha(0) = 0$. We take

$$g(x) = \begin{cases} x^{-1}, & \text{if } e^{-1} \le x \le 1; \\ 0 & \text{if } 0 \le x < e^{-1}; \end{cases}$$

So clearly $g(e^{-t}) = e^t$ for $0 \le t < 1$ and $g(e^{-t}) = 0$ for $t > 1$. Then we have,

$$\chi(y) = \int_0^\infty e^{-yt} g(e^{-yt}) d\alpha(t) = \int_0^{\frac{1}{y}} d\alpha(t) = \alpha(\frac{1}{y})$$

And

$$\int_0^\infty e^{-t} g(e^{-t}) dt = \int_0^\infty dt = 1.$$

Thus, by Theorem 4.3, $\alpha(y^{-1}) \sim Cy^{-1}$ when $y \to 0$, or equivalently, $\alpha(t) \sim Ct$ when $t \to \infty$, with the exception being made in either case of the countable sets of discontinuity. In this specific case we need only worry about one countable set of discontinuities, name the values of α, thus, $\alpha(t) \sim Ct$ as $t \to \infty$ through points of continuity of $\alpha(t)$. And since α increases with t, it is true without special stipulation. □

Now we will demonstrate that Theorem 4.3 implies Theorem 4.2. Let $\alpha(t)$ be a step-function with jumps $a_n \ge 0$ for $t = n$, so

$$I(y) = S(y) = \sum a_n e^{-ny},$$

and $S(y) \sim Cy^{-1}$, which implies $s_n \sim Cn$. This is *very* close to the statement of Theorem 4.3. We just need the same conclusion for $a_n = O_L(1)$. Assume the conditions of Theorem 4.3 are satisfied, and there exists H such that $a_n > -H$. Then let $b_n = a_n + H > 0$. Then

$$\sum b_n x^n = \sum a_n x^n + \frac{H}{1-x} \sim \frac{C+H}{1-x}.$$

Thus, assuming what we've just shown in the previous paragraph,

$$b_0 + b_1 + \ldots + b_n \sim (C+H)n,$$

and so $\sum_{k=0}^n a_k = s_n \sim Cn$, which is Theorem 4.2. □

We will now state and prove the second of the two major Tauberian theorems discussed in this paper. This theorem will describe conditions that, when satisfied, ensure a Cesàro-convergent sequence is convergent in the normal sense.

Theorem 4.6. *If $\sum a_n = s(C,1)$, for a_n real, and $a_n = O_L(n^{-1})$, then $\sum a_n$ converges.*

Proof. We follow the elegant approach of [2]. First, we let

$$s_n = a_0 + \ldots + a_n$$
$$S_n = s_0 + s_1 + \ldots + s_n$$

We consider the following discrete analog of *Taylor's formula* for $h > 0$,

$$S_{n+h} = S_n + hs_n + \frac{1}{2}h(h+1)a_\xi, \tag{4.5}$$

where a_ξ is a number such that

$$\min a_k \le a_\xi \le \max a_k \quad \textit{for} \;\; n < k \le n+h.$$

To prove (4.5), consider

$$S_{n+h} - S_n = (s_0 + \ldots + s_{n+h}) - (s_0 + \ldots + s_n) = (s_{n+1} + s_{n+2} + \ldots + s_{n+h})$$
$$\Rightarrow S_{n+h} - S_n = h(a_0 + a_1 + \ldots a_n) + (ha_{n+1} + (h-1)a_{n+2} \ldots a_{n+h})$$
$$\Rightarrow S_{n+h} - S_n = hs_n + (ha_{n+1} + (h-1)a_{n+2} + \ldots + a_{n+h})$$

The bound on a_ξ follows from

$$\frac{1}{2}h(h+1) \min_{n<k\leq n+h} a_k \leq (ha_{n+1} + (h-1)a_{n+2} + \ldots + a_{n+h}) \leq \frac{1}{2}h(h+1) \max_{n<k\leq n+h} a_k.$$

The remaining expression (4.5) follows from the Intermediate Value Theorem.

We may now assume that $\sum a_n = 0(C,1)$ since, if $\sum a_n = L(C,1)$, then we can replace a_0 by $a_0 - L$ and every partial sum will be decreased by L. So we need only consider $\sum a_n = 0(C,1)$, which implies

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{s_0 + s_1 + \ldots + s_n}{n} = 0$$

We can re-write equation (4.5) in a form that isolates s_n,

$$hs_n = \frac{1}{2}h(h+1)a_\xi + S_{n+h} - S_n$$

Since $h > 0$, we may divide to give

$$s_n = \frac{S_{n+h} - S_n}{h} - \frac{1}{2}(h+1)a_\xi$$

We now use

$$a_n \geq \frac{-K}{n} \text{ for some K} > 0 \text{ and } |S_n| \leq n\varepsilon$$
$$\Rightarrow s_n \leq \frac{2n+h}{h}\varepsilon + K\frac{h+1}{2n}$$

Since this inequality holds for all h, we take $h \approx 2n\sqrt{\varepsilon/K}$, which yields the following arbitrarily small positive upper bound on s_n.

$$s_n < 3\sqrt{K\varepsilon}$$

Now we go through a similar process to obtain a lower bound for s_n, in which we use an analogous form of equation 4.5 for $h < 0$. For the sake of ease, let $h = -i < 0$. Through an entirely similar process as performed above, we see that

$$S_n - S_{n-i} = s_n + s_{n-1} + \ldots + s_{n-i+1}$$

or equivalently,

$$S_n - S_{n-i} = s_n + (s_n - a_n) + (s_n - (a_n + a_{n-1})) + \ldots + (s_n - (a_n + a_{n-1} + \ldots a_{n-i+2}))$$

We can now group all the s_n terms together to give,

$$is_n = S_n - S_{n-i} + \sum_{\nu=n-i+1}^{n} (\nu - (n-i+1))a_\nu$$

which can be further simplified to

$$s_n = \frac{S_n - S_{n-i}}{i} + \frac{1}{2}(i-1)a_\gamma$$

where $\min a_j \leq a_\gamma \leq \max a_j$ for $n-i+1 \leq j \leq n$. Now, we once again use that

$$a_n \geq \frac{-K}{n} \text{ for some K} > 0 \text{ and } |S_n| \leq n\varepsilon$$

to give the lower bound,

$$s_n \geq \frac{-(2n+i)}{i}\varepsilon - K\frac{i-1}{2(n-i)}$$

and taking $i \approx 2n\sqrt{\varepsilon/K}$ we obtain a *lower bound* on s_n, (This may need an added step/additional justification)

$$s_n > -3\sqrt{K\varepsilon}$$

So together with the expression for the upper bound we have

$$-3\sqrt{K\varepsilon} < s_n < 3\sqrt{K\varepsilon}$$

and since ε was arbitrary, we conclude that $s_n \to 0$.

□

Remark 4.7. One can prove (see [3, I.5]) that if $\sum_{n\geq 0}$ is Abel convergent and $a_n = O_L(n^{-1})$ then $s_n = O_L(1)$. Theorem 4.1 then implies that $\sum_{n\geq 0} a_n$ is Cesaro convergent. Invoking Theorem 4.6 we deduce that $\sum_{n\geq 0} a_n$ is convergent in the usual sense. □

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