A STOCHASTIC GAUSS-BONNET-CHERN FORMULA

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ABSTRACT. We prove that a Gaussian ensemble of smooth random sections of a real vector bundle E over compact manifold M canonically defines a metric on E together with a connection compatible with it. Additionally, we prove a refined Gauss-Bonnet theorem stating that if the bundle E and the manifold M are oriented, then the Euler form of the above connection can be identified, as a current, with the expectation of the random current defined by the zero-locus of a random section in the above Gaussian ensemble.

CONTENTS

1. INTRODUCTION

1.1. **Notations and terminology.** Suppose that X is a smooth manifold. We denote by $|\Lambda_X| \to X$ the line bundle of 1-densities on X, [7, 11], so that we have a well defined integration map

$$\int_X : C^\infty(|\Lambda_X|) \to \mathbb{R}, \;\; C^\infty(|\Lambda_X|) \ni \rho \mapsto \int_X \rho(dx).$$

Suppose that F is a smooth vector bundle over X. We have two natural projections

$$\pi_x, \pi_y : X \times X \to X, \;\; \pi_x(x,y) = x, \;\; \pi_y(x,y) = y, \;\; \forall x, y \in X.$$

We set $F \boxtimes F := \pi_x^* F \otimes \pi_y^* F$, so that $F \boxtimes F$ is vector bundle over $X \times X$.

Following [7, Chap.VI,§1], we define generalized section of F to be a continuous linear functional on the space $C_0^\infty(F^* \otimes |\Lambda_X|)$ equipped with the natural locally convex topology. We denote by $C^{-\infty}(F)$ the space of generalized sections of F. We have a natural injection, [7, Chap.VI, §1]

$$i : C^\infty(F) \hookrightarrow C^{-\infty}(F).$$

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Recall that a Borel probability measure μ on $\mathbb{R}$ is called (centered) *Gaussian* if has the form

$$\mu(dx) = \boldsymbol{\gamma}_v(dx) := \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2v}} dx, & v > 0, \\ \delta_0, & v = 0. \end{cases}$$

where δ_0 denotes the Dirac measure concentrated at the origin.

1.2. **Gaussian ensembles of sections and correlators.** The concept of smooth random section of a vector bundle is very similar to the better known concept of random function. Since we were not able to locate a precise references, we present below a coordinate free description of the concept of Gaussian random section of a bundle. This description is implicitly contained in the fundamental work of R.A. Minlos [9], X. Fernique [5] and L. Schwartz [15].

Throughout this paper we fix a smooth compact connected manifold M of dimension m and a smooth real vector bundle $E \to M$ of rank r. The space $C^\infty(E^* \otimes |\Lambda_M|)$ is a nuclear countable Hilbert space in the of [6] and, a such, its dual $C^{-\infty}(E)$ satisfies several useful measure theoretic properties. Their proofs can be found in Appendix A.

Proposition 1.1. (i) *The σ-algebra of weakly Borel subsets of $C^{-\infty}(E)$ is equal with the σ-algebra of strongly Borel subsets. We will refer to this σ-algebra as the* Borel σ-algebra *of $C^{-\infty}(E)$.*

(ii) *Every Borel probability measure on $C^{-\infty}(E)$ is Radon.*

(iii) *Any Borel subset of $C^\infty(E)$ (with its natural topology) belongs to the Borel σ-algebra of $C^{-\infty}(E)$ when viewed as a subset of $C^{-\infty}(E)$.*

□

By definition, any section $\varphi \in C^\infty(E^* \otimes |\Lambda_M|)$ defines a continuous linear map

$$L_\varphi : C^{-\infty}(E) \to \mathbb{R}.$$

Following [3, 6] we define a Gaussian measure on $C^{-\infty}(E)$ to be a Borel measure $\boldsymbol{\Gamma}$ such that, for any section $\varphi \in C^\infty(E^* \otimes |\Lambda_M|)$ the pushforward $(L_\varphi)_\#(\boldsymbol{\Gamma})$ is a centered Gaussian $\boldsymbol{\gamma}_\varphi$ measure on $\mathbb{R}$.

The measure $\boldsymbol{\Gamma}$ is completely determined by its *covariance form* which is the symmetric, nonnegative definite bilinear map

$$\mathcal{K}_{\boldsymbol{\Gamma}} : C^\infty(E^* \otimes |\Lambda_M|) \times C^\infty(E^* \otimes |\Lambda_M|) \to \mathbb{R}$$

given by

$$\mathcal{K}_{\boldsymbol{\Gamma}}(\varphi, \psi) = \boldsymbol{E}_{\boldsymbol{\Gamma}}\big(L_\varphi \cdot L_\psi \big), \;\; \forall \varphi, \psi \in C^\infty(E^* \otimes |\Lambda_M|),$$

and $\boldsymbol{E}_{\boldsymbol{\Gamma}}$ denotes the expectation with respect to the probability measure $\boldsymbol{\Gamma}$.

Results of Fernique [5, Thm.II.2.3 + Thm.II.3.2] imply that $\mathcal{K}_\Gamma$ is separately continuous. According to Schwartz' kernel theorem [6, Chap.I, §3.5] the covariance form can be identified with a linear functional on the topological vector space

$$C_{\boldsymbol{\Gamma}} \in C^\infty\big((E^* \otimes |\Lambda_M|) \boxtimes (E^* \otimes |\Lambda_M|) \big) = C^\infty\big((E^* \boxtimes E^*) \otimes |\Lambda_{M\times M}| \big),$$

i.e., $C_{\boldsymbol{\Gamma}} \in C^{-\infty}(E \boxtimes E)$. We will refer to $C_{\boldsymbol{\Gamma}}$ as the *covariance kernel* of $\boldsymbol{\Gamma}$.

Theorem 1.2 (Minlos, [9])**.** *Given a generalized section $\in C^{-\infty}(E \boxtimes E)$ such that the associated bilinear form*

$$\mathcal{K} : C^\infty(E^* \otimes |\Lambda_M|) \times C^\infty(E^* \otimes |\Lambda_M|) \to \mathbb{R}$$

is symmetric and nonnegative definite, there exists a unique Gaussian measure on $C^{-\infty}(E)$ with covariance kernel C. □

Definition 1.3. A Gaussian measure $\boldsymbol{\Gamma}$ on $C^{-\infty}(E)$ is called *smooth* if $C_{\boldsymbol{\Gamma}}$ is given by a smooth section of $E \boxtimes E$. We will refer to it as the *covariance density*. We will refer to the smooth Gaussian measures on $C^{-\infty}(E)$ as *Gaussian ensemble of smooth sections* of E. □

A smooth section C of $E \boxtimes E$ can be viewed as a smooth family of bilinear maps

$$\tilde{C}_{\boldsymbol{x},\boldsymbol{y}} : E^*_{\boldsymbol{x}} \times E^*_{\boldsymbol{y}} \to \mathbb{R}, \quad \boldsymbol{x}, \boldsymbol{y} \in M,$$

given by

$$\tilde{C}_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{u}^*, \boldsymbol{v}^*) := \big\langle\, \boldsymbol{u}^* \otimes \boldsymbol{v}^*, C_{\boldsymbol{x},\boldsymbol{y}} \,\big\rangle, \ \ \forall \boldsymbol{u}^* \in E^*_{\boldsymbol{x}}, \ \ \boldsymbol{v}^* \in E^*_{\boldsymbol{y}},$$

where $\langle -, - \rangle$ denotes the natural pairing between a vector space and its dual. In the sequel we will identify $C_{\boldsymbol{x},\boldsymbol{y}}$ with the associated bilinear map $\tilde{C}_{\boldsymbol{x},\boldsymbol{y}}$.

The next result, proved in Appendix A, explains the role of the smoothness condition.

Proposition 1.4. *If the Gaussian measure $\boldsymbol{\Gamma}$ on $C^{-\infty}(E)$ is smooth, then $\boldsymbol{\Gamma}\big(\, C^{\infty}(E) \,\big) = 1$. In other words, a random generalized section in the Gaussian ensemble determined by $\boldsymbol{\Gamma}$ is a.s. smooth.* □

Using Propositions 1.1 and 1.4 we deduce that a smooth Gaussian measure on $C^{-\infty}(E)$ induces a Borel probability measure on $C^{\infty}(E)$. Since $C^{\infty}(E)$ is a standard space[1] in the sense of [5] this measure is also a Radon measure. Observe also that for any $\boldsymbol{x} \in M$ the induced map

$$C^{\infty}(E) \to E_{\boldsymbol{x}}, \ \ C^{\infty}(E) \ni \varphi \mapsto \varphi(\boldsymbol{x}) \in E_{\boldsymbol{x}}$$

is Borel measurable. The next result, proved in Appendix A, shows that the collection of random variables $(\, \varphi(\boldsymbol{x}) \,)_{\boldsymbol{x} \in M}$ is Gaussian.

Proposition 1.5. *Suppose that $\boldsymbol{\Gamma}$ is a smooth Gaussian measure on E with covariance density C. Let n be a positive integer. Then for any points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in M$ and any $\boldsymbol{u}^*_i \in E^*_{\boldsymbol{x}_i}$, $i = 1, \dots, n$ the random vector*

$$C^{\infty}(E) \ni \varphi \mapsto \big(\, X_1(\varphi), \dots, X_n(\varphi) \,\big) \in \mathbb{R}^n, \ \ X_i(\varphi) := \langle\, \boldsymbol{u}^*_i, \varphi(\boldsymbol{x}_i) \,\rangle, \ \ i = 1, \dots, n,$$

is Gaussian, Moreover

$$\boldsymbol{E}(X_i X_j) = C_{\boldsymbol{x}_i,\boldsymbol{x}_j}(\boldsymbol{u}^*_i, \boldsymbol{u}^*_j), \ \ \forall i, j. \tag{1.1}$$

□

A section $C \in C^{\infty}(E \boxtimes E)$ is called *symmetric* if

$$C_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{u}^*, \boldsymbol{v}^*) = C_{\boldsymbol{y},\boldsymbol{x}}(\boldsymbol{v}^*, \boldsymbol{u}^*), \ \ \forall \boldsymbol{x}, \boldsymbol{y} \in M, \ \ \forall \boldsymbol{u}^* \in E^*_{\boldsymbol{x}}, \ \ \boldsymbol{v}^* \in E^*_{\boldsymbol{y}}.$$

If C is the covariance density of a smooth Gaussian measure Γ on $C^{-\infty}(E)$, then Proposition 1.5 shows that C is symmetric.

A symmetric section section $C \in C^{\infty}(E \boxtimes E)$ is called *nonnegative/positive definite* if all the symmetric bilinear forms $C_{\boldsymbol{x},\boldsymbol{x}}$ are such. Clearly the covariance density of a smooth Gaussian measure Γ on $C^{-\infty}(E)$ is symmetric and nonnegative definite.

Definition 1.6. (a) A *correlator* on E is a section $C \in C^{\infty}(E \boxtimes E)$ which is symmetric and positive definite.

(b) A Gaussian ensemble of smooth sections of E is called *nondegenerate* if its covariance density is a correlator. □

Lemma 1.7. *There exist nondegenerate Gaussian ensembles of smooth sections of E.*

[1] Fernique's standard spaces are also known as Lusin spaces.

Proof. We follow the ideas in [14]. Fix a finite dimensional subspace $\boldsymbol{U} \in C^\infty(E)$ which is ample, i.e., for any $\boldsymbol{x} \in M$ the evaluation map

$$\mathbf{ev}_{\boldsymbol{x}} : \boldsymbol{U} \to E_{\boldsymbol{x}}, \;\; \boldsymbol{u} \mapsto \boldsymbol{u}(\boldsymbol{x})$$

is onto. By duality we obtain injections

$$\mathbf{ev}_{\boldsymbol{x}}^* : E_{\boldsymbol{x}}^* \to \boldsymbol{U}^*.$$

Fix an Euclidean inner product $(-,-)_{\boldsymbol{U}}$ on $\boldsymbol{U}$ and denote by $\boldsymbol{\gamma}$ the Gaussian measure on $\boldsymbol{U}$ canonically determined by this product. Its covariance pairing $\boldsymbol{U}^* \times \boldsymbol{U}^* \to \mathbb{R}$ coincides with $(-,-)_{\boldsymbol{U}^*}$, the inner product on $\boldsymbol{U}^*$ induced by $(-,-)_{\boldsymbol{U}}$. More precisely, this means that for any $\xi, \eta \in \boldsymbol{U}^*$ we have

$$(\xi,\eta)_{\boldsymbol{U}^*} = \int_{\boldsymbol{U}} \langle \xi, s\rangle\langle \eta, s\rangle \boldsymbol{\gamma}(ds). \tag{1.2}$$

The measure $\boldsymbol{\gamma}$ defines a smooth Gaussian measure $\hat{\boldsymbol{\gamma}}$ on $C^{-\infty}(E)$ such that $\hat{\boldsymbol{\gamma}}(\boldsymbol{U}) = 1$. Concretely, $\hat{\boldsymbol{\gamma}}$ is the pushforward of $\boldsymbol{\gamma}$ via the natyural inclusion $\boldsymbol{U} \hookrightarrow C^\infty(E)$. This is a smooth measure. Its covariance density C is computed as follows: if $\boldsymbol{x}, \boldsymbol{y} \in M$, $\boldsymbol{u}^* \in E_{\boldsymbol{x}}^*$, $\boldsymbol{v}^* \in E_{\boldsymbol{y}}^*$, then

$$C_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{u}^*, \boldsymbol{v}^*) = \int_{\boldsymbol{U}} \langle \boldsymbol{u}^*, s(\boldsymbol{x})\rangle\langle \boldsymbol{v}^*, s(\boldsymbol{y})\rangle \gamma(ds) = \int_{\boldsymbol{U}} \langle \mathbf{ev}_{\boldsymbol{x}}^* \boldsymbol{u}^*, s\rangle\langle \mathbf{ev}_{\boldsymbol{y}}^* \boldsymbol{v}^*, s\rangle \boldsymbol{\gamma}(ds)$$

$$\stackrel{(1.2)}{=} (\mathbf{ev}_{\boldsymbol{x}}^* \boldsymbol{u}^*, \mathbf{ev}_{\boldsymbol{y}}^* \boldsymbol{v}^*).$$

In particular, when $\boldsymbol{x} = \boldsymbol{y}$ we observe that $C_{\boldsymbol{x},\boldsymbol{x}}$ coincides with the restriction to $E_{\boldsymbol{x}}^*$ of the inner product $(-,-)_{\boldsymbol{U}^*}$. In particular, the form $C_{\boldsymbol{x},\boldsymbol{x}}$ is positive definite. □

Definition 1.8. A Gaussian ensemble of smooth sections of E with associated Gaussian measure $\boldsymbol{\Gamma}$ on $C^{-\infty}(E)$ is said to have *finite type* if there exists a finite dimensional subspace $\boldsymbol{U} \subset C^\infty(E)$ such that $\boldsymbol{\Gamma}(\boldsymbol{U}) = 1$. □

The Gaussian ensemble constructed in Lemma 1.7 has finite type. Moreover, all the nondegenerate finite type Gaussian ensembles of smooth sections can be obtained in this fashion. However, there exist nondegenerate gaussian ensembles which are not of finite type.

Definition 1.9. A correlator $C \in C^\infty(E \boxtimes E)$ is called *stochastic* if it is the covariance density of a nondegenerate Gaussian ensemble smooth sections of E. □

Minlos' Theorem1.2 shows that not all correlators are stochastic.

1.3. **Statements of the main results.** The main goal of this paper is to investigate some of the rich geometry of a nondegenerate Gaussian ensemble of smooth sections of E. By definition, the correlator C of such an ensemble defines a metric on the dual bundle E^*, and thus on E as well. Less obvious is that the correlator C also induces a connection ∇^C on E compatible with the above metric. We will refer to this metric/connection as the correlator metric/connection. We prove this fact in Proposition 2.4.

This connection depends only on the first order jet of C along the diagonal of M. Using the corellator metric we can identify the bilinear form $C_{\boldsymbol{x},\boldsymbol{y}}$ with a linear map $T_{\boldsymbol{x},\boldsymbol{y}} : E_{\boldsymbol{y}} \to E_{\boldsymbol{x}}$. The definition of the connection shows that its infinitesimal parallel transport is given by the first order jet of $T_{\boldsymbol{x},\boldsymbol{y}}$ along the diagonal $\boldsymbol{x} = \boldsymbol{y}$. The construction of ∇^C feels very classical, but we were not able to trace any reference.

The inspiration for this construction came from our earlier work [14] where we associated a metric and a compatible connection to an arbitrary *finite type*, nondegenerate Gaussian ensemble of sections of E. As we have explained in [14], the results of Narasimhan and Ramanan [10] show that any metric together with a connection compatible with it are the metric and connection of a correlator associated to a finite type Gaussian ensemble of sections.

In Example 2.1 we show that any embedding of the manifold M in an Euclidean space canonically defines a correlator on TM. The correlator metric is then the metric induced by the embedding and the correlator connection coincides with the Levi-Civita connection. In fact this correlator is a stochastic correlator.

If the correlator C is stochastic, then the connection ∇^C and its curvature can be given a probabilistic interpretations. The equality (2.5) gives a purely probabilistic description of the connection ∇^C. Proposition 2.6 gives a purely probabilistic description of its curvature. This result contains as a special case Gauss' Theorema Egregium.

The connection determined by a Gaussian ensemble through its correlator C has other useful probabilistic properties. In Corollary 2.8 we show that if $\boldsymbol{u}$ is a random section of the ensemble, and ∇^C is the correlator connection, then for any $\boldsymbol{x} \in M$ the random vectors $\nabla^C \boldsymbol{u}(\boldsymbol{x}) \in T^*_{\boldsymbol{x}} M \otimes E_{\boldsymbol{x}}$ and $\boldsymbol{u}(\boldsymbol{x}) \in E_{\boldsymbol{x}}$ are independent. This independence is an immediate consequence of the probabilistic equality (2.5).

In [1, §12.2] Adler and Taylor associate to a Gaussian random function $\boldsymbol{u}$ on a smooth manifold M a Riemann metric and they gave a probabilistic description to the corresponding Levi-Civita connection and its curvature, the Riemann tensor. Their construction is a special case of correlator connection. More precisely, the random function $\boldsymbol{u}$ defines a Gaussian random section $d\boldsymbol{u}$ of T^*M. The induced metrics on TM and T^*M are the metrics associated to the correlator of this random section. Moreover, the correlator connection coincides with the Levi-Civita connection of the correlator metric.

Section 3 contains the main result of this paper. Here we assume that both M and E are oriented, and the rank of E is even and not greater than the dimension of M. Given a nondegenerate Gaussian ensemble of smooth sections of E we obtain as we know a metric and a connection ∇ on E. The Chern-Weil construction associates to this connection an *Euler form* $\boldsymbol{e}(E, \nabla) \in \Omega^r(M)$. This form is closed and, as the name suggests, its cohomology class is the Euler class of E (with real coefficients), We refer to[11, Chap.8] for more details.

If $\boldsymbol{u}$ is a smooth section of E transversal to the zero section, then its zero set $Z_{\boldsymbol{u}}$ is a compact, codimension r-submanifold of M equipped with a canonical orientation. As such it defines a *closed* integration current $[Z_{\boldsymbol{u}}]$ of dimension $(m - r)$ whose cohomology class is independent of of the choice of transversal section $\boldsymbol{u}$. This means that if $\boldsymbol{u}, \boldsymbol{v}$ are two sections of E transversal to the zero section, then for any closed form $\eta \in \Omega^{m-r}(M)$ we have

$$\int_{Z_{\boldsymbol{u}}} \eta = \int_{Z_{\boldsymbol{v}}} \eta.$$

The classical Gauss-Bonnet-Chern theorem gives a geometric description of the cohomology class of the zero-set current $[Z_{\boldsymbol{u}}]$; see cite[Sec. 8.3.2]N1. More precisely, it states that

$$\int_{Z_{\boldsymbol{u}}} \eta = \int_M \eta \wedge \boldsymbol{e}(E, \nabla), \quad \forall \eta \in \Omega^{m-r}(M), \quad d\eta = 0. \tag{1.3}$$

Suppose now that the Gaussian ensemble of sections is *transversal*, i.e., a random section $\boldsymbol{u}$ of this ensemble is a.s. transversal to the zero section. (In Remark 3.3 we describe several conditions on the ensemble guaranteeing transversality.) In Theorem 3.4 we prove a stochastic Gauss-Bonnet formula stating that the expectation of the random current $[Z_{\boldsymbol{u}}]$ is equal to the current defined by the Euler form $\boldsymbol{e}(E,\nabla)$, i.e.,

$$\boldsymbol{E}\left(\int_{Z_{\boldsymbol{u}}}\eta\right)=\int_M \eta\wedge\boldsymbol{e}(E,\nabla),\ \ \forall\eta\in\Omega^{m-r}(M). \tag{1.4}$$

Let us point out that the cohomological formula (1.3) is a consequence of the above equality. However, the stochastic formula is stronger than the cohomological one because the Euler class of E could be zero yet there exist metric connections on E whose associated Euler forms are nonzero.

We prove the stochastic formula (1.4) result by reducing it to the the Kac-Rice formula [2, Thm. 6.1] using a bit of differential geometry and certain Gaussian computations we borrowed from [1]. For the reader's convenience we have included in Appendix B a brief survey of these facts.

In our earlier work [14] we proved a special case of this stochastic Gauss-Bonnet formula for nondegenerate Gausian ensembles of finite type. These are automatically the proof. The proof in [14] is differential geometric in nature and does not extend to the general situation discussed in the present paper.

In [13] we used related probabilistic techniques to prove a cohomological Gauss-Bonnet-Chern formula of the type (1.3) in the special case when $E=TM$, and the connection ∇ is the Levi-Civita connection of a metric on TM. Still in the case $E=TM$, one can used rather different probablistic ideas (Malliavin calculus) to prove the cohomological Gauss-Bonnet; the case when ∇ is the Levi-Civita connection of a metric on M was investigated by E. Hsu [8], while the case of a general metric connection on TM was recently investigated by H. Zhao [16].

2. The differential geometry of correlators

A correlator on a real vector bundle $E\to M$ naturally induces additional geometric structures on E. More precisely, we will show that it induces a metric on E together with a connection compatible with it.

Before we proceed to describing the geometric structures naturally associated to a correlator we want to present a few circumstances that lead to correlators.

Example 2.1. (a) Suppose that M is a properly embedded submanifold of the Euclidean space $\boldsymbol{U}$. Then the inner product $(-,-)_{\boldsymbol{U}}$ on $\boldsymbol{U}$ induces a correlator $C\in C^\infty(T^*M\boxtimes T^*M)$ defined by the equalities

$$C_{\boldsymbol{x},\boldsymbol{y}}(X,Y)=(X,Y)_{\boldsymbol{U}},\ \ \forall\boldsymbol{x},\boldsymbol{y}\in M,\ \ X\in T_{\boldsymbol{x}}M\subset\boldsymbol{U},\ \ Y\in T_{\boldsymbol{y}}M\subset\boldsymbol{U}.$$

(b) For any real vector space $\boldsymbol{U}$ and any smooth manifold M we denote by $\underline{\boldsymbol{U}}_M$ the trivial vector bundle over M with fiber $\boldsymbol{U}$

$$\boldsymbol{U}_M=\boldsymbol{U}\times M\to M.$$

Suppose that $\boldsymbol{U}$ is a real, finite dimensional Euclidean space with inner product $(-,-)$. We denote by $(-,-)_*$ the induced inner product on $\boldsymbol{U}^*$. Suppose that $E\to M$ is a smooth real vector bundle over M and

$$P:\underline{\boldsymbol{U}}\to E$$

is a fiberwise surjective bundle morphism. In other words, E is a quotient bundle of a trivial real metric vector bundle. This induces an injective bundle morphism

$$P^*:E^*\to\underline{\boldsymbol{U}}_M^*.$$

Hence E^* is a subbundle of a trivial metric real vector bundle.

For any $\boldsymbol{x}\in M$ and any $\boldsymbol{u}^*\in E_{\boldsymbol{x}}$ we obtain a vector $P^*_{\boldsymbol{x}}\boldsymbol{u}^*\in\boldsymbol{U}_{\boldsymbol{x}}=$ the fiber of $\underline{\boldsymbol{U}}_M$ at $\boldsymbol{x}\in M$. This allows us to define a correlator $C\in C^\infty(E\boxtimes E)$ given by

$$C_{\boldsymbol{x}_1,\boldsymbol{x}_2}(\boldsymbol{u}_1^*,\boldsymbol{u}_2^*)=\big(\,P^*_{\boldsymbol{x}_1}\boldsymbol{u}_1^*,\ \ P_{\boldsymbol{x}_2}\boldsymbol{u}_2^*\,\big),\ \ \forall\boldsymbol{x}_i\in M,\ \ \boldsymbol{u}_i^*\in E_{\boldsymbol{x}_i}\ \ i=1,2.$$

(c) Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and

$$f : \Omega \times M \to \mathbb{R}, \;\; (\omega, \boldsymbol{x}) \mapsto f_\omega(\boldsymbol{x}),$$

is a Gaussian random function on M such that the sample functions $f_\omega : M \to \mathbb{R}$ are almost surely (a.s.) smooth. Assume for simplicity that

$$\boldsymbol{E}_{\mathbb{P}}\big(f_\omega(\boldsymbol{x})\big) = 0, \;\; \forall \boldsymbol{x} \in M.$$

We define $C \in C^\infty(T^*M \boxtimes T^*M)$ as follows: if $\boldsymbol{x}, \boldsymbol{y} \in M$ and X, Y are smooth vector fields on M, then

$$C_{\boldsymbol{x},\boldsymbol{y}}(X_{\boldsymbol{x}}, Y_{\boldsymbol{y}}) := \boldsymbol{E}_{\mathbb{P}}\big(Xf(\boldsymbol{x}) \cdot Yf(\boldsymbol{u}) \big).$$

This is clearly a symmetric section. Note also that $C_{\boldsymbol{x},\boldsymbol{x}} : T_{\boldsymbol{x}}M \times T_{\boldsymbol{x}}M \to \mathbb{R}$ can be identified with the covariance form of the gaussian vector

$$\Omega \ni \omega \mapsto df_\omega(\boldsymbol{x}) \in T^*_{\boldsymbol{x}}M.$$

We see that C is a correlator on T^*M if and only if the Gaussian random vector $df(\boldsymbol{x})$ is nondegenerate for any $\boldsymbol{x} \in M$. □

Remark 2.2. We want to point out the construction in Example 2.1 (a) is a special case of both (b) and (c). The correlator in (c) is obviously stochastic. The proof of Lemma 1.7 shows that the correlators defined in (a) and (b) are also stochastic. □

Observe that, by definition, a correlator $C \in C^\infty(E \boxtimes E)$ induces a metric on E^* and thus, by duality, a metric on E. We will denote both these metric by $(-,-)_{E^*,C}$ and respectively $(-,-)_{E,C}$. When no confusion is possible will will drop the subscript E or E^* from the notation. To simplify the presentation we adhere to the following conventions.

(i) We will use the Latin letters i, j, k to denote indices in the range $1, \dots, m = \dim M$.
(ii) We will use Greek letters α, β, γ to denote indices in the range $1, \dots, r = \operatorname{rank}(E)$.

Using the metric duality we obtain an induced correlator $C^\dagger$ on E^* defined by bilinear forms

$$C^\dagger_{\boldsymbol{x},\boldsymbol{y}} : E_{\boldsymbol{x}} \times E_{\boldsymbol{y}} \to \mathbb{R}.$$

If $(\boldsymbol{e}_\alpha)$ is a local frame of E defined over an open coordinated neighborhood $\mathcal{O}$ with coordinates (x^i), and $(\boldsymbol{e}^\alpha)$ is the dual frame of E^* defined over the same neighborhood, then the correlator C is described by the matrices

$$C_{x,y} = \big(C^{\alpha,\beta}(x,y) \big)_{1 \leq \alpha,\beta \leq r}, \;\; C^{\alpha,\beta}(x,y) = C_{x,y}(\boldsymbol{e}^\alpha(x), \boldsymbol{e}^\beta(y)), \;\; x, y \in \mathcal{O}.$$

We denote by $C(x)$ the matrix

$$C(x) := C_{x,x} = \big(C^{\alpha,\beta}(x,x) \big)_{1 \leq \alpha,\beta \leq r}.$$

The isomorphism $\boldsymbol{D}_x : E_x \to E^*_x$ induced by the metric $(-,-)_{E,C}$,

$$\big\langle \boldsymbol{D}_x \boldsymbol{u}, \boldsymbol{v} \big\rangle = (\boldsymbol{u}, \boldsymbol{v})_{E,C}, \;\; \forall \boldsymbol{u}, \boldsymbol{v} \in E_x,$$

is described in the bases $(\boldsymbol{e}_\alpha(x))$ and $(\boldsymbol{e}^\beta(x))$ by the inverse matrix

$$C^{-1}_{x,x} = \big(C_{\alpha,\beta}(x) \big)_{1 \leq \alpha,\beta \leq r}, \;\; \sum_\beta C_{\alpha\beta}(x) C^{\beta\gamma}(x,x) = \delta^\gamma_\alpha.$$

More precisely,

$$\boldsymbol{D}_x \boldsymbol{e}_\alpha(x) = \sum_\beta C_{\beta\alpha}(x) \boldsymbol{e}^\beta(x).$$

Note that

$$C^{\dagger}_{\boldsymbol{x},\boldsymbol{y}}\big(\,\boldsymbol{e}_\alpha(\boldsymbol{x}),\boldsymbol{e}_\beta(\boldsymbol{y})\,\big)=C_{\boldsymbol{x},\boldsymbol{y}}\big(\,\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{e}_\alpha(\boldsymbol{x}),\boldsymbol{D}_{\boldsymbol{y}}\boldsymbol{e}_\beta(\boldsymbol{y})\,\big)$$
$$=\sum_{\alpha',\beta'}C_{\alpha'\alpha}(x)C^{\alpha'\beta'}(x,y)C_{\beta'\beta}(y).$$

Taking into account that $C_{x,x}$ is a symmetric matrix we can rewrite the above equality in a more compact form

$$C^{\dagger}_{x,x}=C^{-1}_{x,x}C_{x,y}C^{-1}_{y,y}.$$

In particular

$$C^{\dagger}_{x,x}=C^{-1}_{x,x}.$$

Using the metric $(-,-)_C$ we can identify $C_{\boldsymbol{x},\boldsymbol{y}}\in E_{\boldsymbol{x}}\otimes E_{\boldsymbol{y}}$ with an element of

$$T_{\boldsymbol{x},\boldsymbol{y}}\in E_{\boldsymbol{x}}\otimes E^*_{\boldsymbol{y}}\cong \operatorname{Hom}(E_{\boldsymbol{y}},E_{\boldsymbol{x}}).$$

We will refer to $T_{x,y}$ as the *tunneling map* associated to the correlator C. If $(\boldsymbol{e}_\alpha(x)\,)_{1\leq\alpha\leq r}$ is a local, $(-,-)_C$-*orthonormal* frame, then we have

$$C_{\alpha\beta}(x,x)=\delta_{\alpha\beta}=C^{\alpha\beta}(x,x),$$
$$C_{\boldsymbol{x},\boldsymbol{y}}=\sum_{\alpha,\beta}C^{\alpha,\beta}(x,y)\boldsymbol{e}_\alpha(x)\otimes\boldsymbol{e}_\beta(y),$$

and

$$T_{\boldsymbol{x},\boldsymbol{y}}=\sum_{\alpha,\beta}C^{\alpha,\beta}(x,y)\boldsymbol{e}_\alpha(x)\otimes\boldsymbol{D}_y\boldsymbol{e}_\beta(y)=\sum_{\alpha,\beta,\gamma}C^{\alpha,\beta}(x,y)C_{\gamma\beta}(y)\boldsymbol{e}_\alpha(x)\otimes\boldsymbol{e}^\gamma(y)$$
$$=\sum_{\alpha,\beta,\gamma}C^{\alpha,\beta}(x,y)C_{\beta\gamma}(y)\boldsymbol{e}_\alpha(x)\otimes\boldsymbol{e}^\gamma(y)=\sum_{\alpha,\beta,\gamma}C^{\alpha,\beta}(x,y)\boldsymbol{e}_\alpha(x)\otimes\boldsymbol{e}_\beta(y).$$

We can write this

$$T_{\boldsymbol{x},\boldsymbol{y}}=C_{x,y}.$$

Note that $T_{\boldsymbol{x},\boldsymbol{x}}=\mathbb{1}_{E_{\boldsymbol{x}}}$. If we denote by $T^*_{\boldsymbol{x},\boldsymbol{y}}\in\operatorname{Hom}(E_{\boldsymbol{y}},E_{\boldsymbol{x}})$ the adjoint of $T_{\boldsymbol{x},\boldsymbol{y}}$ with respect to the metric $(-,-)_E$, then the symmetry of C implies that

$$T_{\boldsymbol{y},\boldsymbol{x}}=T^*_{\boldsymbol{x},\boldsymbol{y}}.$$

We recall that for any vector space V and any smooth manifold X we denote by $\underline{V}_X$ the trivial bundle $V\times X\to X$.

Lemma 2.3. *Fix a point $\boldsymbol{p}_0\in M$ and local coordinates $(x^i)_{1\leq i\leq m}$ in a neighborhood $\mathcal{O}$ of $\boldsymbol{p}_0$ in M. Suppose that*

$$\underline{\boldsymbol{e}}(x)=(\boldsymbol{e}_\alpha(x))_{1\leq\alpha\leq r}$$

is a local $(-,-)_C$-orthononomal frame of $E|_{\mathcal{O}}$ which we regard as an isomorphism of metric bundles

$$\underline{\mathbb{R}}^r_{\mathcal{O}}\to E|_{\mathcal{O}}.$$

We obtain a smooth map

$$T(\underline{\boldsymbol{e}}):\mathcal{O}\times\mathcal{O}\to\operatorname{Hom}(\mathbb{R}^r),\ \ (x,y)\mapsto T(\underline{\boldsymbol{e}})_{x,y}=\underline{\boldsymbol{e}}(x)^{-1}T_{x,y}\underline{\boldsymbol{e}}(y).$$

Then for any $i=1,\dots,m$ the operator

$$\partial_{x^i}T(\underline{\boldsymbol{e}})_{x,y}|_{x=y}:\underline{\mathbb{R}}^r_y\to\underline{\mathbb{R}}^r_y,$$

is skew-symmetric.

Proof. We identify $\mathcal{O}\times\mathcal{O}$ with an open neighborhood of $(0,0)\in\mathbb{R}\times\mathbb{R}$ with coordinates (x^i,y^j). Introduce new coordinates

$$z^i = x^i - y^i,\;\; s^j = x^j + y^j,$$

so that

$$\partial_{x^i} = \partial_{z^i} + \partial_{s^i}.$$

We view the map $T(\underline{\boldsymbol{e}})$ as depending on the variables z,s. Note that

$$T(\underline{\boldsymbol{e}})_{0,s} = \mathbb{1},\;\; T(\underline{\boldsymbol{e}})_{-z,s} = T(\underline{\boldsymbol{e}})^*_{z,s},\;\; \forall z,s.$$

We deduce that

$$\begin{gathered}\partial_{s^i}T(\underline{\boldsymbol{e}})|_{0,s} = \partial_{s^i}T(\underline{\boldsymbol{e}})|^*_{0,s} = 0,\\ \partial_{x^i}T(\underline{\boldsymbol{e}})|_{0,s} = \partial_{z^i}T(\underline{\boldsymbol{e}})|_{0,s} + \partial_{s^i}T(\boldsymbol{e})|_{0,s} = \partial_{z^i}T(\underline{\boldsymbol{e}})|_{0,s},\\ \big(\,\partial_{x^i}T(\underline{\boldsymbol{e}})|_{0,s}\,\big)^* = \partial_{x^i}T(\underline{\boldsymbol{e}})^*|_{0,s} = -\partial_{z_i}T(\underline{\boldsymbol{e}})|_{0,s} + \partial_{s^i}T(\underline{\boldsymbol{e}})|_{0,s} = -\partial_{x^i}T(\underline{\boldsymbol{e}})|_{0,s}.\end{gathered}$$

□

Given a coordinate neighborhood with coordinates (x^i) and a local isomorphism of metric vector bundles (local orthonormal frame)

$$\underline{\boldsymbol{e}} : \underline{\mathbb{R}}^r_{\mathcal{O}} \to E|_{\mathcal{O}}$$

as above, we define skew-symmetric endomorphisms

$$\Gamma_i(\underline{\boldsymbol{e}}) : \underline{\mathbb{R}}^r_{\mathcal{O}} \to \underline{\mathbb{R}}^r_{\mathcal{O}},\;\; i=1,\dots,m=\dim M,\;\; \Gamma_i(\underline{\boldsymbol{e}})_y = -\partial_{x^i}T_{x,y}|_{x=y}. \tag{2.1}$$

We obtain a 1-form with mattix coefficients

$$\Gamma(\underline{\boldsymbol{e}}) = \sum_i \Gamma_i(\underline{\boldsymbol{e}})dy^i.$$

The operator

$$\nabla^{\underline{\boldsymbol{e}}} = d + \Gamma(\underline{\boldsymbol{e}}) \tag{2.2}$$

is then a connection on $\underline{\mathbb{R}}^r_{\mathcal{O}}$ compatible with the metric natural metric on this trivial bundle. The isomorphism $\underline{\boldsymbol{e}}$ induces a metric connection $\underline{\boldsymbol{e}}_*\nabla^{\underline{\boldsymbol{e}}}$ on $E|_{\mathcal{O}}$.

Suppose that $\underline{\boldsymbol{f}} : \underline{\mathbb{R}}^r_{\mathcal{O}} \to E|_{\mathcal{O}}$ is another orthonormal frame of $E_{\mathcal{O}}$ related to $\underline{\boldsymbol{e}}$ via a transition map

$$g : \mathcal{O} \to O(r),\;\; \underline{\boldsymbol{f}} = \underline{\boldsymbol{e}}\cdot g.$$

Then

$$T(\underline{\boldsymbol{f}})_{x,y} = g^{-1}(x)T(\underline{\boldsymbol{e}})_{x,y}g(y).$$

We denote by d_x the differential with respect to the x variable. We deduce

$$\begin{aligned}\Gamma(\underline{\boldsymbol{f}})_y &= -d_xT(\underline{\boldsymbol{f}})_{x,y}|_{x=y}\\ &= -\big(\,d_xg^{-1}(x)\,\big)_{x=y}\cdot\underbrace{T(\underline{\boldsymbol{e}})_{y,y}}_{=\mathbb{1}}\cdot g(y) - g^{-1}(y)\big(\,d_xT(\underline{\boldsymbol{e}})_{x,y}\big)|_{x=y}g(y)\end{aligned}$$

$(d(g^{-1}) = -g^{-1}\cdot dg\cdot g^{-1})$

$$= g^{-1}(y)dg(y)g^{-1}(y)\cdot g(y) + g^{-1}(y)\Gamma(\underline{\boldsymbol{e}})_yg(y) = g(y)^{-1}dg(y) + g^{-1}(y)\Gamma(\underline{\boldsymbol{e}})_yg(y).$$

Thus

$$\Gamma(\underline{\boldsymbol{e}}\cdot g) = g^{-1}dg + g^{-1}\Gamma(\underline{\boldsymbol{e}})g.$$

This shows that for any local orthonormal frames $\underline{\boldsymbol{e}}$, $\underline{\boldsymbol{f}}$ of $E|_{\mathcal{O}}$ we have

$$\underline{\boldsymbol{e}}_*\nabla^{\underline{\boldsymbol{e}}} = \underline{\boldsymbol{f}}_*\nabla^{\underline{\boldsymbol{f}}}.$$

We have thus proved the following result.

Proposition 2.4. *If $E \to M$ is a smooth real vector bundle, then any correlator C on M induces a canonical metric $(-,-)_C$ on E and a connection ∇^C compatible with this metric. More explicitly, if $\mathcal{O} \subset M$ is an coordinate neighborhood on M and $\underline{\boldsymbol{e}} : \underline{\mathbb{R}}^r_{\mathcal{O}} \to E|_{\mathcal{O}}$ is an orthogonal trivialization , then ∇^C is described by*

$$\nabla^C = d + \sum_i \Gamma_i(\underline{\boldsymbol{e}})dx^i,$$

where the skew-symmetric $r \times r$-matrix $\Gamma_i(\underline{\boldsymbol{e}})$ is given by (2.1). □

Remark 2.5. (a) In [14] we have shown that any pair metric on E + connection on E compatible with the metric is determined by the correlator of a finite type Gaussian ensemble of smooth sections of E.

(b) In the special case described in Example 2.1, the connection associated to the corresponding correlator coincides with the Levi-Civita connection of the metric induced by the correlator.

The metric and the connection associated to the correlator described in Example 2.1(c) where discussed in great detail by Adler and Taylor, [1, §12.2]. In this case the connection determined by the correlator coincides with the Levi-Civita connection of the metric determined by the correlator.

(c) Suppose that we fix local coordinates (x^i) near a point $\boldsymbol{p}_0$ such that $x^i(\boldsymbol{p}_0) = 0$. We denote by $P_{x,0}$ the parallel transport of ∇^C from 0 to X along the line segment from 0 to x. Then

$$P_{0,0} = \mathbb{1}_{E_0} = T_{0,0}, \;\; \partial_{x^i} P_{x,0}|_{x=0} = -\Gamma_i(0) = \partial_{x^i,0} T_{x,0}|_{x=0}.$$

We see that the tunneling map $T_{x,0}$ is a first order approximation at 0 of the parallel transport map $P_{x,0}$ of the connection ∇^C. □

For later use, we want to give a more explicit description of the curvature of the connection ∇^C in the special case when the correlator C *stochastic* and thus it is the covariance density of a nondegenerate Gaussian ensemble of smooth sections of E.

Proposition 2.6. *Suppose that C is a stochastic correlator on E defined by the nondegenerate Gaussian ensemble smooth random sections of E. Denote by $\boldsymbol{u}$ a random section in this ensemble. Fix a point $\boldsymbol{p}_0$, local coordinates (x^i) on M near $\boldsymbol{p}_0$ such that $x^i(\boldsymbol{p}_0) = 0 \; \forall i$, and a local $(-,-)_C$-orthonormal frame $\big(\boldsymbol{e}_\alpha(x)\big)_{1 \le \alpha \le r}$ of E in a neighborhood of $\boldsymbol{p}_0$ which is is synchronous at $\boldsymbol{p}_0$,*

$$\nabla^C \boldsymbol{e}_\alpha|_{\boldsymbol{p}_0} = 0, \;\; \forall \alpha.$$

Denote by F the curvature of ∇^C,

$$F = \sum_{ij} F_{ij}(x) dx^i \wedge dx^j, \;\; F_{ij}(x) \in \operatorname{End}(E_{\boldsymbol{p}_0}).$$

Then $F_{ij}(0)$ is the endomorphism of $E_{\boldsymbol{p}_0}$ which in the frame $\boldsymbol{e}_\alpha(\boldsymbol{p}_0)$ is described by the $r \times r$ matrix with entries

$$F_{\alpha\beta|ij}(0) := \boldsymbol{E}\big(\partial_{x^i} u_\alpha(x) \partial_{x^j} u_\beta(x)\big)|_{x=0} - \boldsymbol{E}\big(\partial_{x^j} u_\alpha(x) \partial_{x^i} u_\beta(x)\big)|_{x=0}, \;\; 1 \le \alpha, \beta \le r, \tag{2.3}$$

where $u_\alpha(x)$ is the random function

$$u_\alpha(x) := \big(\boldsymbol{u}(x), \boldsymbol{e}_\alpha(x)\big)_C.$$

Proof. The random section $\boldsymbol{u}$ has the local description

$$\boldsymbol{u} = \sum_\alpha u_\alpha(x) \boldsymbol{e}_\alpha(x).$$

Then $T(x,y)$ is a linear map $E_y \to E_x$ given by the $r \times r$ matrix

$$T(x,y) = \big(\, T_{\alpha\beta}(x,y)\,\big)_{1\leq \alpha,\beta\leq r},\;\; T_{\alpha\beta}(x,y) = \boldsymbol{E}(\, u_\alpha(x)u_\beta(y)\,\big).$$

The coefficients of the connection 1-form $\Gamma = \sum_i \Gamma_i dx^i$ are endomorphisms of E_x given by $r \times r$ matrices

$$\Gamma_i(x) = \big(\,\Gamma_{\alpha\beta|i}(x)\,\big)_{1\leq\alpha,\beta\leq r}.$$

More precisely, we have

$$\Gamma_{\alpha\beta|i}(x) = -\boldsymbol{E}\big(\,\partial_{x^i}u_\alpha(x)u_\beta(x)\,\big). \tag{2.4}$$

Because the frame $\big(\,\boldsymbol{e}_\alpha(x)\,\big)$ is synchronous at $x=0$ we deduce that, *at* $\boldsymbol{p}_0$, we have $\Gamma_i(0)=0$ and

$$F(\boldsymbol{p}_0) = \sum_{i<j} F_{ij}(x)dx^i\wedge dx^j \in \operatorname{End}(E_{\boldsymbol{p}_0})\otimes \Lambda^2 T^*_{\boldsymbol{p}_0}M,\;\; F_{ij} = \partial_{x^i}\Gamma_j(\boldsymbol{p}_0) - \partial_{x^j}\Gamma_i(\boldsymbol{p}_0).$$

The coefficients $F_{ij}(x)$ are $r\times r$ matrices with entries $F_{\alpha\beta|ij}(x)$, $1\leq\alpha,\beta\leq r$. Moreover

$$F_{\alpha\beta|ij}(0) = \partial_{x^i}\Gamma_{\alpha\beta|j}(0) - \partial_{x^j}\Gamma_{\alpha\beta|i}(0)$$

$$\begin{aligned}
&\stackrel{(2.4)}{=} \partial_{x^j}\boldsymbol{E}\big(\,\partial_{x^i}u_\alpha(x)u_\beta(x)\,\big)|_{x=0} - \partial_{x^i}\boldsymbol{E}\big(\,\partial_{x^j}u_\alpha(x)u_\beta(x)\,\big)|_{x=0}\\
&= \boldsymbol{E}\big(\,\partial^2_{x^jx^i}u_\alpha(x)u_\beta(x)\,\big)|_{x=0} + \boldsymbol{E}\big(\,\partial_{x^i}u_\alpha(x)\partial_{x^j}u_\beta(x)\,\big)|_{x=0}\\
&\quad -\boldsymbol{E}\big(\,\partial^2_{x^ix^j}u_\alpha(x)u_\beta(x)\,\big)|_{x=0} - \boldsymbol{E}\big(\,\partial_{x^j}u_\alpha(x)\partial_{x^i}u_\beta(x)\,\big)|_{x=0}\\
&= \boldsymbol{E}\big(\,\partial_{x^i}u_\alpha(x)\partial_{x^j}u_\beta(x)\,\big)|_{x=0} - \boldsymbol{E}\big(\,\partial_{x^j}u_\alpha(x)\partial_{x^i}u_\beta(x)\,\big)|_{x=0}.
\end{aligned}$$

□

Remark 2.7. When C is the stochastic correlator defined in Example 2.1, Proposition 2.6 specializes to Gauss' Theorema Egregium. □

Corollary 2.8. *Suppose that $\boldsymbol{u}$ is a nondegenerate, Gaussian smooth random section of E with covariance density $C\in C^\infty(E\boxtimes E)$. Denote by $(-,-)_C$ and respectively ∇^C the metric and respectively the connection on E defined by C. Then for any $\boldsymbol{p}_0\in M$ the random variables $\boldsymbol{u}(\boldsymbol{p}_0)$ and $\nabla^C\boldsymbol{u}(\boldsymbol{p}_0)$ are independent.*

Proof. We continue to use the same notations as in the proof of Proposition 2.6. Observe first that

$$\big(\,\boldsymbol{u}(\boldsymbol{p}_0),\nabla\boldsymbol{u}(\boldsymbol{p}_0)\,\big)\in E_{\boldsymbol{p}_0}\oplus E_{\boldsymbol{p}_0}\otimes T^*_{\boldsymbol{p}_0}M,$$

is a Gaussian random vector. The section $\boldsymbol{u}$ has the local description

$$\boldsymbol{u}(x) = \sum_\beta u_\alpha(x)\boldsymbol{e}_\beta(x).$$

Then

$$\nabla^C_{x^i}\boldsymbol{u}(\boldsymbol{p}_0) = \sum_\alpha \partial_{x^i}u_\alpha(0)\boldsymbol{e}_\alpha(0),$$

and

$$0 = \Gamma_{\alpha\beta|i}(x) \stackrel{(2.4)}{=} -\boldsymbol{E}\big(\,\partial_{x^i}u_\alpha(0)u_\beta(0)\,\big).$$

Since $(u_\beta(0),\partial_{x^i}u_\alpha(0)\,)$ is a Gaussian vector, we deduce that the random variables $u_\beta(0),\partial_{x^i}u_\alpha(0)$ are independent. □

Remark 2.9. The local definition of the connection coefficients Γ_i shows that the above independence result is a special case of a well known fact in the theory of Gaussian random vectors: if X, Y are finite dimensional Gaussian vectors such that the dirrect sum $X \oplus Y$ is also Gaussian, then for a certain deterministic linear operator A the random vector $X - AY$ is independent of Y; see e.g. [2, Prop. 1.2]. More precisely this happens when

$$A = \boldsymbol{cov}(X, Y) \cdot \boldsymbol{cov}(Y)^{-1}.$$

Corollary 2.8 follows from this fact applied in the special case $X = d\boldsymbol{u}(0)$ and $Y = \boldsymbol{u}(0)$.

If we use local coordinates (x^i) and a local orthonormal frame $(\boldsymbol{e}_\alpha)$ in a neighborhood $\mathcal{O}$, then we can view $\mathcal{O}$ as an open subset $\mathbb{R}^m$ and $\boldsymbol{u}$ as a map

$$\boldsymbol{u} : \mathcal{O} \to \mathbb{R}^r.$$

as such, it has a differential $d\boldsymbol{u}(x)$ at any $x \in \mathcal{O}$. The formula (2.4) defining the coefficients of the correlator connection ∇^C and the classical regression formula [2, Prop.1.2] yield the following a.s. equality: for any point $x \in \mathcal{O}$ we have

$$\nabla^C \boldsymbol{u}(x) = d\boldsymbol{u}(x) - \boldsymbol{E}\big(d\boldsymbol{u}(x) \mid \boldsymbol{u}(x)\,\big). \tag{2.5}$$

Above, the notation $\boldsymbol{E}(\,\mathbf{var} \mid \mathbf{cond}\,)$ stands for the conditional expectation of the variable $\mathbf{var}$ given the conditions $\mathbf{cond}$. The above equality implies immediately that the random vectors $d\boldsymbol{u}(x)$ and $\boldsymbol{u}(x)$ are independent. □

3. KAC-RICE IMPLIES GAUSS-BONNET-CHERN

In this section we will prove a refined Gauss-Bonnet equality involving a nondegenerate Gaussian ensembles of smooth sections of E satisfying certain ampleness condition. We will make the following additional assumption.

- The manifold M is oriented.
- The bundle E are oriented and its rank is even, $r = 2h$.
- $r \leq m = \dim M$.

3.1. The setup.

We denote by $\Omega_k(M)$ the space of k-dimensional currents, i.e., the space of linear maps $\Omega^k(M) \to \mathbb{R}$ which are continuous with respect to the natural locally convect topology on the space of smooth k-forms on M. If C is a k-current and k is a smooth k-form, we denote by $\langle \eta, C\rangle$ the value of C at η.

Suppose that we are given a metric on E and a connection ∇ compatible with the metric. Observe that if $\boldsymbol{u} : M \to E$ is a smooth section of E transversal to the zero section, then its zero set $Z_{\boldsymbol{u}}$ is a smooth codimension r submanifold of M and there is a canonical adjunction isomorphism

$$\mathfrak{a}_{\boldsymbol{u}} : T_{Z_{\boldsymbol{u}}} M \to E|_{Z_{\boldsymbol{u}}},$$

where $T_{Z_{\boldsymbol{u}}} M = TM|_{Z_{\boldsymbol{u}}}/TZ_{\boldsymbol{u}}$ is the normal bundle of$Z_{\boldsymbol{u}} \hookrightarrow M$. For more details about this map we refer to [14, Sec. 2]. From the orientability of M and E and from the adjunction induced isomorphism

$$TM|_{Z_{\boldsymbol{u}}} \cong E|_{Z_{\boldsymbol{u}}} \oplus TZ_{\boldsymbol{u}}$$

we deduce that $Z_{\boldsymbol{u}}$ is equipped with a natural orientation uniquely determined by the equalities

$$\boldsymbol{or}(TM|_{Z_{\boldsymbol{u}}}) = \boldsymbol{or}(E|_{Z_{\boldsymbol{u}}}) \wedge \boldsymbol{or}(TZ_{\boldsymbol{u}}) \overset{(r\in 2\mathbb{Z})}{=} \boldsymbol{or}(TZ_{\boldsymbol{u}}) \wedge \boldsymbol{or}(E|_{Z_{\boldsymbol{u}}}).$$

Thus the zero set $Z_{\boldsymbol{u}}$ with this induced orientation defines an integration current $[Z_{\boldsymbol{u}}] \in \Omega_{m-r}(M)$

$$\Omega^{m-r}(M) \ni \eta \mapsto \langle \eta, [Z_{\boldsymbol{u}}]\rangle := \int_{Z_{\boldsymbol{u}}} \eta.$$

To a metric $(-,-)$ on E and a connection ∇ compatible with the metric we can associate a closed form

$$\boldsymbol{e}(E,\nabla)\in\Omega^r(M).$$

Its construction involves the concept of *Pfaffian* discussed in great detail in Appendix B and it goes as follows.

Denote by F the curvature of ∇ and set

$$\boldsymbol{e}(E,\nabla):=\frac{1}{(2\pi)^h}\,\mathbf{Pf}(-F)\in\Omega^r(M),$$

where the Pfaffian $\mathbf{Pf}(-F)$ has the following local description. Fix a positively oriented, local orthonormal frame $\boldsymbol{e}_1(x),\dots,\boldsymbol{e}_r(x)$ of E defined on some open coordinate neighborhood $\mathcal{O}$ of M. Then $F|_{\mathcal{O}}$ is described by a skew-symmetric $r\times r$ matrix $(F_{\alpha\beta})_{1\leq\alpha,\beta\leq r}$, where

$$F_{\alpha\beta}\in\Omega^2(\mathcal{O}),\;\;\forall\alpha,\beta.$$

If we denote by $\mathcal{S}_r$ the group of permutations of $\{1,\dots,r=2h\}$, then

$$\mathbf{Pf}\big(-F\big)=\frac{1}{2^h h!}\sum_{\sigma\in\mathcal{S}_r}\epsilon(\sigma)F_{\sigma_1\sigma_2}\wedge\cdots\wedge F_{\sigma_{2h-1}\sigma_{2h}}\in\Omega^{2h}(\mathcal{O}),\tag{3.1}$$

where $\epsilon(\sigma)$ denotes the signature of the permutation $\sigma\in\mathcal{S}_r$.

Remark 3.1. The $r\times r$-matrix $(F_{\alpha\beta})$ depends on the choice of positively oriented local orthonormal frame $(\boldsymbol{e}_\alpha(x))$. However, the Pfaffian $\mathbf{Pf}(-F)$ is a degree r-form on $\mathcal{O}$ *that is independent of the choice of positively oriented local orthonormal frame*. □

As explained in [11, Chap.8], the degree $2h$-form $\boldsymbol{e}(E,\nabla)$ is closed and it is called the *Euler form* of the connection E. Moreover, its DeRham cohomology class is independent of the choice of the metric connection ∇. The Euler form defines an $(m-r)$-dimensional current

$$\boldsymbol{e}(E,\nabla)^\dagger\in\Omega_{m-r}(M),\;\;\Omega^{m-r}(M)\ni\eta\mapsto\langle\eta,\boldsymbol{e}(E,\nabla)^\dagger\rangle:=\int_M\eta\wedge\boldsymbol{e}(E,\nabla).$$

Definition 3.2. We say that a Gaussian ensemble of smooth sections of E is *transversal* if a random section in this ensemble is a.s. transversal to the zero section of E. □

Remark 3.3. (a) As explained in [14, Lemma 2.2], any finite type nondegenerate Gaussian ensemble of smooth sections of E is transversal.

(b) If $m=\dim M=\operatorname{rank}E=r$, then [2, Prop. 6.5] shows that any Gaussian ensemble of smooth sections of E is ample.

(c) Consider a nondegenerate Gaussian ensemble of smooth sections of E with associated correlator C. Denote by $\boldsymbol{u}$ a random section in this ensemble. In [2, Prop. 6.12] it is shown that a sufficient condition for the transverselity of the ensemble is the requirement that for any $\boldsymbol{x}\in M$ the Gaussian random vector $\nabla^C\boldsymbol{u}(\boldsymbol{x})\in E_{\boldsymbol{x}}\otimes T^*_{\boldsymbol{x}}M$ is nondegenerate.

(d) We are inclined to believe that any nondegenerate Gaussian ensemble of smooth sections is transversal, but at this moment we do not have a proof for this claim. □

3.2. **A stochastic Gauss-Bonnet-Chern theorem.** We can now state the main theorem of this paper.

Theorem 3.4 (Stochastic Gauss-Bonnet-Chern)**.** *Assume that the manifold M is oriented, the bundle E is oriented and has even rank $r = 2h \leq m = \dim M$. Fix a transversal, nondegenerate Gaussian ensemble of smooth sections of E. Denote by $\boldsymbol{u}$ a random section of this ensemble, by C the correlator of this Gaussian ensemble, by $(-,-)_C$ the metric on E induced by C and by ∇ the connection on E determined by this corellator. Then the expectation of the random $(m-r)$-dimensional current $[Z_{\boldsymbol{u}}]$ is equal to the current $\boldsymbol{e}(E,\nabla)^\dagger$, i.e.,*

$$\boldsymbol{E}\big(\,\langle \eta, [Z_{\boldsymbol{u}}]\rangle\,\big) = \int_M \eta \wedge \boldsymbol{e}(E,\nabla),\;\;\forall \eta \in \Omega^{m-r}(M) \tag{3.2}$$

Proof. The linearity in η of (3.2) shows that it suffices to prove this equality in the special case when η is compactly supported on an coordinate neighborhood $\mathcal{O}$ of a point $\boldsymbol{p}_0 \in M$. Fix coordinates $x^1,\dots,x^m$ on $\mathcal{O}$ with the following properties.

- $x^i(\boldsymbol{p}_0) = 0, \forall i = 1,\dots,m.$
- The orientation of M along $\mathcal{O}$ is given by the top degree form $\omega_{\mathcal{O}} := dx^1 \wedge \cdots \wedge dx^m$.

Invoking again the linearity in η of (3.2) we deduce that it suffices to prove it in the special case when η has the form

$$\eta = \eta_0 dx^{r+1} \wedge \cdots \wedge dx^m,\;\; \eta_0 \in C_0^\infty(\mathcal{O}).$$

In other words we have to prove the equality

$$\boldsymbol{E}\big(\,\big\langle\, \eta_0 dx^{r+1} \wedge \cdots \wedge dx^m, [Z_{\boldsymbol{u}}]\,\big\rangle\,\big) = \int_{\mathcal{O}} \eta_0 dx^{r+1} \wedge \cdots \wedge dx^m \wedge \boldsymbol{e}(E,\nabla),\;\;\forall \eta_0 \in C_0^\infty(\mathcal{O}). \tag{3.3}$$

For any subset

$$I = \{i_1 < \cdots < i_k\} \subset \{1,\dots,m\}$$

we write

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We set

$$I_0 := \{1,\dots,r\},\;\; J_0 := \{r+1,\dots,m\}.$$

We can rewrite (3.3) in the more compact form

$$\boldsymbol{E}\big(\,\big\langle\, \eta_0 dx^{J_0}, [Z_{\boldsymbol{u}}]\,\big\rangle\,\big) = \int_{\mathcal{O}} \eta_0 dx^{J_0} \wedge \boldsymbol{e}(E,\nabla),\;\;\forall \eta_0 \in C_0^\infty(\mathcal{O}). \tag{3.4}$$

Fix a local, *positively oriented*, $-(,-)_C$-orthonormal frame $(\boldsymbol{e}_\alpha(x))_{1\leq\alpha\leq r}$ of $E|_{\mathcal{O}}$. The restriction to $\mathcal{O}$ of the curvature F of ∇ is then a skew-symmetric $r \times r$-matrix

$$F = (F_{\alpha\beta})_{1\leq \alpha,\beta\leq r},\;\; F_{\alpha\beta} \in \Omega^2(\mathcal{O}),\;\; \forall \alpha,\beta.$$

Each of the 2-forms $F_{\alpha\beta}$ admits a unique decomposition

$$F_{\alpha\beta} = \sum_{1\leq i<j\leq m} F_{\alpha\beta|ij}\, dx^i \wedge dx^j.$$

For each subset $I \subset \{1,\dots m\}$ we write

$$F^I_{\alpha\beta} := \sum_{\substack{i<j\\ i,j\in I}} F_{\alpha\beta|ij}\, dx^i \wedge dx^j \in \Omega^2(\mathcal{O}).$$

We denote by F^I the skew-symmetric $r \times r$ matrix with entries $F^I_{\alpha\beta}$.

The degree r-form $\mathbf{Pf}(-F)$ admits a canonical decomposition

$$\mathbf{Pf}(-F) = \sum_{|I|=r} \mathbf{Pf}\big(-F^I\big) = \sum_{|I|=r} \mathbf{pf}(-F)_I\, dx^I, \;\; \mathbf{pf}(-F)_I \in C^\infty(\mathcal{O}).$$

The equality (3.4) is then equivalent to the equality

$$\boldsymbol{E}\big(\big\langle\, \eta_0 dx^{J_0}, [Z_{\boldsymbol{u}}]\,\big\rangle\big) = \frac{1}{(2\pi)^h}\int_{\mathcal{O}} \eta_0\, \mathbf{pf}(-F)_{I_0}\omega_{\mathcal{O}}, \;\; \forall \eta_0 \in C_0^\infty(\mathcal{O}), \tag{3.5}$$

where we recall that $\omega_{\mathcal{O}} = dx^1\wedge\cdots\wedge dx^m$. To prove the above equality we will use the following two-step strategy.

Step 1. Invoke the Kac-Rice formula to express the left-hand side of (3.5) as an integral over $\mathcal{O}$

$$\boldsymbol{E}\big(\big\langle\, \eta_0 dx^{J_0}, [Z_{\boldsymbol{u}}]\,\big\rangle\big) = \int_{\mathcal{O}} \eta_0(x)\rho(x)\omega_{\mathcal{O}},$$

where $\rho(x)$ is a certain smooth function on $\mathcal{O}$.

Step 2. Use the Gaussian computations in Appendix B to show that

$$\rho(x) = \mathbf{pf}(-F)_{I_0}(x), \;\; \forall x \in \mathcal{O}.$$

Let us know implement this strategy. We view $\mathcal{O}$ as an open neighborhood of the origin in $\mathbb{R}^m$ equipped with the canonical Euclidean metric and the orientation given by $\omega_{\mathcal{O}}$. Denote by E_0 the fiber of E over the origin. Using the oriented, orthonormal local frame $(\boldsymbol{e}_\alpha)$ we can view the restriction to $\mathcal{O}$ of the random section $\boldsymbol{u}$ as Gaussian smooth random map

$$\boldsymbol{u} : \mathcal{O} \to E_0 \cong \mathbb{R}^r, \;\; x \mapsto (u_\alpha(x))_{1\leq\alpha\leq r},$$

where again $\mathbb{R}^r$ is equipped with the canonical Euclidean metric and orientation given by the volume form

$$\omega_E = du_1\wedge\cdots\wedge du_r.$$

The fact that the frame $(\,\boldsymbol{e}_\alpha(x)\,)$ is orthonormal with respect to the metric $(-,-)_C$ implies that for any $x\in\mathcal{O}$ the probability distribution of random vector $\boldsymbol{u}(x)$ is the standard Gaussian measure on the Euclidean space $\mathbb{R}^r$. We denote by $p_{\boldsymbol{u}(x)}$ the probability density of this vector so that

$$p_{\boldsymbol{u}(x)}(u) = \frac{1}{(2\pi)^h} e^{-\frac{1}{2}|u|^2}, \tag{3.6}$$

where $|-|$ denotes the canonical Euclidean norm on $\mathbb{R}^r$.

The zero set $Z_{\boldsymbol{u}}$ is a.s. a submanifold of $\mathcal{U}$ and as such it is equipped with an Euclidean metric and an induced volume density $|dV_{Z_{\boldsymbol{u}}}|$.

Recall that if $T : U \to V$ is a linear map between two Euclidean spaces such that $\dim U \geq \dim V$, then its *Jacobian* is the scalar

$$\mathrm{Jac}_T := \sqrt{\det(TT^*)}.$$

We define the *Jacobian at* $x \in \mathcal{O}$ of a smooth map $F : \mathcal{O} \to E_0$ to be the scalar

$$J_F(x) = \mathrm{Jac}_{dF(x)} = \sqrt{\det dF(x)dF(x)^*},$$

where $dF(x) : \mathbb{R}^m \to E_0$ is the differential $dF(x)$ of F at x. We set

$$\mathcal{T} := \mathrm{Hom}(\mathbb{R}^m, E_0),$$

so that have a Gaussian random map

$$d\boldsymbol{u} : \mathcal{O} \to \mathcal{T}, \;\; x \mapsto d\boldsymbol{u}(x).$$

This random map is a.s. smooth and the random map

$$\mathcal{O} \to E_0 \times \mathcal{T}, \quad x \mapsto \big(\boldsymbol{u}(x), d\boldsymbol{u}(x)\big),$$

is also a Gaussian random map. Suppose that $g : \mathcal{T} \to \mathbb{R}$ is a bounded continuous function. We then have the following Kac-Rice formula, [2, Thm. 6.10].

Theorem 3.5 (Kac-Rice)**.** *Let $g : \mathcal{T} \to \mathbb{R}$ be a bounded continuous function. Then, for any $\lambda_0 \in C_0(\mathcal{O})$, the random variable*

$$\boldsymbol{u} \mapsto \int_{Z_{\boldsymbol{u}}} \lambda_0(x) g\big(\, d\boldsymbol{u}(x)\,\big) |dV_{Z_{\boldsymbol{u}}}(x)|$$

is integrable and

$$\boldsymbol{E}\left(\int_{Z_{\boldsymbol{u}}} \lambda_0(x) g\big(\, d\boldsymbol{u}(x)\,\big) |dV_{Z_{\boldsymbol{u}}}(x)|\right) = \int_{\mathcal{O}} \lambda_0(x) \boldsymbol{w}(x) \omega_{\mathcal{O}}(x), \quad \forall \eta_0 \in C_0(\mathcal{O}), \tag{3.7a}$$

$$\begin{aligned} \boldsymbol{w}(x) &= \boldsymbol{E}\Big(\, J_{\boldsymbol{u}}(x) g(\, d\boldsymbol{u}(x)\,) \;\big|\; \boldsymbol{u}(x) = 0\Big) p_{\boldsymbol{u}(x)}(0) \\ &\stackrel{(3.6)}{=} \frac{1}{(2\pi)^h} \boldsymbol{E}\Big(\, J_{\boldsymbol{u}}(x) g(\, d\boldsymbol{u}(x)\,) \;\big|\; \boldsymbol{u}(x) = 0\Big). \end{aligned} \tag{3.7b}$$

In particular the function $x \mapsto \lambda_0(x)\rho(x)$ is also integrable. □

The above equality extends to more general g's.

Definition 3.6. We say that a bounded measurable function $g : \mathcal{T} \to \mathbb{R}$ is *admissible* if there exists a sequence of bounded continuous functions with the following properties.

(i) The sequence g_n converges a.e. to g.
(ii) $\sup_n \|g_n\|_{L^\infty} < \infty$.

□

Lemma 3.7. *Theorem 3.5 continues to hold if g is an admissible function $\mathcal{T} \to \mathbb{R}$.*

Proof. Fix an admissible function $g : W \to \mathbb{R}$ and a sequence of bounded measurable functions $g_n : W \to \mathbb{R}$ satisfying the conditions in Definition 3.6. Set

$$K := \sup_n \|g_n\|_{L^\infty}.$$

Then

$$\left|\int_{Z_{\boldsymbol{u}}} \lambda_0(x) g_n\big(\, d\boldsymbol{u}(x)\,\big) |dV_{Z_{\boldsymbol{u}}}(x)|\right| \leq K \int_{Z_{\boldsymbol{u}}} |\lambda_0(x)||dV_{Z_{\boldsymbol{u}}}(x)|.$$

The random variable

$$\boldsymbol{u} \mapsto K \int_{Z_{\boldsymbol{u}}} |\lambda_0(x)||dV_{Z_{\boldsymbol{u}}}(x)|$$

is integrable according the Theorem 3.5 in the special case when $g \equiv K$ and $\lambda_0 = |\lambda_0|$. The dominated converge theorem implies that

$$\lim_{n\to\infty} \boldsymbol{E}\left(\int_{Z_{\boldsymbol{u}}} \lambda_0(x) g_n\big(\, d\boldsymbol{u}(x)\,\big) |dV_{Z_{\boldsymbol{u}}}(x)|\right) = \boldsymbol{E}\left(\int_{Z_{\boldsymbol{u}}} \lambda_0(x) g\big(\, d\boldsymbol{u}(x)\,\big) |dV_{Z_{\boldsymbol{u}}}(x)|\right).$$

A similar argument shows that

$$\lim n \to \infty \boldsymbol{E}\Big(\, J_{\boldsymbol{u}}(x) g(\, d\boldsymbol{u}(x)\,) \;\big|\; \boldsymbol{u}(x) = 0\Big) = \boldsymbol{E}\Big(\, J_{\boldsymbol{u}}(x) g(\, d\boldsymbol{u}(x)\,) \;\big|\; \boldsymbol{u}(x) = 0\Big).$$

□

To apply the above Kac-Rice formula we need to express the integral over $Z_{\boldsymbol{u}}$ of a form as an integral of a function with respect to the volume density. More precisely, we seek an equality of the type

$$\int_{Z_{\boldsymbol{u}}} dx^{J_0} = \int_{Z_{\boldsymbol{u}}} \eta_0(x) g\big(\, d\boldsymbol{u}(x)\,\big) |dV_{Z_{\boldsymbol{u}}}(x)|,$$

for some admissible function g. This is achieved in the following technical result whose proof can be found in Appendix A.

Lemma 3.8. *Suppose that* 0 *is a regular value of* $\boldsymbol{u}$. *Set* $u_\alpha(x) = (\boldsymbol{u}, \boldsymbol{e}_\alpha(x)\,)$. *Then*

$$dx^{J_0}|_{Z_{\boldsymbol{u}}} = \frac{\Delta_{I_0}(d\boldsymbol{u})}{J_{\boldsymbol{u}}},$$

where $J_{\boldsymbol{u}} : \mathcal{O} \to \mathbb{R}_{\geq 0}$ *is the Jacobian of* $\boldsymbol{u}$ *and* $\Delta_{I_0}(d\boldsymbol{u})$ *is the determinant of the* $r \times r$ *matrix* $\frac{\partial \boldsymbol{u}}{\partial x^{I_0}}$ *with entries*

$$\frac{\partial u_\alpha}{\partial x^j}, \;\; \alpha, j \in I_0. \qquad \square$$

Any linear map $T \in \mathcal{T} = \mathrm{Hom}(\mathbb{R}^m, E_0)$ is represented by an $r \times m$ matrix. For any subset J of $\{1, \dots, m\}$ we denote by $\Delta_J(T)$ the determinant of the $r \times r$ minor T_J determined by the columns indexed by J.

Denote by $\mathcal{T}^*$ the subset of $\mathcal{T}$ consisting of surjective linear maps $\mathbb{R}^m \to E_0$. The complement $\mathcal{T} \setminus \mathcal{T}^*$ is a negligible subset of $\mathcal{T}$ Observe that

$$T \in \mathcal{T}^* \Longleftrightarrow \mathrm{Jac}_T \neq 0.$$

Define

$$G : \mathcal{T} \to \mathbb{R}, \;\; g(T) = \begin{cases} \frac{\Delta_{I_0}(T)}{\mathrm{Jac}_T}, & T \in \mathcal{T}^*, \\ 0, & T \in W \setminus W^*. \end{cases}$$

Lemma 3.8 shows that that if 0 is a regular value of $\boldsymbol{u}$, then

$$\int_{Z_{\boldsymbol{u}}} \eta_0 dx^{J_0} = \int_{Z_{\boldsymbol{u}}} \eta_0(x) G(d\boldsymbol{u}(x)) |dV_{Z_{\boldsymbol{u}}}(x)|, \;\; \forall \eta_0 \in C_0(\mathcal{O}).$$

Lemma 3.9. *The measurable function* $G : \mathcal{T} \to \mathbb{R}$ *is admissible.*

Proof. We first prove that G is bounded on $\mathcal{T}^*$. This follows from the classical identity

$$\mathrm{Jac}_T^2 = \sum_{|J|=r} \Delta_J(T)^2.$$

This proves that

$$\left| \frac{\Delta_{I_0}(T)}{\mathrm{Jac}_T} \right| \leq 1.$$

Now define

$$G_n(T) := \frac{\Delta_{I_0}(T)}{\sqrt{n^{-2} + \mathrm{Jac}_T^2}}, \;\; \forall T \in \mathcal{T}.$$

Observe that $G_n(T) \nearrow G(T)$ on $\mathcal{T}^*$ as $n \to \infty$ and

$$\sup_n \| G_n \|_{L^\infty} \leq 1.$$

$\square$

We deduce that

$$\begin{aligned}\boldsymbol{E}\Big(\big\langle\,\eta_0(x)dx^{J^0},[Z_{\boldsymbol{u}}]\,\big) &= \boldsymbol{E}\left(\int_{Z_{\boldsymbol{u}}}\eta_0(x)G\big(\,d\boldsymbol{u}(x)\,\big)|dV_{Z_{\boldsymbol{u}}}(x)|\right)\\ &\stackrel{(3.7a)}{=}\frac{1}{(2\pi)^h}\int_{\mathcal{O}}\eta_0(x)\boldsymbol{E}\Big(\,J_{\boldsymbol{u}}(\boldsymbol{x})G(\,d\boldsymbol{u}(x)\,)\;\Big|\;\boldsymbol{u}(x)=0\Big)\omega_{\mathcal{O}}\\ &=\frac{1}{(2\pi)^h}\int_{\mathcal{O}}\eta_0(x)\;\underbrace{\boldsymbol{E}\Big(\,\Delta_{I_0}\big(\,d\boldsymbol{u}(x)\,\big)\;\Big|\;\boldsymbol{u}(x)=0\Big)}_{=:\rho(x)}\omega_{\mathcal{O}}.\end{aligned}$$

We have thus proved the equality

$$\boldsymbol{E}\Big(\big\langle\,\eta_0(x)dx^{J^0}\,,\,[Z_{\boldsymbol{u}}]\,\big\rangle\Big)=\frac{1}{(2\pi)^h}\int_{\mathcal{O}}\rho(x)\omega_{\mathcal{O}},\;\;\forall\eta_0\in C_0^\infty(\mathcal{O}).\tag{3.8}$$

The density $\rho(x)$ in the right-hand-side of the above equality could appriori depend on the choice of the $(-,-)_C$-orthonormal frame because it involved the frame dependent matrix $\frac{\partial\boldsymbol{u}}{\partial x^{I_0}}$. On the other hand, the left-hand-side of the equality (3.8) is plainly frame independent. This shows that the density ρ is also frame independent. To prove (3.5) and thus Theorem 3.4 it suffices to show that

$$\boldsymbol{E}\Big(\,\Delta_{I_0}\big(\,d\boldsymbol{u}(x)\,\big)\;\Big|\;\boldsymbol{u}(x)=0\Big)=\mathbf{pf}(-F)_{I_0}(x),\;\;\forall x\in\mathcal{O}.\tag{3.9}$$

We will prove the above equality for $x=0$. Both sides are frame invariant and thus we are free to choose the frame $(\,\boldsymbol{e}_\alpha(x)\,)$ as we please. We assume that it is synchronous at $x=0$, i.e.,

$$\nabla\boldsymbol{e}_\alpha(0),\;\;\forall\alpha.$$

Then $\nabla^C\boldsymbol{u}(0)=d\boldsymbol{u}(0)$. Corollary 2.8 now implies that the Gaussian vectors $d\boldsymbol{u}(0)$ and $\boldsymbol{u}(0)$ are independent. Hence

$$\boldsymbol{E}\Big(\,\Delta_{I_0}\big(\,d\boldsymbol{u}(0)\,\big)\;\Big|\;\boldsymbol{u}(x)=0\Big)=\boldsymbol{E}\Big(\,\Delta_{I_0}\big(\,d\boldsymbol{u}(0)\,\big)\,\Big),$$

and thus we have to prove that

$$\boldsymbol{E}\Big(\,\Delta_{I_0}\big(\,d\boldsymbol{u}(0)\,\big)\,\Big)=\mathbf{pf}(-F)_{I_0}(0).\tag{3.10}$$

The random variable $\Delta_{I_0}\big(\,d\boldsymbol{u}(0)\,\big)$ is the determinant of the $r\times r$ Gaussian matrix $S:=\frac{\partial\boldsymbol{u}}{\partial x^{I_0}}$ with entries

$$S_{\alpha i}:=\partial_{x^i}u_\alpha(0),\;\;1\leq\alpha,i\leq r.$$

Its statistics are determined by the covariances

$$K_{\alpha i|\beta j}:=\boldsymbol{E}\big(\,S_{\alpha i}S_{\beta j}\,\big)=\boldsymbol{E}\big(\,\partial_{x^i}u_\alpha(0)\partial_{x^j}u_\beta(0)\,\big).$$

As in Appendix B we consider the $(2,2)$-double form

$$\boldsymbol{\Xi}_K=\sum_{\alpha<\beta,\;i<j}\boldsymbol{\Xi}_{\alpha\beta|ij}\boldsymbol{v}^\alpha\wedge\boldsymbol{v}^\beta\otimes\boldsymbol{v}^i\wedge\boldsymbol{v}^j\in\Lambda^{2,2}\boldsymbol{V}^*,$$

where

$$\boldsymbol{\Xi}_{\alpha\beta|ij}:=\big(K_{\alpha i|\beta j}-K_{\alpha j|\beta i}\big),\;\;\forall 1\leq\alpha,\beta\leq r,\;\;1\leq i,j\in I_0.$$

Then

$$\boldsymbol{E}\Big(\,\Delta_{I_0}\big(\,d\boldsymbol{u}(0)\,\big)\,\Big)\stackrel{(B.10)}{=}\frac{1}{h!}\operatorname{tr}\boldsymbol{\Xi}_K^{\wedge h}.\tag{3.11}$$

Now observe that (2.3) implies that

$$\boldsymbol{\Xi}_{\alpha\beta|ij}=F_{\alpha\beta|ij}(0)=\forall 1\leq\alpha,\beta\leq r,\;\;1\leq i,j\in I_0.$$

We deduce that

$$\Xi_K = \Omega_{-F^{I_0}(0)} \stackrel{(B.2)}{:=} \sum_{\substack{\alpha<\beta,\, i<j,\\ i,j\in I_0}} F_{\alpha\beta|ij} du_\alpha \wedge du_\beta \otimes dx^i \wedge dx^j$$

Using (B.6) and (B.9) we deduce

$$\begin{aligned}\mathbf{Pf}(-F^{I_0})_{x=0} = \mathbf{pf}(-F)_{I_0}(0)\, dx^{I_0} &= \frac{1}{h!}\Big(\operatorname{tr} \Omega^{\wedge h}_{-F^{I_0}(0)}\Big) dx^{I_0}\\ &= \frac{1}{h!}\Big(\operatorname{tr} \Xi_K^{\wedge h}\Big) dx^{I_0} \stackrel{(3.11)}{=} \boldsymbol{E}\Big(\Delta_{I_0}\big(d\boldsymbol{u}(0)\big)\Big) dx^{I_0}.\end{aligned}$$

This proves (3.10) and thus completes the proof of Theorem 3.4. □

Remark 3.10. When the rank of E odd, the Euler class with real coefficients is trivial. In this case if $\boldsymbol{u}$ is a section of E transversal to the zero section, then we have the equality of currents

$$[Z_{-\boldsymbol{u}}] = -[Z_{\boldsymbol{u}}].$$

If $\boldsymbol{u}$ is a random section of a smooth, ample Gaussian ensemble, then the above equality implies

$$\boldsymbol{E}([Z_{\boldsymbol{u}}]) = 0.$$ □

APPENDIX A. PROOFS OF VARIOUS TECHNICAL RESULTS

Proof of Proposition 1.1. Following Fernique [5] we say that a topological space X is *standard* if it admits a continuous bijection $f : P \to X$, where P is a Polish space, i.e., complete, separable metric space. (L. Schwartz [15] refers to Fernique's standard spaces as *Lusin spaces*.)

(i) Since M is compact, the space $C^\infty(E)$ is a Fréchet-Montel space. Using [5, Thm.I.5.1] we deduce that $C^{-\infty}(E)$ with the strong topology is a standard space. The claim now follows from [5, Thm.I.2.5].

(ii) This follows from [5, Thm. I.3.2] or [15, Thm.9, p.122].

(iii) The space $C^\infty(E)$ with its natural topology is a separable Fréchet space. It is thus standard according to [5, Thm. I.5.1]. Thus any Borel subset $\mathcal{B}$ of $C^\infty(E)$ is a standard space with the induced topology. The inclusion

$$i : C^\infty(E) \hookrightarrow C^{-\infty}(E)$$

is continuous with respect to the weak topology on $C^{-\infty}(E)$. Since $C^{-\infty}(E)$ with the weak topology is standard, we deduce from [5, Thm. Prop. I.2.2 (b)] that the image $i(\mathcal{B})$ is a standard subspace of $C^{-\infty}(E)$ with the weak topology. We can now invoke [5, Thm. Prop. I.2.2 (a)] to conclude that $i(\mathcal{B})$ is a weak Borel subset of $C^{-\infty}(E)$. From part(i) we deduce that $i(\mathcal{B})$ is also a strong Borel subset of $C^{-\infty}(E)$. □

Proof of Proposition 1.4. Fix a metric g on M, a metric and a compatible connection on E. For each nonnegative integer k we can define the Sobolev spaces $\mathcal{H}_k$ consisting of L^2-sections of E whose generalized derivatives up to order k are L^2-sections. We have a decreasing sequence of Hilbert spaces

$$\mathcal{H}_0 \supset \mathcal{H}_1 \supset \cdots$$

whose intersection is $C^\infty(E)$. The natural locally convex topology on $C^\infty(E)$ is then the projective limit of this family of Hilbert spaces. For $k \geq 0$ we denote by $\mathcal{H}_{-k}$ the topological dual of $\mathcal{H}_k$ so that we have a decreasing family of Hilbert spaces

$$\cdots \subset \mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H}_{-1} \subset \cdots .$$

The union of this family of spaces is $C^{-\infty}(E)$, and the strong topology on $C{-}\infty(E)$ is the locally convex inductive limit of this family. Arguing as in the proof of Proposition 1.1 we deduce that each of the subsets $\mathcal{H}_k \subset C^{-\infty}(E)$, $k \in \mathbb{Z}$, is a Borel subset. Using Minlos's theorem [9, Sec. 4, Thm. 2] we deduce that if the covariance kernel $C_{\boldsymbol{\Gamma}}$ is smooth then

$$\boldsymbol{\Gamma}(\mathcal{H}_k) = 1, \ \ \forall k \in \mathbb{Z}. \qquad \square$$

Proof of Proposition 1.5. Fix a Riemann metric g on M. For each $i = 1, \dots, n$ choose a sequence $(\delta_{\nu,i})_{\nu \geq 0}$ of smooth functions on M supported in a coordinate neighborhood of $\boldsymbol{x}_i$ such that

$$\lim_{\nu \to \infty} \delta_{\nu,i} |dV_g| = \delta_{\boldsymbol{x}_i} = \text{the Dirac measure concentrated at } x_i.$$

Fix trivializations of E near each $\boldsymbol{x}_i$. Let $t_1, \dots, t_n$. Now define

$$\Phi_\nu = \sum_{i=1}^n t_i \boldsymbol{u}_i^* \otimes \delta_{\nu,i} |dV_g| \in C^\infty(E^* \otimes |\Lambda_M|),$$

an form the random variable

$$C^{-\infty}(E) \ni \varphi \mapsto Y_\nu = Y_\nu(\varphi) = L_{\Phi_\nu}(\varphi).$$

This is a Gaussian random variable with variance

$$\boldsymbol{E}_\Gamma(Y_\nu^2) = \mathcal{K}_\Gamma(\Phi_\nu, \Phi_\nu) = \sum_{i,j} t_i t_j \int_{M \times M} C_{x,y}(\boldsymbol{u}_i^*, \boldsymbol{u}_j^*) \delta_{\nu,i}(x) \delta_{\nu,j}(y) |dV_g(x) dV_g(y)|.$$

Now observe that

$$\lim_{\nu \to \infty} Y_n(\varphi) = \sum_{i=1}^n t_i X_i(\varphi).$$

We deduce that Y_ν converges in law to $\sum_{i=1}^n t_i X_i$. In particular, this random variable is Gaussian and its variance is

$$\lim_{\nu \to \infty} \boldsymbol{E}(Y_\nu^2) = \lim_{\nu \to \infty} \sum_{i,j} t_i t_j \int_{M \times M} C_{x,y}(\boldsymbol{u}_i^*, \boldsymbol{u}_j^*) \delta_{\nu,i}(x) \delta_{\nu,j}(y) |dV_g(x) dV_g(y)|$$

$$= \sum_{i,j} t_i t_j C_{\boldsymbol{x}_i, \boldsymbol{x}_j}(\boldsymbol{u}_i^*, \boldsymbol{u}_j^*).$$

This completes the proof of Proposition 1.5. $\square$

Proof of Lemma 3.8. We follow a strategy similar to the one used in the proof of [12, Cor. 2.11]. Fix a point $p_0 \in Z_{\boldsymbol{u}}$. Now choose local coordinates $(t^1, \dots, t^m)$ on $\mathcal{O}$ near p_0 and local coordinates $y^1, \dots, y^r$ on E_0 near $0 \in E_0$ with the following properties.

- In the (t, y)-coordinates the map $\boldsymbol{u}$ is given by the linear projection
$$y^j = t^j, \ \ j = 1, \dots, r.$$
- The orientation of E_0 is given by $dy = dy^1 \wedge \cdots \wedge dy^r$.

We set

$$dt^{J_0} := dt^{r+1} \wedge \cdots \wedge dt^r, \ \ dt^{I_0} := dt^1 \wedge \cdots \wedge dt^r.$$

The coordinates t^{J_0} can be used as local coordinates on $Z_{\boldsymbol{u}}$ near p_0 and we assume that dt^{J_0} defines the induced orientation of $Z_{\boldsymbol{u}}$. We can then write

$$\omega_{\mathcal{O}} = \rho_{\mathcal{O}} dt^{J_0} \wedge dt^{I_0}, \ \ \omega_E = \rho_E dy = \rho_E dy^1 \wedge \cdots \wedge dy^r, \ \ dV_{Z_{\boldsymbol{u}}} = \rho_{\boldsymbol{u}} dt^{J_0}, \tag{A.1}$$

where $\rho_{\mathcal{O}}$, ρ_E and $\rho_{\boldsymbol{u}}$ are positive smooth functions on their respective domains.

In the t-coordinates we have

$$dx^{J_0} = \lambda dt^{J_0} + \text{other exterior monomials},$$

where λ is the determinant of the $(m-r)\times(m-r)$ matrix $\frac{\partial x^{J_0}}{\partial t^{J_0}}$ with entries

$$\frac{\partial x^i}{\partial t^j},\ \ i,j\in J_0.$$

Thus

$$dx^{J_0}|_{Z_{\boldsymbol{u}}} = \lambda dt^{J_0} \stackrel{(A.1)}{=} \frac{\lambda}{\rho_{\boldsymbol{u}}} dV_{Z_{\boldsymbol{u}}}.$$

We have

$$dx^{J^0}\wedge \boldsymbol{u}^*\omega_E = \varphi\omega_{\mathcal{O}},\ \ \varphi = \varphi = \Delta_{I_0}(d\boldsymbol{u}) = \det\left(\frac{\partial \boldsymbol{u}}{\partial x^{I_0}}\right). \tag{A.2}$$

On the other hand,

$$dx^{J^0}\wedge \boldsymbol{u}^*\omega_E = \rho_E dx^{J_0}\wedge dt^{I_0} \stackrel{(A.1)}{=} \lambda\rho_E dt^{J_0}\wedge dt^{I_0} = \frac{\lambda\rho_E}{\rho_{\mathcal{O}}}\omega_{\mathcal{O}}. \tag{A.3}$$

Using this in (A.2) we deduce

$$\varphi = \frac{\lambda\rho_E}{\rho_{\mathcal{O}}}. \tag{A.4}$$

Now observe that along $Z_{\boldsymbol{u}}$ we have

$$\frac{\varphi}{J_{\boldsymbol{u}}} dV_{Z_{\boldsymbol{u}}} \stackrel{(A.1)}{=} \frac{\varphi}{J_{\boldsymbol{u}}}\rho_{\boldsymbol{u}} dt^{J_0} \stackrel{(A.4)}{=} \frac{\lambda\rho_E\rho_{\boldsymbol{u}}}{J_{\boldsymbol{u}}\rho_{\mathcal{O}}} dt^{J^0} = \frac{\rho_E\rho_{\boldsymbol{u}}}{J_{\boldsymbol{u}}\rho_{\mathcal{O}}} dx^{J^0}|_{Z_{\boldsymbol{u}}}.$$

On the other hand, [12, Lemma 1.2] shows that $\frac{\rho_E\rho_{\boldsymbol{u}}}{J_{\boldsymbol{u}}\rho_{\mathcal{O}}} = 1$ which proves that

$$dx^{J^0}|_{Z_{\boldsymbol{u}}} = \frac{\varphi}{J_{\boldsymbol{u}}} dV_{Z_{\boldsymbol{u}}} \stackrel{(A.2)}{=} \frac{\Delta_{I_0}(d\boldsymbol{u})}{J_{\boldsymbol{u}}}.$$

□

APPENDIX B. PFAFFIANS AND GAUSSIAN COMPUTATIONS

We collect here a few facts about Pfaffians need in the main body of the paper.

Fix a positive even integer $r = 2h > 0$. Given a commutative $\mathbb{R}$-algebra $\mathcal{A}$ we denote by $\mathrm{Skew}_r(\mathcal{A})$ the space of skew-symmetric $r\times r$-matrices with entries in $\mathcal{A}$. The Pfaffian of a matrix $F\in\mathrm{Skew}_r(\mathcal{A})$ is a certain universal homogeneous polynomial of degree $h = r/2$ in the entries of F. More precisely, if we denote by $\mathcal{S}_r$ the group of permutations of $\{1,\dots,r=2h\}$, then

$$\mathbf{Pf}\big(F\big) = \frac{(-1)^h}{2^h h!}\sum_{\sigma\in\mathcal{S}_r}\epsilon(\sigma)F_{\sigma_1\sigma_2}\cdots F_{\sigma_{2h-1}\sigma_{2h}}\in\mathcal{A}, \tag{B.1}$$

where $\epsilon(\sigma)$ denotes the signature of the permutation $\sigma\in\mathcal{S}_r$. The Pfaffian can be given an equivalent alternate description.

Fix an *oriented* real, r-dimensional Euclidean space E and an *oriented* orthonormal basis $e_1,\dots,e_r$ in V. Denote by $e^1,\dots,e^r$ the dual basis of V^* and consider the $\mathcal{A}$-valued 2-form

$$\Omega^E_F = -\sum_{1\le\alpha<\beta} F_{\alpha\beta}\otimes e^\alpha\wedge e^\beta\in\mathcal{A}\otimes\Lambda^2E^*, \tag{B.2}$$

then the Pfaffian of F is uniquely determined by the equality, [11, Sec. 2.2.4],

$$\mathbf{Pf}(F)e^1\wedge\cdots\wedge e^r = \frac{1}{h!}(\Omega^E_F)^{\wedge h}\in\mathcal{A}\otimes\Lambda^{2h}E^*. \tag{B.3}$$

We are interested only in a certain special case when

$$\mathcal{A} = \Lambda^{\mathrm{even}}\boldsymbol{V}^* = \bigoplus_{2k\leq m}\Lambda^{2k}\boldsymbol{V}^*,$$

where $\boldsymbol{V}$ is a real Euclidean space of dimension $m\geq r$ and $F_{\alpha\beta}\in\Lambda^2\boldsymbol{V}^*$, $\forall 1\leq\alpha,\beta\leq r$. In this case $\mathbf{Pf}(F)\in\Lambda^r\boldsymbol{V}^*$ and has the following alternate description.

Fix an *orthonormal* basis $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m)$ of $\boldsymbol{V}$. For $1\leq\alpha_1,\alpha_2\leq r$ and $1\leq j_1,j_2\leq m$ we set

$$F_{\alpha_1\alpha_2|j_1j_2} := F^E_{i_1i_2}(\boldsymbol{v}_{j_1},\boldsymbol{v}_{j_2}). \tag{B.4}$$

Denote by $\mathcal{S}_r'$ the subset of $\mathcal{S}_r$ consisting of permutations $(\sigma_1,\dots,\sigma_{2h})$ such that

$$\sigma_1<\sigma_2,\ \sigma_3<\sigma_4,\ \dots,\ \sigma_{2h-1}<\sigma_{2h}.$$

Then

$$\mathbf{Pf}\big(F\big)\big(\,\boldsymbol{v}_1,\cdots,\boldsymbol{v}_r\,\big) = \frac{(-1)^h}{h!}\sum_{\varphi,\sigma\in\mathcal{S}_r'}\epsilon(\sigma\varphi)F_{\sigma_1\sigma_2|\varphi_1\varphi_2}\cdots F_{\sigma_{2h-1}\sigma_{2h}|\varphi_{2h-1}\varphi_{2h}}. \tag{B.5}$$

For every subset $I=\{i_1<\cdots<i_r\}\subset\{1,\dots,m\}$ we write

$$\boldsymbol{v}^{\wedge I} = \boldsymbol{v}^{i_1}\wedge\cdots\wedge\boldsymbol{v}^{i_r},$$

where $\{\boldsymbol{v}^1,\dots,\boldsymbol{v}^m\}$ is the orthonormal basis of $\boldsymbol{V}^*$ dual to $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\}$.

$$\mathbf{Pf}(F) = \sum_{|I|=r}\mathbf{pf}(F)_I\boldsymbol{v}^{\wedge I}.$$

For and ordered multi index I we denote by $\boldsymbol{V}_I$ the subspace spanned by $\boldsymbol{v}_i$, $i\in I$, and by $F^I_{\alpha\beta}$ the restriction of $F_{\alpha\beta}$ to $\boldsymbol{V}_I$, i.e.,

$$F^I_{\alpha\beta} = \sum_{\substack{i<j\\ i,j\in I}} F^I_{\alpha\beta|ij}\boldsymbol{v}^i\wedge\boldsymbol{v}^j\in\Lambda^2\boldsymbol{V}_I^*.$$

We denote by F^I the $r\times r$ skew-symmetric matrix with entries $(F^I_{\alpha\beta})_{1\leq\alpha,\beta\leq r}$. Note that for any subset $I\subset\{1,\dots,m\}$ of cardinality r we have

$$\mathbf{pf}(F)_I\boldsymbol{v}^I = \mathbf{Pf}(F^I). \tag{B.6}$$

This shows that the computation of the Pfaffians reduces to the case when $\dim\boldsymbol{V}=r$. This is what we will assume in the remainder of this section. We fix an orthonormal basis $\boldsymbol{v}_1,\dots,\boldsymbol{v}_r$ of $\boldsymbol{V}$ and we denote by $\boldsymbol{v}^1,\dots,\boldsymbol{v}^r$ the dual basis of $\boldsymbol{V}^*$.

To proceed further we need to introduce some more terminology. A *double form* on the above Euclidean space V is, by definition, an element of the vector space

$$\Lambda^{p,q}\boldsymbol{V}^* := \Lambda^p\boldsymbol{V}^*\otimes\Lambda^q\boldsymbol{V}^*,\ \ p,q\in\mathbb{Z}_{\geq 0}.$$

We have an associative product

$$\wedge : \Lambda^{p,q}\boldsymbol{V}^*\times\Lambda^{p',q'}\boldsymbol{V}^*\to\Lambda^{p+p',q+q'}\boldsymbol{V}^*$$

given by

$$(\omega\otimes\eta)\wedge(\omega'\otimes\eta') := \omega\wedge\omega')\otimes(\eta\wedge\eta'),$$

for any $\omega\in\Lambda^p\boldsymbol{V}^*$, $\eta\in\Lambda^q\boldsymbol{V}^*$, $\omega'\in\Lambda^{p'}\boldsymbol{V}^*$, $\eta'\in\Lambda^{q'}\boldsymbol{V}^*$.

Observe that the metric on $\boldsymbol{V}$ produces an isomorphism

$$\Lambda^{j,j}\boldsymbol{V}^*\cong\mathrm{End}\big(\,\Lambda^j\boldsymbol{V}^*,\Lambda^j\boldsymbol{V}^*\,\big),$$

and thus we have a well defined trace

$$\operatorname{tr}:\Lambda^{j,j}\boldsymbol{V}^*\to\mathbb{R},\ \ \forall j=0,1,\dots,r.$$

Observe that an endomorphism T of $\boldsymbol{V}$ can be identified with the $(1,1)$ double form

$$\omega_T=\sum_{1\leq\alpha,i\leq r}T_{\alpha i}\boldsymbol{v}^\alpha\otimes\boldsymbol{v}^i,\ \ T_{\alpha i}=(\boldsymbol{v}_\alpha,T\boldsymbol{v}_i)_{\boldsymbol{V}}.$$

We the have the equality

$$\det T=\frac{1}{r!}\operatorname{tr}\omega_T^{\wedge r}. \tag{B.7}$$

Let us specialize (B.2) to the case when $E=\boldsymbol{V}$ and $e^\alpha=\boldsymbol{v}^\alpha$. In particular, this implies that $\boldsymbol{V}$ is oriented by the volume form

$$\Omega_{\boldsymbol{V}}:=\boldsymbol{v}^1\wedge\cdots\wedge\boldsymbol{v}^r.$$

If we write

$$\Omega_F=-\sum_{\alpha<\beta}F_{\alpha\beta}\otimes\boldsymbol{v}^\alpha\wedge\boldsymbol{v}^\beta \tag{B.8}$$

then we observe that $\Omega_F\in\Lambda^{2,2}\boldsymbol{V}^{*,*}$ and that the equality (B.3) can be rewritten in the the more compact form

$$\mathbf{Pf}(F)=\frac{1}{h!}\big(\operatorname{tr}\Omega_F^{\wedge h}\big)\Omega_{\boldsymbol{V}}. \tag{B.9}$$

As explained in [1, §12.3] the formalism of double forms and Pfaffians makes its appearance in certain Gaussian computation.

Suppose that S is an random Gaussian endomorphism of $\boldsymbol{V}$ with entries

$$S_{\alpha i}:=(\boldsymbol{v}_\alpha,S\boldsymbol{v}_i)_{\boldsymbol{V}},\ \ \alpha,i=1,\dots,r,$$

centered Gaussian random variables with covariances

$$K_{\alpha i|\beta j}:=\boldsymbol{E}\big(S_{\alpha i}S_{\beta j}\big),\ \ \forall\alpha,\beta,i,j=1,\dots,r.$$

We regard S as $(1,1)$ double form

$$S=\sum_{\alpha,i}S_{\alpha i}\boldsymbol{v}^\alpha\otimes\boldsymbol{v}^i,$$

and we get a random (r,r) double form

$$S^{\wedge r}\in\Lambda^{r,r}\boldsymbol{V}^*.$$

Its expectation can be given a very compact description. Define the $(2,2)$ double form

$$\boldsymbol{\Xi}_K=\sum_{\alpha<\beta,\ i<j}\boldsymbol{\Xi}_{\alpha\beta|ij}\boldsymbol{v}^\alpha\wedge\boldsymbol{v}^\beta\otimes\boldsymbol{v}^i\wedge\boldsymbol{v}^j\in\Lambda^{2,2}\boldsymbol{V}^*,$$

where

$$\boldsymbol{\Xi}_{\alpha\beta|ij}:=\big(K_{\alpha i|\beta j}-K_{\alpha j|\beta i}\big),\ \ \forall\alpha,\beta,i,j.$$

We then have the following equalities, [1, Lemma 12.3.1],

$$\frac{1}{r!}\boldsymbol{E}\big(S^{\wedge r}\big)=\frac{1}{h!}\boldsymbol{\Xi}_K^{\wedge h},\ \ \boldsymbol{E}\big(\det S\big)=\frac{1}{h!}\operatorname{tr}\boldsymbol{\Xi}_K^{\wedge h} \tag{B.10}$$

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