THE GAUSS-BONNET-CHERN THEOREM: A PROBABILISTIC PERSPECTIVE

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ABSTRACT. We prove that the Euler form of a metric connection on a real oriented vector bundle E over a compact oriented manifold M can be identified, as a current, with the expectation of the random current defined by the zero-locus of a certain random section of the bundle. We also explain how to reconstruct probabilistically the metric and the connection on E from the statistics of random sections of E.

CONTENTS

1. INTRODUCTION

1.1. **The Gauss-Bonnet-Chern theorem.** We begin by recalling the classical Gauss-Bott-Chern theorem [5, 13, 20]. Suppose that $E \to M$ is a real *oriented* vector bundle of even rank $r = 2h$ over the smooth, compact oriented manifold M of dimension m. Fix a metric $(-,-)_E$ on E and a connection ∇^E compatible with the metric. We denote by F^E the curvature of the connection ∇^E on E. The *Euler form* of (E, ∇^E) is the closed form

$$\boldsymbol{e}(E, \nabla^E) := \frac{1}{(2\pi)^h} \mathbf{Pf}\big(-F^E\big) \in \Omega^r(M), \quad r = 2h, \tag{1.1}$$

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where $\mathbf{Pf}$ denotes the Pfaffian construction, [13, §8.2.4].

More explicitly, if we fix a local, *positively oriented* orthonormal frame $\boldsymbol{e}_1, \dots, \boldsymbol{e}_r$ of E defined on some open set $\mathcal{O} \subset M$, then the curvature F^E is represented by a skew-symmetric $r \times r$ matrix

$$F^E = (F^E_{\alpha\beta})_{1 \leq \alpha,\beta \leq r}, \;\; F_{\alpha\beta} \in \Omega^2(\mathcal{O}).$$

If we denote by $\mathcal{S}_r$ the group of permutations of $\{1, \dots, r = 2h\}$, then

$$\mathbf{Pf}\big(-F^E\big) = \frac{1}{2^h h!} \sum_{\sigma \in \mathcal{S}_r} \epsilon(\sigma) F^E_{\sigma_1\sigma_2} \wedge \cdots \wedge F^E_{\sigma_{2h-1}\sigma_{2h}} \in \Omega^{2h}(\mathcal{O}), \tag{1.2}$$

where $\epsilon(\sigma)$ denotes the signature of the permutation $\sigma \in \mathcal{S}_r$.

Suppose additionally that we have local coordinates $(x^1, \dots, x^m)$ on $\mathcal{O}$. For $1 \leq \alpha_1, \alpha_2 \leq r$ and $1 \leq j_1, j_2 \leq m$ we set

$$F^E_{\alpha_1\alpha_2|j_1j_2} := F^E_{i_1i_2}(\partial_{x^{j_1}}, \partial_{x^{j_2}}). \tag{1.3}$$

Denote by $\mathcal{S}'_r$ the subset of $\mathcal{S}_r$ consisting of permutations $(\sigma_1, \dots, \sigma_{2h})$ such that

$$\sigma_1 < \sigma_2, \;\sigma_3 < \sigma_4, \;\dots, \;\sigma_{2h-1} < \sigma_{2h}.$$

We deduce from (1.2) that

$$\mathbf{Pf}\big(-F^E\big)\big(\,\partial_{x^1}, \cdots, \partial_{x^r}\,\big) = \frac{1}{h!} \sum_{\varphi,\sigma \in \mathcal{S}'_r} \epsilon(\sigma\varphi) F^E_{\sigma_1\sigma_2|\varphi_1\varphi_2} \cdots F^E_{\sigma_{2h-1}\sigma_{2h}|\varphi_{2h-1}\varphi_{2h}}. \tag{1.4}$$

We denote by $\Omega_k(M)$ the space of k-dimensional currents on M, i.e., the topological dual of the space $\Omega^k(M)$ of smooth k-forms on M. By definition, we have a pairing

$$\langle -, - \rangle : \Omega^k(M) \times \Omega_k(M) \to \mathbb{R}, \;\; (\eta, C) \mapsto \langle \eta, C \rangle.$$

The orientation of M defines a natural Poincaré duality map

$$\Omega^{m-k}(M) \ni \omega \mapsto \omega^\dagger \in \Omega_k(M), \;\; \langle \eta, \omega^\dagger \rangle := \int_M \eta \wedge \omega, \;\; \forall \eta \in \Omega^k(M).$$

Given $\omega \in \Omega^{m-k}(M)$ we will refer to $\omega^\dagger \in \Omega_k(M)$ as the *current determined by the form* ω. By duality we obtain a boundary map

$$\partial : \Omega_k(M) \to \Omega_{k-1}(M), \;\; \langle \eta, \partial C \rangle := \langle d\eta, C \rangle, \;\; \forall C \in \Omega_k(M), \;\; \eta \in \Omega^{k-1}(M).$$

A current C is called closed if $\partial C = 0$.

A generic section $\boldsymbol{u}$ of E is transversal to the zero section, $u \pitchfork 0$, and its zero locus is a smooth submanifold $Z_{\boldsymbol{u}} \subset M$ of dimension $m - r$ equipped with a natural orientation. The integration along this oriented submanifold defines a closed current $[Z_{\boldsymbol{u}}] \in \Omega_{m-r}(M)$.

The most general version of the Gauss-Bonnet-Chern theorem states that for a generic section $\boldsymbol{u}$ the $(m-r)$-dimensional closed currents $[Z_{\boldsymbol{u}}]$ and the Poincaré dual $\boldsymbol{e}(E, \nabla^E)^\dagger$ are homologous, i.e.,

$$\forall \boldsymbol{u} \in C^\infty(E) : \;\; \boldsymbol{u} \pitchfork 0 \Rightarrow [Z_{\boldsymbol{u}}] - \boldsymbol{e}(E, \nabla^E) \in \partial\Omega_{m-r-1}(M). \tag{1.5}$$

In view of DeRham's theorem [7, §22 Thm. 17′], this is equivalent with the statement

$$\forall \boldsymbol{u} \in C^\infty(E), \;\; \boldsymbol{u} \pitchfork 0 \Rightarrow \langle \eta, [Z_{\boldsymbol{u}}] \rangle = \int_M \eta \wedge \boldsymbol{e}(E, \nabla^E), \;\; \forall \eta \in \Omega^r(M), \;\; d\eta = 0. \tag{1.6}$$

1.2. **Overview of the paper.** The first goal of this paper is to provide a probabilistic proof and a refinement of (1.6). Let us first observe that if $\boldsymbol{u}, \boldsymbol{v}$ are two generic smooth sections of E, then the corresponding currents are homologous, i.e.,

$$[Z_{\boldsymbol{u}}] - [Z_{\boldsymbol{v}}] \in \partial\Omega_{m-r-1}(M) \iff \langle \eta, [Z_{\boldsymbol{u}}]\rangle = \langle \eta, [Z_{\boldsymbol{v}}]\rangle, \;\; \forall \eta \in \Omega^{m-r}(M), \;\; d\eta = 0.$$

This shows that if $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n$ are generic sections of E and $p_1, \dots, p_n$ are positive weights such that $p_1 + \cdots + p_n = 1$, then the average

$$p_1[Z_{\boldsymbol{u}_1}] + \cdots + p_n[Z_{\boldsymbol{u}_n}]$$

is a closed current homologous to each of the currents $[Z_{\boldsymbol{u}_k}]$. More generally, if $\boldsymbol{P}$ is a probability measure on $C^\infty(E)$ such that $\boldsymbol{P}$-almost surely a section $\boldsymbol{u}$ intersects the zero section transversally, then the expected current

$$\boldsymbol{E}_{\boldsymbol{P}}([Z_{\boldsymbol{u}}]) := \int [Z_{\boldsymbol{u}}] \boldsymbol{P}(d\boldsymbol{u})$$

is a current *homologous to* the current defined by the zero locus of any generic section $\boldsymbol{u}_0$, i.e.

$$\int \langle \eta, [Z_{\boldsymbol{u}}]\rangle \boldsymbol{P}(d\boldsymbol{u}) = \langle \eta, [Z_{\boldsymbol{u}_0}]\rangle, \;\; \forall \eta \in \Omega^{m-r}(M), \;\; d\eta = 0. \tag{1.7}$$

The first main result of this paper shows that there exist probability measures $\boldsymbol{P}$ on $C^\infty(E)$ such that

- a section $\boldsymbol{u} \in C^\infty(E)$ is $\boldsymbol{P}$-almost surely transversal to the zero section, and
- the expected current $\boldsymbol{E}_{\boldsymbol{P}}([Z_{\boldsymbol{u}}])$ is *equal to* the current determined by Euler form $\boldsymbol{e}(E, \nabla^E)$ associated to the metric $(-,-)_E$ and the connection ∇^E, i.e.,

$$\big\langle\, \eta\,,\, \boldsymbol{E}_{\boldsymbol{P}}([Z_{\boldsymbol{u}}])\,\big\rangle = \int_{C^\infty(E)} \langle \eta, [Z_{\boldsymbol{u}}]\rangle \boldsymbol{P}(d\boldsymbol{u}) = \int_M \eta \wedge \boldsymbol{e}(E, \nabla^E), \;\; \forall \eta \in \Omega^{m-r}(M).$$

We will refer to such probability a measure as *adapted to the metric and connection on E*.

The first step in our program is to produce a large supply of examples of metrics $(-,-)_E$ and compatible connections ∇^E for which we can *explicitly* construct adapted probability measures $\boldsymbol{P}$ on $C^\infty(E)$. In the sequel, we will refer to a pair consisting of a metric on a vector bundle and a connection compatible with it as a *(metric,connection)-pair*.

Fix a finite dimensional real oriented vector space $\boldsymbol{U}$ equipped with an Euclidean inner product $(-,-)_{\boldsymbol{U}}$. We form the trivial real vector bundle

$$\underline{\boldsymbol{U}}_M := \boldsymbol{U} \times M.$$

Assume that $E \to M$ is a an oriented subbundle of rank r of $\underline{\boldsymbol{U}}_M$. The metric $(-,-)_{\boldsymbol{U}}$ on $\boldsymbol{U}$ induces a metric $(-,-)_E$ on E. For each $\boldsymbol{x} \in M$ we denote by $P_{\boldsymbol{x}}$ the orthogonal projection $\boldsymbol{U} \to E_{\boldsymbol{x}}$. The trivial connection d on $\underline{\boldsymbol{U}}_M$ induces a connection $\nabla^E = Pd$ on E. We will call *special* a (metric, connection)-pair $\big(\, (-,-)_E, \nabla^E\,\big)$ constructed as above, via an embedding of E in a trivial vector bundle equipped with a trivial metric and the trivial connection.

Any $\boldsymbol{u} \in \boldsymbol{U}$ defines a section $S^E_{\boldsymbol{u}}$ of E given by

$$S^E_{\boldsymbol{u}}(\boldsymbol{x}) = P_{\boldsymbol{x}} u, \;\; \forall \boldsymbol{x} \in M.$$

We thus get a linear map $S^E : \boldsymbol{U} \to C^\infty(E)$, $\boldsymbol{u} \mapsto S^E_{\boldsymbol{u}}$, whose range is the finite dimensional space

$$\widehat{\boldsymbol{U}} := \big\{\, S^E_{\boldsymbol{u}};\;\; \boldsymbol{u} \in \boldsymbol{U}\,\big\} \subset C^\infty(E).$$

The metric on $\boldsymbol{U}$ induces a Gaussian probability measure (3.1) on $\widehat{\boldsymbol{U}}$. Its pushforward by S^E is a Gaussian probability measure $\gamma_{\boldsymbol{U}}$ on $\widehat{\boldsymbol{U}} \subset C^\infty(E)$. We can view $\gamma_{\boldsymbol{U}}$ as a measure on $C^\infty(E)$ supported on $\widehat{\boldsymbol{U}}$.

Theorem 2.1(i) shows that, $\gamma_{\boldsymbol{U}}$-almost surely, a section $\hat{\boldsymbol{u}} \in \widehat{\boldsymbol{U}}$ intersects transversally the zero section of E. We denote by $[Z_{\hat{\boldsymbol{u}}}]$ the current of integration defined by zero locus of $\hat{\boldsymbol{u}}$.

The key integral formula (2.1) in Theorem 2.1 shows that the expectation of the random current $[Z_{\hat{\boldsymbol{u}}}]$ is equal to the current determined by $\boldsymbol{e}(E,\nabla^E)$, i.e.,

$$\big\langle\, \eta\,,\, \boldsymbol{E}_{\gamma_{\boldsymbol{U}}}([Z_{\hat{\boldsymbol{u}}}])\,\big\rangle = \int_{\widehat{\boldsymbol{U}}} \langle \eta, [Z_{\hat{\boldsymbol{u}}}]\rangle \gamma_{\boldsymbol{U}}(d\hat{\boldsymbol{u}}) = \int_M \eta\wedge \boldsymbol{e}(E,\nabla^E),\ \ \forall \eta\in\Omega^{m-r}(M). \tag{1.8}$$

In other words, the Gaussian measure $\gamma_{\boldsymbol{U}}$ is adapted to the pair $\big(\,(-,-)_E, \nabla^E\big)$.

Obviously the above equality implies (1.6) for special (metric, connection)-pairs on E. Since the Euler form is gauge invariant, we see that (1.8) is valid if we replace the special connection ∇^E with a connection that is gauge equivalent to it. Here the gauge group is the group of orientation preserving, metric preserving automorphisms of E. On the other hand, we have the following result.

Proposition 1.1. *Any (metric, connection)-pair $(\boldsymbol{\sigma},\nabla)$ on an oriented vector bundle $E\to M$ is gauge equivalent to a special pair.*

Proof. The proof is carried out in two steps.

1. The pulback of a special (metric, connection)-pair is a special (metric connection)-pair. Suppose that $(\boldsymbol{\sigma},\nabla)$ is a special (metric, connection)-pair on the subbundle $E\to M$ of the trivial bundle $\underline{\boldsymbol{U}}_M$.

If X is a smooth manifold and $\Phi: X\to M$ is a smooth map, then we get a bundle Φ^*E with metric $\Phi^*\boldsymbol{\sigma}$ and compatible connection $\Phi^*\nabla$. The bundle Φ^*E is a subbundle of the trivial vector bundle

$$\Phi^*\underline{\boldsymbol{U}}_M = \underline{\boldsymbol{U}}_X$$

equipped with a metric h. Then $\Phi^*\boldsymbol{\sigma}$ is the induced metric on Φ^*E as a subbundle of the metric bundle $\underline{\boldsymbol{U}}_X$ and $\Phi^*\nabla$ is the connection induced via orthogonal projection from the trivial connection on $\underline{\boldsymbol{U}}_X$.

2. Consider the Grassmannian $\mathbf{Gr}_r^+(\boldsymbol{U})$ of r-dimensional oriented subspaces of $\boldsymbol{U}$. Denote by $\mathcal{T}_r(\boldsymbol{U})\to\mathbf{Gr}_r^+(\boldsymbol{U})$ the associated tautologial oriented vector bundle. A metric h on $\boldsymbol{U}$ induces a metric $\boldsymbol{\sigma}_h$, and a compatible connection ∇^h on $\mathcal{T}_r(\boldsymbol{U})$. The pair (σ_h,∇^h) is special.

In [12, Thm. 1, 2] Narasimhan and Ramanan have shown that for any smooth, real oriented vector bundle $E\to M$ and any (metric, connection)-pair $(\boldsymbol{\sigma},\nabla)$ on M there exists a finite dimensional Euclidean space $(\boldsymbol{U},h)$ and a smooth map $\Phi: M\to\mathbf{Gr}_r^+(\boldsymbol{U})$ such that

$$E=\Phi^*\mathcal{T}_t(\boldsymbol{U}),\ \ \boldsymbol{\sigma}=\Phi^*\boldsymbol{\sigma}_h$$

and the connection ∇ is gauge equivalent to $\Phi^*\nabla^h$. □

Putting together all of the above we obtain the first main result of this paper.

Theorem 1.2. *Suppose that $E\to M$ is a smooth real oriented vector bundle of rank $r=2h$ over a smooth compact oriented manifold M of dimension m. For any metric $\boldsymbol{\sigma}$ on E and any connection ∇ on E compatible with $\boldsymbol{\sigma}$ there exists a finite dimensional subspace $\widehat{\boldsymbol{U}}\subset C^\infty(E)$ and a Gaussian measure γ on $\widehat{\boldsymbol{U}}$ such that, γ-almost surely, a section $\hat{\boldsymbol{u}}\in\widehat{\boldsymbol{U}}$ is transversal to the zero section and the expectation of the random zero-locus-cycle*

$$\widehat{\boldsymbol{U}}\ni\hat{\boldsymbol{u}}\mapsto [Z_{\hat{\boldsymbol{u}}}]\in\Omega_{m-r}(M)$$

is equal to the current determined by the Euler form of ∇. □

Clearly the above result implies the classical Gauss-Bonnet-Chern theorem, but it has a glaring æsthetic flaw: its formulation includes extrinsic objects, the space $\widehat{\boldsymbol{U}}$ and the Gaussian measure on it, whose relationships to the geometry of (E, σ, ∇) are shrouded in mystery. They depend on many rather noncanonical choices: a description of E via local trivializations and a gluing cocycle and correspondingly, a description of ∇ as a collection of locally defined $\underline{so}(n)$-valued 1-forms. The dependence of $\widehat{\boldsymbol{U}}$ on these choices is hidden in the details of the proofs of [12, Thm. 1,2].

The second goal of the paper is to address this issue and describe more canonical and explicit descriptions for the sample space $\widehat{\boldsymbol{U}}$ with the properties in Theorem 1.2. To formulate our second main result we need to describe an alternate way of producing special (metric, connection)-pairs.

Suppose that $\boldsymbol{U} \to C^\infty(E)$ is a finite dimensional space of sections of E large enough so that it satisfies the *ampleness condition*

$$\operatorname{span}\big\{ \boldsymbol{u}(\boldsymbol{x});\;\; \boldsymbol{u} \in \boldsymbol{U} \big\} = E_{\boldsymbol{x}},\;\; \forall \boldsymbol{x} \in M. \tag{1.9}$$

In particular, for any $\boldsymbol{x} \in M$ the evaluation map

$$\mathbf{ev}_{\boldsymbol{x}} : \boldsymbol{U} \to E_{\boldsymbol{x}},\;\; \boldsymbol{u} \mapsto \mathbf{ev}_{\boldsymbol{x}}\, \boldsymbol{u} := \boldsymbol{u}(\boldsymbol{x})$$

is onto, so that its dual $\mathbf{ev}^*_{\boldsymbol{x}} : E^*_{\boldsymbol{x}} \to \boldsymbol{U}^*$ is one-to-one. Thus, the dual bundle E^* is naturally a subbundle of $\underline{\boldsymbol{U}}^*_M$.

If we fix an inner product $(-,-)_{\boldsymbol{U}}$ on $\boldsymbol{U}$, then we can identify $\boldsymbol{U}$ with $\boldsymbol{U}^*$ and we can view E as a subbundle of the trivial bundle $\underline{\boldsymbol{U}}_M$. Observe that fixing an Euclidean metric on $\boldsymbol{U}$ is equivalent with fixing a nondegenerate Gaussian probability measure $\gamma_{\boldsymbol{U}}$ on $\boldsymbol{U}$. This discussion shows that to any nondegenerate Gaussian probability measure on an ample subspace $\boldsymbol{U} \subset C^\infty(E)$ we can cannonically associate a special (metric, connection)-pair on E.

We define a *sample subspace* of $C^\infty(E)$ to be a pair $(\boldsymbol{U}, \gamma)$, where $\boldsymbol{U} \subset C^\infty(E)$ is an ample finite dimensional subspace and γ is a nondegenerate Gaussian measure on $\boldsymbol{U}$. The space $\boldsymbol{U}$ is called the *support* of the sample space. Thus, to any sample subspace $(\boldsymbol{U}, \gamma)$ of $C^\infty(E)$ we can associate a special (metric, connection)-pair on E. Theorem 2.1 shows that the expectation of the random current defined by the zero-locus of a random $\boldsymbol{u} \in \boldsymbol{U}$ is equal to the current determined by the Euler form of the associated special (metric, connection)-pair.

In Theorem 3.1 we show that any (metric, connection)-pair $(\boldsymbol{\sigma}_0, \nabla^0)$ on E can be approximated by special (metric,connection)-pairs associated to sample subspaces canonically and explicitly determined by $(\boldsymbol{\sigma}_0, \nabla^0)$.

More precisely, in Theorem 3.1 we produce *explicitly* a family of sample spaces $(\boldsymbol{U}_\varepsilon, \gamma_\varepsilon)_{\varepsilon>0}$ with associated special (metric,connection)-pairs $(\boldsymbol{\sigma}_\varepsilon, \nabla^\varepsilon)$ satisfying the following properties.

$$\varepsilon_1 < \varepsilon_2 \Rightarrow \boldsymbol{U}_{\varepsilon_1} \supset \boldsymbol{U}_{\varepsilon_2}, \tag{1.10a}$$

$$\bigcup_{\varepsilon>0} \boldsymbol{U}_\varepsilon \;\text{ is dense in } C^\infty(E), \tag{1.10b}$$

$$\|\|\boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_0\|_{C^0} + \|\nabla^\varepsilon - \nabla^0\|_{L^{1,p}} + \|F^\varepsilon - F^0\|_{C^0} = O(\varepsilon) \;\text{ as } \varepsilon \to 0,\;\; \forall p \in (1, \infty) \tag{1.10c}$$

where $L^{1,p}$ denotes the Sobolev space of distributions with first order derivatives in L^p while F^ε denotes the curvature of ∇^ε.

Let us observe that these facts also imply the Gauss-Bonnet-Theorem for the pair $(\boldsymbol{\sigma}_0, \nabla^0)$ but without appealing to the results of Narasimhan and Ramanan [12]. Indeed, (1.8) implies that for any $\varepsilon > 0$ and any $\eta \in \Omega^{n-r}(M)$ we have

$$\int_{\boldsymbol{U}_\varepsilon} \langle \eta, [Z_{\boldsymbol{u}}] \rangle \gamma_\varepsilon(d\boldsymbol{u}) = \int_M \eta \wedge \boldsymbol{e}(E, \nabla^\varepsilon) \iff \boldsymbol{E}\big([Z_{\boldsymbol{u}}] | \boldsymbol{u} \in \boldsymbol{U}_\varepsilon\big) = \boldsymbol{e}(E, \nabla^\varepsilon)^\dagger.$$

We let $\varepsilon \to 0$ and we conclude from (1.10c) that,

$$\lim_{\varepsilon \to 0} \int_{\boldsymbol{U}_\varepsilon} \langle \eta, [Z_{\boldsymbol{u}}] \rangle \gamma_\varepsilon(d\boldsymbol{u}) = \int_M \eta \wedge \boldsymbol{e}(E, \nabla^0), \quad \forall \eta \in \Omega^{m-r}(M). \tag{1.11}$$

On the other hand, (1.7) shows that for any generic section $\boldsymbol{u}_0$ of E, any *closed* form $\eta \in \Omega^{m-r}(M)$ and any $\varepsilon > 0$ we have

$$\langle \eta, [Z_{\boldsymbol{u}_0}] \rangle = \int_{\boldsymbol{U}_\varepsilon} \langle \eta, [Z_{\boldsymbol{u}}] \rangle \gamma_\varepsilon(d\boldsymbol{u}).$$

As we mentioned earlier, the spaces $\boldsymbol{U}_\varepsilon$ can be constructed *explicitly*. Their description depends on two additional choices.

Fix a Riemann metric g on M and form the covariant Laplacian

$$\Delta_0 = (\nabla^0)^* \nabla^0 : C^\infty(E) \to C^\infty(E).$$

Next, *fix a compactly supported, smooth, even function* $w : \mathbb{R} \to [0, \infty)$ such that $w(0) > 0$. For any $\varepsilon > 0$ we have a smoothing operator

$$W_\varepsilon := w\big(\varepsilon \sqrt{\Delta_0}\,\big) : L^2(E) \to L^2(E).$$

The operator W_ε is symmetric, nonnegative definite and has finite dimensional range $\boldsymbol{U}_\varepsilon := \operatorname{Range} W_\varepsilon$. Clearly the family $(\boldsymbol{U}_\varepsilon)_{\varepsilon > 0}$ satisfies (1.10a) and (1.10b). In particular, this shows that $\boldsymbol{U}_\varepsilon$ is ample if ε is sufficiently small.

The space $\boldsymbol{U}_\varepsilon$ is also a W_ε-invariant subspace of $L^2(E)$ and the restriction of W_ε to $\boldsymbol{U}_\varepsilon$ is invertible because $w(0) \neq 0$. The Gaussian measure γ_ε is then defined by

$$\gamma_\varepsilon(d\boldsymbol{u}) = \frac{1}{\sqrt{\det 2\pi W_\varepsilon}} e^{-\frac{1}{2}(W_\varepsilon^{-1}\boldsymbol{u}, \boldsymbol{u})_0} |d\boldsymbol{u}|_0,$$

where $(-,-)_0$ denotes the L^2-inner product on $\boldsymbol{U}_\varepsilon$ and $|d\boldsymbol{u}|_0$ denotes the associated Lebesgue measure.

The sample space $(\boldsymbol{U}_\varepsilon, \gamma_\varepsilon)$ has a simple classical probabilistic interpretation. Suppose

$$\operatorname{spec}(\Delta_0) = \lambda_1 \leq \lambda_2 \leq \cdots$$

and assume that $(\Psi_n)_{n \geq 1}$ is a complete orthonormal family of $L^2(E)$ consisting of eigensections of Δ_0,

$$\Delta_0 \Psi_n = \lambda_n \Psi_n.$$

A random section $\boldsymbol{u}_\varepsilon \in \boldsymbol{U}_\varepsilon$ is then a random linear superposition

$$\boldsymbol{u}_\varepsilon = \sum_n X_n^\varepsilon \Psi_n,$$

where the coefficients X_n^ε are independent normal random variables with mean 0 and variances

$$\boldsymbol{var}(X_n^\varepsilon) = w(\varepsilon \sqrt{\lambda_n}).$$

If $w = 1$ in a neighborhood of 0, then as $\varepsilon \to 0$ the above random linear superposition formally converges to a random series

$$\sum_n X_n^0 \Psi_n,$$

where the coefficients X_n^0 are independent standard normal random variables, This is very similar to the classical scalar white noise. For this reason we will refer to the $\varepsilon \to 0$ limit as the white-noise limit. Thus, the differential geometry of $(E, \boldsymbol{\sigma}_0, \nabla^0)$ is determined by the white-noise-limit statistics of the random section $\boldsymbol{u}_\varepsilon$. Observe also that the equality (1.11) shows that the expectation of the random current $[Z_{\boldsymbol{u}_\varepsilon}]$ converges in the white-noise limit to the current $\boldsymbol{e}(E, \nabla^E)^\dagger$.

1.3. **Related work.** The results in this paper take place on real manifolds and real vector bundles. In the holomorphic context there is an alternate way of investigating zeros of holomorphic sections. In [25], S. Zelditch has investigated random sections of an ample hermitian holomorphic line bundle L over a compact Kähler manifold M. Among many other things he showed that as $n \to \infty$, the statistics of random sections of L^n determine the differential geometry of the line bundle L. This was later extended to symplectic manifolds by Schifmann and Zelditch in [19].

The statistics of the zero sets of random sections of L^n were thoroughly investigated by Schifmann and Zelditch [18]. Our equality (1.11) has the same flavor as [18, Thm. 1.1, Prop. 4.4] .

For certain classes of noncompact the Kähler manifolds M the statistics of the zero sets of random sections of L^n was investigated by Coman and Marinescu [6] and Dinh, Marinescu, Schmidt [8].

The large n limit is conceptually similar to the white noise limit we employ in this paper although the technical details are quite different. In particular, [25, Cor.3] or [19, Thm. 2] produce C^∞-approximations the curvature of the (almost)hermitian line bundle L. In Theorem 3.1 we produce only C^0-approximations of the curvature of the vector bundle. However, in the special case when $E = TM$, $\boldsymbol{\sigma}_0$ is a Riemannian metric on M and ∇^0 is the associated Levi-Civita connection, then the results in [2, 16] imply that (1.10c) can be refined to a C^∞-convergence of $\boldsymbol{\sigma}_\varepsilon$ to the Riemann metric $\boldsymbol{\sigma}_0$.

In [15] the first author has investigated critical sets of random functions on a compact Riemann manifold. The critical points of a functions are zeros of rather special sections of the cotangent bundles, namely zeros of exact 1-forms. In [15, Thm.1.7] it was shown that the geometry of a Riemann manifold is determined by the statistics of the differentials of random functions on it. This is similar in flavor with Theorem 3.1 in the present paper. However [15, Thm. 1.7] does not follow from the apparently more general Theorem 3.1 in this paper.

1.4. **Organization of the paper.** The main body of the paper consists of two sections. In Section 2 we prove our main integral formula Theorem 2.1 which states that if $(\boldsymbol{U}, \gamma)$ is a sample space of $C^\infty(E)$, then the expectation of the zero-locus-current of a random section $\boldsymbol{u} \in \boldsymbol{U}$ is equal to the current determined by the Euler form of the special connection on E induced by this sample space. The proof relies on the ubiquitous double-fibration trick. We evaluate the various intervening integrals using the theory of orthogonal invariants like in Weyl's proof of his tube formula [24].

Section 3 contains the proof of our probabilistic reconstruction result, Theorem 3.1. It boils down to a detailed understanding of the Schwartz kernel of the smoothing operator $w(\varepsilon\sqrt{\Delta_0})$.

We approach this problem using the wave kernel technique pioneered by L. Hörmander [11]. The fact that our operators are not scalar makes the identification of various terms in the asymptotic expansion of this kernel a bit more challenging. We achieve this by gradually reducing the computation of these terms to the special case involving the heat kernel.

2. A finite dimensional integral formula

2.1. **The setup.** Suppose that M is a compact oriented smooth manifold of dimension m and $E \to \mathbb{R}$ is a real, oriented vector bundle of even rank $r = 2h$. We fix a finite dimensional space $\boldsymbol{U} \subset C^\infty(E)$,

$$\dim \boldsymbol{U} = N.$$

Any $\boldsymbol{x} \in M$ defines a linear evaluation map

$$\mathbf{ev}_{\boldsymbol{x}} : \boldsymbol{U} \to E_{\boldsymbol{x}}, \;\; \boldsymbol{U} \ni \boldsymbol{u} \mapsto \boldsymbol{u}(\boldsymbol{x}).$$

We assume that $\boldsymbol{U}$ satisfies the ampleness condition (1.9). The dual map $\mathbf{ev}_{\boldsymbol{x}}^* : E_{\boldsymbol{x}}^* \to \boldsymbol{U}^*$ is an injection and the family $(\mathbf{ev}_{\boldsymbol{x}}^*)_{\boldsymbol{x}\in M}$ describes an inclusion of E^* as a subbundle of the trivial vector bundle $\underline{\boldsymbol{U}}_M^*$.

We fix an Euclidean metric $(-,-)_{\boldsymbol{U}}$ on $\boldsymbol{U}$. It induces a metric $(-,-)_{\boldsymbol{U}^*}$ on $\boldsymbol{U}^*$. The inclusion

$$\mathbf{ev}^* : E^* \to \underline{\boldsymbol{U}}_M^*$$

induces a metric $(-,-)_{E^*}$ on the bundle E^* and, by duality, a metric $(-,-)_E$ on E.

The evaluation map $\mathbf{ev}_{\boldsymbol{x}} : \boldsymbol{U} \to E_{\boldsymbol{x}}$ can be identified with the orthogonal projection. To emphasize this aspect, we will use the alternate notation $P = P_{\boldsymbol{x}} := \mathbf{ev}_{\boldsymbol{x}}$. We also set $Q = Q_{\boldsymbol{x}} = \mathbb{1} - P_{\boldsymbol{x}}$.

If we choose an orthonormal basis $(\Psi_k)_{1\leq k\leq N}$ of $\boldsymbol{U}$, then we can describe the projection $P_{\boldsymbol{x}}$ in the concrete form

$$P_{\boldsymbol{x}}\boldsymbol{u} = \sum_{k=1}^N (\boldsymbol{u}, \Psi_k)_{\boldsymbol{U}} \Psi_k(\boldsymbol{x}).$$

Let us point a confusing fact. A *fixed* vector $\boldsymbol{u} \in \boldsymbol{U}$ can be viewed as a constant section of the trivial bundle $\underline{\boldsymbol{U}}_M$ and also, by definition, as a section of E. As such it is given by the smooth map

$$S_{\boldsymbol{u}}^E : M \to \boldsymbol{U}, \;\; S_{\boldsymbol{u}}^E(\boldsymbol{x}) = \mathbf{ev}_{\boldsymbol{x}}\,\boldsymbol{u} = P_{\boldsymbol{x}}\boldsymbol{u}.$$

We denote by K the subbundle of $\underline{\boldsymbol{U}}_M$ defined by the kernels of the above projections, $K := \ker P$. Note that

$$E = K^\perp, \;\; E \oplus K \cong \underline{\boldsymbol{U}}_M = \boldsymbol{U} \times M.$$

If we denote by d the trivial connection on $\underline{\boldsymbol{U}}_M$, then we obtain a connection on ∇^E on E compatible with the metric $(-,-)_E$,

$$\nabla^E := PdP.$$

We denote by F^E the curvature of the connection ∇^E on E and by $\boldsymbol{e}(E,\nabla^E)$ the associated Euler form defined as in (1.1)

$$\boldsymbol{e}(E,\nabla^E) = \frac{1}{(2\pi)^h}\,\mathbf{Pf}(-F^E) \in \Omega^r(M), \;\; r = 2h.$$

If a section $\boldsymbol{u} \in \boldsymbol{U}$ is transversal to the zero section, $\boldsymbol{u} \pitchfork 0$, then its zero set

$$Z_{\boldsymbol{u}} := \big\{\, \boldsymbol{x} \in M;\;\; \boldsymbol{u}(x) = 0 \,\big\}$$

is a compact submanifold of M of codimension p. We denote by $T_{Z_{\boldsymbol{u}}}M$ its normal bundle in M,

$$T_{Z_{\boldsymbol{u}}}M := TM|_{Z_{\boldsymbol{u}}}/TZ_{\boldsymbol{u}}.$$

Given any connection ∇ on E we obtain a linear map

$$\nabla_{\bullet}\boldsymbol{u} : (TM)|_{Z_{\boldsymbol{u}}} \to E|_{Z_{\boldsymbol{u}}}$$

which vanishes along $TZ_{\boldsymbol{u}}$ and thus induces a bundle morphism

$$\mathfrak{a}_{\boldsymbol{u}} : T_{Z_{\boldsymbol{u}}}M \to E|_{Z_{\boldsymbol{u}}}$$

that is independent of the choice of ∇. We will refer to $\mathfrak{a}_{\boldsymbol{u}}$ as the *ajunction morphism*.

The transversality $\boldsymbol{u} \pitchfork 0$ is equivalent to the fact that $\mathfrak{a}_{\boldsymbol{u}}$ is a bundle isomorphism. The orientation on E induces via the adjunction morphism an orientation in the normal bundle $(TM)|_{Z_{\boldsymbol{u}}}$ and thus an orientation on $Z_{\boldsymbol{u}}$ uniquely determined by the requirement

$$\text{orientation}\, TM|_{Z_{\boldsymbol{u}}} = \text{orientation}\,(Z_{\boldsymbol{u}}) \wedge \text{orientation}\,(T_{Z_{\boldsymbol{u}}}M).$$

Let us point out that since $Z_{\boldsymbol{u}}$ has *even* codimension we have

$$\text{orientation}\,(Z_{\boldsymbol{u}}) \wedge \text{orientation}\,(T_{Z_{\boldsymbol{u}}}M) = \text{orientation}\,(T_{Z_{\boldsymbol{u}}}M) \wedge \text{orientation}\,(Z_{\boldsymbol{u}}).$$

We denote by $[Z_{\boldsymbol{u}}] \in \Omega_{m-r}(M)$ the integration current defind by the submanifold $Z_{\boldsymbol{u}}$ equipped with the above orientation.

Theorem 2.1. *Let $E \to M$ be a real oriented, smooth vector bundle of rank $r = 2h$ over the compact oriented smooth manifold M. Fix a subspace $\boldsymbol{U} \subset C^\infty(E)$ of dimension $\dim \boldsymbol{U} = N < \infty$ satisfying the ampleness condition (1.9). Fix an Euclidean inner product $(-,-)_{\boldsymbol{U}}$ on $\boldsymbol{U}$ and denote by $\gamma_{\boldsymbol{U}}$ the Gaussian measure on $\boldsymbol{U}$ determined by this inner product,*

$$\gamma_{\boldsymbol{U}}(d\boldsymbol{u}) := \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|\boldsymbol{u}|^2}{2}} d\boldsymbol{u}.$$

Then the following hold.

(i) *A section $\boldsymbol{u} \in \boldsymbol{U}$ almost surely intersects transversally the zero section of E and thus we obtain a random current*

$$\boldsymbol{U} \ni \boldsymbol{u} \mapsto [Z_{\boldsymbol{u}}] \in \Omega_{m-p}(M).$$

(ii) *The expectation of this random current is the current determined by the Euler form $\boldsymbol{e}(E, \nabla^E)$*

$$\boldsymbol{E}_{\gamma_{\boldsymbol{U}}}([Z_{\boldsymbol{u}}]) = \boldsymbol{e}(E, \nabla^E)^\dagger.$$

More precisely,

$$\int_{\boldsymbol{U}} \langle \eta, [Z_{\boldsymbol{u}}] \rangle d\gamma_{\boldsymbol{U}}(du) = \frac{1}{(2\pi)^{\frac{r}{2}}} \int_M \eta \wedge \mathbf{Pf}(-F^E), \quad \forall \eta \in \Omega^{m-r}(M). \tag{2.1}$$

The proof of the the integral formula (2.1) is based on Gelfand's double fibration trick, [1, 9]. Its formulation relies on two versions of the coarea formula. We describe these versions below.

2.2. **The coarea formula.** Suppose that X, Y are *oriented* smooth manifolds of dimensions

$$\dim X = N \geq n = \dim Y.$$

Asume further that that we are given a smooth map $\pi : X \to Y$. For any regular value $y \in Y$ of π the fiber $X_y := \pi^{-1}(y)$ is a smooth submanifold of X of codimension n and its conormal bundle $T^*_{X_y}X$ is naturally isomorphic with $\pi^*T^*Y|_{X_y}$ and thus it has a natural orientation. We orient X_y using the *fiber-first convention*, i.e.,

$$\text{orientation}\,(X) = \text{orientation}\,(X_y) \wedge \text{orientation}\, T^*_{X_y}X.$$

Suppose that $\omega_Y \in \Omega^n(Y)$ is a volume form on Y, i.e., a nowhere vanishing top-degree form on Y. Fix a smooth function $\rho_Y : Y \to \mathbb{R}$ and a form $\eta \in \Omega^{N-n}(X)$ such that

$$-\infty < \int_{X_y} \eta < \infty$$

for any regular value y of π. Sard's theorem implies that y is a regular value of π for almost all $y \in Y$.

The first version of the coarea formula states that the function

$$Y \ni y \mapsto \int_{X_y} \eta \in \mathbb{R}$$

is Lebesgue measurable and

$$\int_Y \left(\int_{X_y} \eta \right) \rho_Y(y)\omega_Y = \int_X \eta \wedge \pi^*(\rho_Y \omega_Y), \;\; \eta \in \Omega^c(X). \tag{2.2}$$

For the second version of the coarea formula we choose a top degree form $\alpha \in \Omega^N(X)$. If $y_0 \in Y$ is a regular value of π, then there is an induced *Gelfand-Leray residue form*

$$\frac{\alpha}{\pi^*\omega_Y} \in \Omega^{N-n}(X_{y_0}).$$

It is locally constructed as follows. Fix a point $p_0 \in X_y$ and local coordinates $(x^1, \dots, x^N)$ on X in a neighborhood U of p_0 and coordinates $(y^1, \dots, y^n)$ on Y in a neighborhood V of $y_0 = \pi(p_0)$ such that, in these coordinates, the smooth map π is linear and described by the functions

$$y^i(x) = x^{N-n+i}, \;\; \forall i = 1, \dots, n.$$

In the coordinates (y^i) the volume form ω_Y has the form

$$\omega_Y = a(y) dy^1 \wedge \cdots \wedge dy^n,$$

where $a \in C^\infty(V)$ is a nowhere vanishing function. Now choose a form $\beta \in \Omega^{N-n}(U)$ such that

$$\beta \wedge a\big(x^{N-n+1}, \dots, x^N \big) dx^{N-n+1} \wedge \cdots \wedge dx^N = \alpha.$$

Ther restriction of β to $X_{y_0} \cap Y$ is an $(N-n)$-form on $X_{y_0} \cap U$ that is independent of all the choices and it is the Gelfand-Leray residue $\frac{\alpha}{\pi^*\omega_Y}$.

The second version of the coarea formula that we will need takes the form

$$\int_X \alpha = \int_Y \left(\int_{X_y} \frac{\alpha}{\pi^*\omega_Y} \right) \omega_Y. \tag{2.3}$$

For a an explanation of why the more traditional coarea formula implies (2.2) and (2.3) we refer to [14, Cor. 2.11].

2.3. **The double fibration trick.** Consider the incidence set

$$\mathcal{X} := \big\{ (\boldsymbol{u}, \boldsymbol{x}) \in \boldsymbol{U} \times M;\ \ \boldsymbol{u}(x) = 0 \big\}.$$

It comes equipped with two natural projections

$$\boldsymbol{U} \xleftarrow{\pi_-} \mathcal{X} \xrightarrow{\pi_+} M,$$

$$\pi_+(\boldsymbol{u}, \boldsymbol{x}) = \boldsymbol{x},\ \ \pi_-(\boldsymbol{u}, \boldsymbol{x}) = \boldsymbol{u},\ \ \forall (\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{X}.$$

For any subset $A \subset M$ and $B \subset \boldsymbol{U}$ we set

$$\mathcal{X}^+_A := \pi_+^{-1}(A),\ \ \mathcal{X}^-_B := \pi_-^{-1}(B).$$

Lemma 2.2. *(a) The incidence set $\mathcal{X}$ has a natural structure of smooth manifold diffeomorphic to the total space of the vector bundle $K \to M$.*
(b) If $\boldsymbol{u} \neq 0$ is a regular value of π_-, then $\boldsymbol{u} \pitchfork 0$.

Proof. (a) Note that

$$(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{X} \Longleftrightarrow P_{\boldsymbol{x}} \boldsymbol{u} = \mathbf{ev}_x\, \boldsymbol{u} = 0 \Longleftrightarrow \boldsymbol{u} = K_{\boldsymbol{x}}.$$

This proves the first claim.

(b) Suppose that $\boldsymbol{u}_0 \in \boldsymbol{U} \setminus 0$ is a regular value of β. We will show that for any $\boldsymbol{x}_0 \in M$ such that $\boldsymbol{u}_0(\boldsymbol{x}_0) = 0$, the adjunction map $\mathfrak{a}_{\boldsymbol{u}_0}$ defines an isomorphisn

$$(T_{Z_{\boldsymbol{u}_0}} M)_{\boldsymbol{x}_0} \to E_{\boldsymbol{x}_0}.$$

Fix a small open coordinate neighborhood $\mathcal{O} \subset M$ of $\boldsymbol{x}_0$ in M with locall coordinates $(x^1, \dots, x^m)$. We assume that via these coordinates $\mathcal{O}$ is identified with a ball $B \subset \mathbb{R}^m$ centered at 0 and $\boldsymbol{x}_0$ is identified with the center of the ball, $x^i(\boldsymbol{x}_0)$, $\forall i = 1, \dots, m$.

Both bundles E and K are trivializable over B. We can therefore find smooth maps

$$\boldsymbol{e}_1, \dots, \boldsymbol{e}_N : \mathcal{O} \to \boldsymbol{U}$$

such that the following hold.

$$\text{For any } \boldsymbol{x} \in \mathcal{O} \text{ the collection } \{\boldsymbol{e}_a(\boldsymbol{x})\}_{1 \leq a \leq N} \text{ is an orthonormal basis of } \boldsymbol{U}. \tag{2.4}$$

$$\operatorname{span}\{\boldsymbol{e}_i(\boldsymbol{x}),\ \ 1 \leq i \leq r\} = E_{\boldsymbol{x}},\ \forall \boldsymbol{x} \in \mathcal{O}. \tag{2.5}$$

$$\operatorname{span}\{\boldsymbol{e}_\alpha(\boldsymbol{x}),\ \ r < \alpha \leq N\} = K_{\boldsymbol{x}},\ \forall \boldsymbol{x} \in \mathcal{O}. \tag{2.6}$$

$$\nabla^E \boldsymbol{e}_i(\boldsymbol{x}_0) = 0,\ \ \forall i = 1, \dots, r, \tag{2.7}$$

We will use the following conventions frequently encountered in integral geometry.

- We will use the Latin letters a, b, c to denote indices in the range $1, \dots, N$.
- We will use the Latin letters i, j, k, ℓ to denote indices in the range $1, \dots, r = \operatorname{rank}(E)$.
- We will use the Greek letters α, β, γ to denote indices in the range $r+1, \dots, N$.

The map

$$\mathbb{R}^N \times B \ni (t, x) \mapsto \left(\sum_a t^a \boldsymbol{e}_a(x), x \right) \in \boldsymbol{U} \times \mathcal{O}$$

is a diffeomorphism. The set $\mathcal{X}^+_{\mathcal{O}} \subset \underline{\boldsymbol{U}}_{\mathcal{O}}$ can be identified with the set

$$\big\{ (t^1, \dots, t^N, \underbrace{x^1, \dots, x^m}_{x}) \in \mathbb{R}^N \times \mathbb{R}^m;\ \ x \in B,\ \ t^j = 0,\ \ \forall j \leq r \big\}.$$

We write

$$t := (t^i)_{1\leq i\leq r},\ \ \tau := (t^\alpha)_{r<\alpha\leq N},\ \ \tilde{t} := (t,\tau).$$

Thus the pair (τ, x) defines local coordinates on $\mathfrak{X}_0^+$. In these coordinates the pair $(\boldsymbol{u}_0, \boldsymbol{x}_0)$ is identified with the pair $(\tau_0, 0) \in \mathbb{R}^{N-r} \times \mathbb{R}^m$,

$$\tau_0 = (\tau_0^{r+1}, \dots, \tau_0^N).$$

Moreover, the map π_- is given by

$$(\tau, x) \mapsto \pi_-(\tau, x) = \sum_\alpha t^\alpha \boldsymbol{e}_\alpha(x) \in \boldsymbol{U}.$$

We set

$$u^a(x) := \big(\, \boldsymbol{u}_0, \boldsymbol{e}_a(x)\,\big)_{\boldsymbol{U}},\ \ \forall a = 1, \dots, N,$$

so that

$$\boldsymbol{u}_0 = \sum_a u^a(x)\boldsymbol{e}_a(x),\ \ \forall x \in B. \tag{2.8}$$

Above, we think of $\boldsymbol{u}_0$ as a constant section of the trivial bundle $\underline{\boldsymbol{U}}_M$. The functions $u^a(x)$ are the coordinates of this section in the moving frame $(\boldsymbol{e}_a(x))$. Note that

$$S^E_{\boldsymbol{u}_0} = \sum_i u^i(x)\boldsymbol{e}_i(x). \tag{2.9}$$

The fiber $\mathfrak{X}^-_{\boldsymbol{u}_0} = \pi_-^{-1}(\boldsymbol{u}_0)$ is described in the coordinates (τ, x) by the equalities

$$u^i(x) = 0,\ \ t^\alpha = u^\alpha(x),\ \ \forall 1 \leq i \leq r,\ \ \forall \alpha > r.$$

This shows that the section

$$Q\boldsymbol{u}_0 : M \to K,\ \ x \mapsto Q_x\boldsymbol{u}_0, \tag{2.10}$$

induces a diffeomorphism from $Z_{\boldsymbol{u}_0}$ to the fiber $\mathfrak{X}^-_{\boldsymbol{u}_0}$.

The differential of π_- at $(\tau_0, 0) \in \mathfrak{X}^-_{\boldsymbol{u}_0}$ is

$$d\pi_-|_{\tau_0,0} = \sum_\alpha dt^\alpha \boldsymbol{e}_\alpha|_{\tau=\tau_0} + \sum_\alpha \tau_0^\alpha d\boldsymbol{e}_\alpha|_{x=0}.$$

Since $\boldsymbol{u}_0$ is a regular value of π_-, the differential $d\pi_-$ at any point in $\mathfrak{X}^-_{\boldsymbol{u}_0}$ is surjective. In particular, the induced linear map

$$Pd\pi_-|_{\tau_0,0} = \sum_\alpha \tau_0^\alpha Pd\boldsymbol{e}_\alpha(x)|_{\boldsymbol{x}=0} : T_{\boldsymbol{x}_0}M \to E_{\boldsymbol{x}_0}$$

must be surjective. From (2.9) we deduce that

$$\nabla^E S^E_{\boldsymbol{u}_0} = Pd\left(\sum_i u^i(x)\boldsymbol{e}_i(x)\right) = \sum_i du^i\boldsymbol{e}_i + \sum_i u^i Pd\boldsymbol{e}_i$$

At $\boldsymbol{x}_0$ we have $u^i(\boldsymbol{x}_0) = 0$ and we conclude that

$$\big(\, \nabla^E S^E_{\boldsymbol{u}_0}\,\big)|_{\boldsymbol{x}_0} = \sum_i du^i\boldsymbol{e}_i.$$

On the other hand, from (2.8) we deduce that

$$0 = d\left(\sum_a u^a(x)\boldsymbol{e}_a(x)\right) \Rightarrow Pd\left(\sum_a u^a(x)\boldsymbol{e}_a(x)\right) = 0$$

$$\Rightarrow \sum_i du^i \boldsymbol{e}_i + \sum_i u^i P d\boldsymbol{e}_i = -\sum_\alpha u^\alpha P d\boldsymbol{e}_\alpha.$$

At $\boldsymbol{x}_0$ we have $u^i(\boldsymbol{x}_0) = 0$, $u^\alpha(\boldsymbol{x}_0) = \tau_0^\alpha$ and we deduce

$$\left(\nabla^E S^E_{\boldsymbol{u}_0}\right)|_{\boldsymbol{x}_0} = \sum_i du^i \boldsymbol{e}_i = -\sum_\alpha \tau_0^\alpha P d\boldsymbol{e}_\alpha(x)|_{x=0} = -P d\pi_-|_{\tau_0,0}.$$

This proves that the adjunction map

$$\mathfrak{a}_{\boldsymbol{u}_0}|_{\boldsymbol{x}_0} = \left(\nabla^E S^E_{\boldsymbol{u}_0}\right)|_{\boldsymbol{x}_0} = -P d\pi_-|_{\tau_0,0} : T_{\boldsymbol{x}_0} M \to E_{\boldsymbol{x}_0} \tag{2.11}$$

is surjective. Since

$$\sum_i du^i \boldsymbol{e}_i = -P d\pi_-|_{\tau_0,0}$$

we deduce that near $\boldsymbol{x}_0$ the zero set $Z_{\boldsymbol{u}_0}$ is cut out transversally by the equations $u^i(x) = 0$, $i = 1, \dots, r$. □

Observe that it suffices to prove (2.1) only for forms η supported in some coordinate neighborhood $\mathcal{O}$ of some point $\boldsymbol{x}_0 \in M$. We continue to use the notations and the conventions introduced in the proof of Lemma 2.2. We have a double fibration

$$\boldsymbol{U} \xleftarrow{\pi_-} \mathcal{X}|_{\mathcal{O}} \xrightarrow{\pi_+} \mathcal{O}.$$

Assume that the volume form

$$\omega_{\mathcal{O}} = dx^1 \wedge \cdots \wedge dx^m \in \Omega^m(\mathcal{O})$$

defines the given orientation of M. Clearly, the equality (2.1) is linear in η so it suffices to prove it in the special case when

$$\eta = f_M dx^{r+1} \wedge \cdots \wedge dx^m, \quad f_M \in C_0^\infty(\mathcal{O}).$$

We fix an orientation on $\boldsymbol{U}$ and consider the volume form

$$\omega_{\boldsymbol{U}} = \rho_{\boldsymbol{U}} dV_{\boldsymbol{U}}, \quad \rho_{\boldsymbol{U}} = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|\boldsymbol{u}|^2}{2}},$$

where $dV_{\boldsymbol{U}}$ denotes the Euclidean volume form on $\boldsymbol{U}$ determined by the given orientation.

The orientation on $\boldsymbol{U}$ defines an orientation on the trivial bundle $\underline{\boldsymbol{U}}_M$. Coupled with the orientation on E it induces an orientation on the vector bundle K uniquely determined by the requirements

$$\text{orientation}\,(\underline{\boldsymbol{U}}_M) = \text{orientation}\,(E) \wedge \text{orientation}\,(K) = \text{orientation}\,(K) \wedge \text{orientation}\,(E).$$

Finally, the orientation on K induces an orientation on the total space $\mathcal{X}$ via the fiber-first convention. We will refer to this orientation as the *natural orientation* on $\mathcal{X}$.

For any regular value $\boldsymbol{u}_0$ of π_-, the fiber $\mathcal{X}^-_{\boldsymbol{u}_0}$ caries an orientation given by the fiber-first convention applied to the fibration $\pi_- : \mathcal{X} \to \boldsymbol{U}$.

Lemma 2.3. *The natural orientation orientation of $\mathcal{X}|_{\mathcal{O}}$ has the property that for any regular value $\boldsymbol{u}_0$ of π_-, the natural isomorphism*

$$Q\boldsymbol{u}_0 : Z_{\boldsymbol{u}_0} \to \mathcal{X}^-_{\boldsymbol{u}_0}$$

defined in (2.10) has degree $(-1)^{Nm}$ and thus changes the orientation by the factor $(-1)^{Nm}$.

Proof. The fiber $\mathcal{X}^-_{\boldsymbol{u}_0}$ is the image of $Z_{\boldsymbol{u}_0}$ via the section $\Psi = Q\boldsymbol{u}_0$ of $\mathcal{X} \to M$. The map Ψ identifies the normal bundle $T_{Z_{\boldsymbol{u}_0}}M$ of $Z_{\boldsymbol{u}_0}$ in M with the normal bundle $T_{\mathcal{X}^-_{\boldsymbol{u}_0}}\Psi(M)$ of $\mathcal{X}^-_{\boldsymbol{u}_0}$ in $\Psi(M)$.

The equality (2.11) shows that the restriction of $d\pi_-$ to $T_{\mathcal{X}^-_{\boldsymbol{u}_0}}\Psi(\mathcal{O})$ can be identified up to a sign with the opposite of the adjunction map. This sign is not important for orientations purposes since the bundles involved have even rank. Now observe that at $(\boldsymbol{u}_0, \boldsymbol{x}_0) \in \mathcal{X}$ we have

$$\begin{aligned} \text{orientation}\,(\mathcal{X}) &= \text{orientation}\,(K_{\boldsymbol{x}_0}) \wedge \text{orientation}\,\Psi(M) \\ &= \text{orientation}\,(K_{\boldsymbol{x}_0}) \wedge \text{orientation}\,(Z_{\boldsymbol{u}_0}) \wedge \text{orientation}\,(E_{\boldsymbol{x}_0}) \\ &= (-1)^{Nm}\text{orientation}\,(Z_{\boldsymbol{u}_0}) \wedge \text{orientation}\,(E_{\boldsymbol{x}_0}) \wedge \text{orientation}\,(K_{\boldsymbol{x}_0}). \end{aligned}$$

On the other hand

$$\begin{aligned} \text{orientation}\,(\mathcal{X}) &= \text{orientation}\,(\mathcal{X}^-_{\boldsymbol{u}_0}) \wedge \text{orientation}\,\boldsymbol{U} \\ &= \text{orientation}\,(\mathcal{X}^-_{\boldsymbol{u}_0}) \wedge \text{orientation}\,(E_{\boldsymbol{x}_0}) \wedge \text{orientation}\,(K_{\boldsymbol{x}_0}). \end{aligned}$$

□

The first coarea formula (2.2) coupled with Lemma 2.3 imply that

$$\int_{\boldsymbol{U}} \left(\int_{Z_u} \eta \right) \rho_{\boldsymbol{U}} dV_{\boldsymbol{U}} = (-1)^{Nm} \int_{\boldsymbol{U}} \left(\int_{\mathcal{X}^-_{\boldsymbol{u}}} \eta \right) \rho_{\boldsymbol{U}} dV_{\boldsymbol{U}} = (-1)^{Nm} \int_{\mathcal{X}^+_{\mathcal{O}}} \pi^*_+\eta \wedge \pi^*_-\omega_{\boldsymbol{U}}.$$

Hence

$$\int_{\boldsymbol{U}} \left(\int_{Z_u} \eta \right) \rho_{\boldsymbol{U}} dV_{\boldsymbol{U}} = \int_{\mathcal{X}^+_{\mathcal{O}}} \pi^*_-\omega_{\boldsymbol{U}} \wedge \pi^*_+\eta. \tag{2.12}$$

Recalling that $\pi_+^{-1}(x) = K_x$, $\forall x \in \mathcal{O}$, we deduce from (2.12) and the second coarea formula (2.3) that

$$\int_{\boldsymbol{U}} \left(\int_{Z_u} \eta \right) \rho_{\boldsymbol{U}} dV_{\boldsymbol{U}} = \int_{\mathcal{O}} \left(\int_{K_x} \frac{\pi^*_-\omega_{\boldsymbol{U}} \wedge \pi^*_+\eta}{\pi^*_+\omega_{\mathcal{O}}} \right) \omega_{\mathcal{O}}. \tag{2.13}$$

This is Gelfand's double fibration trick. To prove (2.1) we need to show that

$$\left(\int_{K_x} \frac{\pi^*_-\omega_{\boldsymbol{U}} \wedge \pi^*_+\eta}{\pi^*_+\omega_{\mathcal{O}}} \right) \omega_{\mathcal{O}} = \frac{1}{(2\pi)^h} \eta \wedge \mathbf{Pf}\big(-F^E\big) = \frac{1}{(2\pi)^h} \mathbf{Pf}\big(-F^E\big) \wedge \eta \;\text{ on }\; \mathcal{O}. \tag{2.14}$$

2.4. **Proof of (2.14).** Suppose that $\big(\boldsymbol{e}_a(0)\big)_{1 \le a \le N}$ is a positively oriented basis of $\boldsymbol{U}$ and $(\boldsymbol{e}_i(0))_{1 \le i \le r}$ is a positively oriented basis of $E_{\boldsymbol{x}_0}$. We set

$$y_{ab}(x) := \big(\boldsymbol{e}_a(0), \boldsymbol{e}_b(x)\big)_{\boldsymbol{U}}, \;\; \forall 1 \le a, b \le N.$$

The $N \times N$ matrix $Y(x) = (y_{ab}(x))$ is orthogonal and $Y(0) = \mathbb{1}$. Moreover

$$\boldsymbol{e}_a(x) = \sum_b y_{ba}(x)\boldsymbol{e}_b(0), \;\; \boldsymbol{e}_a(0) = \sum_b y_{ab}(x)\boldsymbol{e}_b(x), \;\; \forall a. \tag{2.15}$$

We deduce

$$P_x\boldsymbol{e}_a(0) = \sum_i y_{ai}(x)\boldsymbol{e}_i(x) = \sum_{i,b} y_{ai}(x)y_{bi}(x)\boldsymbol{e}_b(0).$$

Hence

$$\nabla^E\boldsymbol{e}_j(x) = P_x d\sum_b y_{bj}(x)\boldsymbol{e}_b(0) = \sum_b dy_{bj}(x)P\boldsymbol{e}_b(0) = \sum_{i,b} y_{bi}(x)dy_{bj}(x)\boldsymbol{e}_i(x).$$

Thus, in the local orthonormal frame $(\, \boldsymbol{e}_i(x)\,)$ the connection ∇^E is described by the matrix-valued 1-form

$$\Gamma = (\Gamma_{ij}(x))_{1\leq i,j\leq p},\;\; \Gamma_{ij}(x) = \sum_b y_{bi}(x)\wedge dy_{bj}(x).$$

The curvature of ∇^E is $F^E = d\Gamma + \Gamma\wedge\Gamma$. Note that

$$d\Gamma_{ij}(x) = \sum_b dy_{bi}(x)\wedge dy_{bj}(x).$$

At $\boldsymbol{x}_0$, the constraint (2.7) on the frame $\boldsymbol{e}_i(x)$ implies that $\nabla^E\boldsymbol{e}_j|_{\boldsymbol{x}_0} = 0$, $\forall j$. Thus

$$0 = \Gamma_{ij}(\boldsymbol{x}_0) = \sum_b y_{bi}(0)dy_{bj}(0) = \sum_b \delta_{bi}dy_{bj}(0) = dy_{ij}(0),\;\; \forall i,j. \tag{2.16}$$

Hence

$$F^E|_{\boldsymbol{x}_0} = d\Gamma = (F_{ij})_{1\leq i,j\leq p},$$
$$F_{ij} = \sum_b dy_{bi}(0)\wedge dy_{bj}(0) = \sum_\beta dy_{\beta i}(0)\wedge dy_{\beta j}(0) \in \Lambda^2 T^*_{\boldsymbol{x}_0}M.$$

On the other hand, the $N\times N$ Maurer-Cartan matrix $Y^{-1}(x)dY(x)$ is skew-symmetric for any x. At $x=0$ we have $Y(0)=1$ and we deduce

$$dy_{\beta i}(0) = -dy_{i\beta}(0),\;\; \forall i,\beta.$$

We conclude that

$$F^E|_{\boldsymbol{x}_0} = d\Gamma = (F_{ij})_{1\leq i,j\leq r},\;\; F_{ij} = \sum_\beta dy_{i\beta}(0)\wedge dy_{j\beta}(0). \tag{2.17}$$

Define

$$y_a : \boldsymbol{U}\to\mathbb{R}\;\;,\;\; y_a(\boldsymbol{u}) = \big(\,\boldsymbol{u}, \boldsymbol{e}_a(0)\,\big)_{\boldsymbol{U}},\;\; 1\leq a\leq N.$$

The Euclidean volume form on $\boldsymbol{U}$ is then

$$dV_{\boldsymbol{U}} = dy_1\wedge\cdots\wedge dy_N.$$

Note that

$$y_a\big(\,\pi_-(\tau, x^1,\dots,x^m)\,\big) = y_a\left(\sum_\alpha t^\alpha \boldsymbol{e}_\alpha(x)\right) = \sum_\alpha t^\alpha\big(\,\boldsymbol{e}_a(0),\,\boldsymbol{e}_\alpha(x)\,\big)_{\boldsymbol{U}} = \sum_\alpha t^\alpha y_{a\alpha}(x).$$

We set

$$\xi_a(x) = \xi_a(\tau,x) := \sum_\alpha t^\alpha y_{a\alpha}(x),$$

so that

$$\pi_-(\tau,x) = \sum_a \xi_a(x)\boldsymbol{e}_A(0),$$

and

$$\pi_-^* dV_{\boldsymbol{U}} = d\xi_1\wedge\cdots\wedge d\xi_N.$$

We view this as a form on the space $\mathbb{R}^{N-r}\times\mathcal{O}$ with coordinates (τ,x). We have

$$d\xi_a = \sum_\alpha dt^\alpha y_{a\alpha} + \sum_\alpha t^\alpha dy_{a\alpha}(x).$$

Observe that at $(\tau_0, 0)$ we have

$$y_{ab}(0) = \delta_{ab},\;\; t^\alpha = \tau_0^\alpha,$$

so

$$d\xi_a(0) := d\xi_a|_{x=0} = \sum_\alpha \delta^a_\alpha dt^\alpha + \sum_\alpha \tau_0^\alpha dy^a_\alpha(0).$$

Hence

$$d\xi_i(0) = \sum_\alpha \tau_0^\alpha dy_{i\alpha}(0), \;\; d\xi_\beta = dt^\beta + \sum_\alpha \tau_0^\alpha dy_{\beta\alpha}(0),$$

so that

$$\pi_-^*\omega_{\boldsymbol{U}} = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|\tau|^2}{2}} d\xi_1 \wedge \cdots d\xi_N.$$

Now observe that

$$d\xi_1 \wedge \cdots d\xi_N = \underbrace{(dt^{r+1} \wedge \cdots \wedge dt^N)}_{=:d\tau} \underbrace{\bigwedge_i \sum_{\alpha_i} \tau_0^{\alpha_i} dy_{i\alpha_i}(0)}_{=:\Omega(\tau_0)} + \mathcal{L},$$

where $\mathcal{L}$ incorporates all the other terms that have degrees $< N-r$ in the dt^α variables, and

$$\Omega(\tau_0) \in \Lambda^r T^*_{\boldsymbol{x}_0} M.$$

Since the terms collected in $\mathcal{L}$ have degrees $> r$ in the variables $(x^1, \dots, x^m)$ we deduce

$$d\xi_1 \wedge \cdots d\xi_N \wedge \pi^*\eta = f_M d\tau \wedge \Omega(\tau_0) \wedge dx^{r+1} \wedge \cdots \wedge dx^m.$$

Denote by $\Omega(\tau_0)_{1,\dots,r}$ the coefficient of $dx^1 \wedge \cdots \wedge dx^r$ in the decomposition of $\Omega(\tau_0)$ with respect to the basis $\{ dx^{j_1} \wedge \cdots \wedge dx^{j_r} \}_{1 \le j_1 < \cdots , j_r \le m}$ of $\Lambda^r T^*_{\boldsymbol{x}_0} M$. If we set

$$\gamma_K(d\tau) := \frac{1}{(2\pi)^{\frac{N-r}{2}}} e^{-\frac{|\tau|^2}{2}} d\tau \in \Omega^{N-r}(K_{\boldsymbol{x}_0}),$$

then we deduce that

$$\frac{\pi_-^*\omega_{\boldsymbol{U}} \wedge \pi_+^*\eta}{dx^1 \wedge \cdots \wedge dx^m} = \frac{1}{(2\pi)^{\frac{p}{2}}} \gamma_K \wedge f_M(\boldsymbol{x}_0) \Omega(\tau_0)_{1,\dots,r}. \tag{2.18}$$

Hence

$$\int_{K_{\boldsymbol{x}_0}} \frac{\pi_-^*\omega_{\boldsymbol{U}} \wedge \pi_+^*\eta}{dx^1 \wedge \cdots \wedge dx^m} = \frac{f_M(\boldsymbol{x}_0)}{(2\pi)^{\frac{r}{2}}} \int_{K_{\boldsymbol{x}_0}} \Omega(\tau)_{1,\dots,r} \gamma_K(d\tau). \tag{2.19}$$

In the sequel we will denote by $\bullet$ the inner product in the space $K_{\boldsymbol{x}_0}$ Our choice of local frames amounts to a metric isomorphism $K_{\boldsymbol{x}_0} \cong \mathbb{R}^{N-r}$.

For every $i = 1, \dots, r$ and $\tau \in K_{\boldsymbol{x}_0}$ we set

$$\Phi_i := \begin{bmatrix} dy_{ir+1}(0) \\ \vdots \\ dy_{iN}(0) \end{bmatrix} \in T^*_{\boldsymbol{x}_0} M \otimes K_{\boldsymbol{x}_0}, \;\; \omega_i(\tau) = \Phi_i \bullet \tau := \sum_\alpha t^\alpha dy_{i\alpha}(0) \in T^*_{\boldsymbol{x}_0} M.$$

Let us point out that the $(N-r) \times r$ matrix with columns $\Phi_1, \dots, \Phi_r$ describes the differential at $\boldsymbol{x}_0$ of the Gauss map

$$M \ni \boldsymbol{x} \mapsto E_{\boldsymbol{x}} \in \mathbf{Gr}_r(\boldsymbol{U}) = \text{the Grassmannian of } r\text{-planes in } \boldsymbol{U}.$$

We have

$$\Omega(\tau) = \omega_1(\tau) \wedge \cdots \wedge \omega_r(\tau).$$

For every $j=1,\dots,m$ and $\tau\in K_{\boldsymbol{x}_0}$ we set

$$\Phi_{ij} := \partial_{x^j}\lrcorner\,\Phi_i = \left[\begin{array}{c} \frac{\partial y_{ir+1}}{\partial x^j}(0)\\ \vdots \\ \frac{\partial y_{iN}}{\partial x^j}(0)\end{array}\right] \in K_{\boldsymbol{x}_0},\ \ \omega_{ij}(\tau) = (\Phi_{ij},\tau)_{\boldsymbol{U}} = \Phi_{ij}\bullet\tau\in\mathbb{R}.$$

We denote by $A(\tau)$ the $r\times r$ matrix with entries

$$A(\tau)_{ij} = \omega_{ij}(\tau),\ \ 1\leq i,j\leq r.$$

Then

$$\omega_i(\tau) = \sum_{j=1}^m \omega_{ij}(\tau)dx^j,\ \ \forall i=1,\dots,r,\ \ \Omega(\tau)_{1,\dots,r} = \det A(\tau).$$

We set

$$\overline{\Omega}_{1,\dots,r} := \int_{K_{\boldsymbol{x}_0}} \det A(\tau)\gamma_K(d\tau). \tag{2.20}$$

Using (2.19) we deduce

$$\int_{K_{\boldsymbol{x}_0}} \frac{\pi_-^*\omega_{\boldsymbol{U}}\wedge\pi_+^*\eta}{dx^1\wedge\cdots\wedge dx^m} = \frac{f_M(\boldsymbol{x}_0)}{(2\pi)^{\frac{p}{2}}}\overline{\Omega}_{1,\dots,r}. \tag{2.21}$$

To compute the Gaussian average (2.20) we use the theory of orthogonal invariants [23] as in Weyl's proof of his tube formula [10, §4.4], [13, §9.3.3], [24].

Let us first observe that for $1\leq i_1\neq i_2\leq r$ and $1\leq j_1<j_2\leq m$ we have

$$\Phi_{i_1j_1}\bullet\Phi_{i_2j_2} - \Phi_{i_1j_2}\bullet\Phi_{i_2j_1} = \sum_\alpha\left(\frac{\partial y_{i_1\alpha}}{\partial x^{j_1}}\frac{\partial y_{i_2\alpha}}{\partial x^{j_2}} - \frac{\partial y_{i_1\alpha}}{\partial x^{j_2}}\frac{\partial y_{i_2\alpha}}{\partial x^{j_1}}\right)$$

$$= \left(\sum_\alpha dy_{i_1\alpha}\wedge dy_{i_2\alpha}\right)(\partial_{x^{j_1}},\partial_{x^{j_2}}).$$

Using (2.17) and the notation (1.3) we deduce

$$F^E_{i_1i_2|j_1j_2} = \Phi_{i_1j_1}\bullet\Phi_{i_2j_2} - \Phi_{i_1j_2}\bullet\Phi_{i_2j_1},\ \ \forall 1\leq i_1,i_2\leq r,\ \ 1\leq j_1,j_2\leq m. \tag{2.22}$$

For any collection of vectors $\boldsymbol{u}_{ij}\in K_{\boldsymbol{x}_0}$, $1\leq i,j\leq r$ and any $\tau\in K_{\boldsymbol{x}_0}$ we define the $r\times r$ matrix

$$A(\tau,\boldsymbol{u}_{ij}) := \big(\,\boldsymbol{u}_{ij}\bullet\tau\,\big)_{1\leq i,j\leq r},$$

and we consider the average

$$\mu(\boldsymbol{u}_{ij}) := \int_{K_{\boldsymbol{x}_0}} \det A(\tau,\boldsymbol{u}_{ij})\gamma_K(d\tau).$$

The average $\mu(\boldsymbol{u}_{ij})$ is a polynomial in the variables $\boldsymbol{u}_{ij}\in K_{\boldsymbol{x}_0}$, $1\leq i,j,\leq r$, and it is invariant with respect to the action of the group $O(N-r)$ of orthogonal transformations of $K_{\boldsymbol{x}_0}$. Note that when $\boldsymbol{u}_{ij}=\Phi_{ij}$ we have

$$\mu(\Phi_{ij}) = \overline{\Omega}_{1,\dots,r}.$$

We recall that $r=2h$ and we denote by $\mathcal{S}_r=\mathcal{S}_{2h}$ the group of permutations of $\{1,2,\dots,2h\}$. As in [13, §9.3.3] we define

$$Q_{\sigma,\varphi}(\boldsymbol{u}_{ij}) := \prod_{j=1}^h\big(\boldsymbol{u}_{\varphi_{2j-1}\sigma_{2j-1}}\bullet\boldsymbol{u}_{\varphi_{2j}\sigma_{2j}}\big),\ \ Q=Q(\boldsymbol{u}_{ij}) := \sum_{\sigma,\varphi\in\mathcal{S}_r}\epsilon(\sigma\varphi)Q_{\sigma,\varphi}(\boldsymbol{u}_{ij}).$$

Lemma 9.3.9 in [13] shows that there exists a constant Z such that

$$\mu(\boldsymbol{u}_{ij}) = ZQ(\boldsymbol{u}_{ij}),\ \ \forall \boldsymbol{u}_{ij}.$$

To find the constant Z we choose the variables $\boldsymbol{u}_{ij}\in K_{\boldsymbol{x}_0}$ judiciously. More precisely, we set

$$\boldsymbol{u}^*_{ij} := \begin{cases} \boldsymbol{e}_N(0), & i=j,\\ 0, & i\neq 0.\end{cases}$$

In this case

$$A(\tau,\boldsymbol{u}^*_{ij}) = \mathrm{Diag}(\underbrace{t^N,\dots,t^N}_{2h}),\ \ \det A(\tau,\boldsymbol{u}_{ij}) = |t^N|^{2h},$$

$$\mu\big(\,\boldsymbol{u}^*_{ij}\,\big) = \int_{K_{\boldsymbol{x}_0}} \big|\,t^N\,\big|^{2h}\gamma_K(d\tau) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} s^{2h}e^{-\frac{s^2}{2}}ds = \prod_{j=1}^h(2j-1) = (2h-1)!!.$$

On the other hand,

$$Q_{\sigma,\varphi}\big(\,\boldsymbol{u}^*_{ij}\,\big) = \begin{cases} 1, & \sigma=\varphi,\\ 0, & \sigma\neq\varphi,\end{cases}$$

and we deduce that $Q\big(\,\boldsymbol{u}^*_{ij}\,\big) = (2h)!$. Thus

$$Z = \frac{(2h-1)!!}{(2h)!} = \frac{1}{2^h h!},\ \ \mu(\Phi_{ij}) = \frac{1}{2^h h!}Q(\Phi_{ij}).$$

Denote by $\mathcal{S}'_r$ the set of permutations φ of $\{1,2,\dots,2h\}$ such that

$$\varphi_1<\varphi_2,\ \ \varphi_3<\varphi_4,\dots,\varphi_{2h-1}<\varphi_{2h}.$$

Using (2.22) we deduce as in the proof of [13, Eq. (9.3.11)] that

$$Q(\Phi_{ij}) = 2^h\sum_{\sigma,\varphi\in\mathcal{S}'_r}\prod_{j=1}^h\epsilon(\sigma\varphi)F^E_{\varphi_{2j-1}\varphi_{2j}|\sigma_{2j-1}\sigma_{2j}}. \tag{2.23}$$

Thus

$$\mu(\Phi_{ij}) = \frac{1}{h!}\sum_{\sigma,\varphi\in\mathcal{S}'_r}\prod_{j=1}^h\epsilon(\sigma\varphi)F^E_{\varphi_{2j-1}\varphi_{2j}|\sigma_{2j-1}\sigma_{2j}} \stackrel{(1.4)}{=} \mathbf{Pf}\big(-F^E\big)(\partial_{x^1},\cdots,\partial_{x^r}).$$

$$\overline{\Omega}_{1,\dots,r} = \mu(\Phi_{ij}) = \mathbf{Pf}\big(-F^E\big)(\partial_{x^1},\dots,\partial_{x^r}). \tag{2.24}$$

Using (2.21) and (2.24) we conclude that

$$\left(\int_{K_{\boldsymbol{x}_0}}\frac{\pi_-^*\omega_{\boldsymbol{U}}\wedge\pi_+^*\eta}{dx^1\wedge\cdots\wedge dx^m}\right)dx^1\wedge\cdots\wedge dx^m = \frac{f_M(\boldsymbol{x}_0)}{(2\pi)^{\frac{r}{2}}}\overline{\Omega}_{1,\dots,p}dx^1\wedge\cdots\wedge dx^m$$

$$= \frac{1}{(2\pi)^h}\mathbf{Pf}\big(-F^E\big)\wedge\eta.$$

This proves (2.14). ∎

3. The white noise limit

3.1. **Gaussian measures.** Recall [4] that a *centered Gaussian measure* on a finite dimensional real vector space $\boldsymbol{U}$ is a probability measure γ on $\boldsymbol{U}$ such that for any linear functional $\xi \in \boldsymbol{U}^* = \mathrm{Hom}(\boldsymbol{U}, \mathbb{R})$ the pushforward $\xi_\# \gamma$ is Gaussian measure on $\mathbb{R}$

$$\xi_\# \gamma = \boldsymbol{\gamma}_v := \frac{1}{\sqrt{2\pi v}} e^{-\frac{\xi^2}{2v}} d\xi, \;\; v \geq 0.$$

Above, when $v = 0$, we define γ_v to the Dirac delta-measure concentrated at 0.

A centered Gaussian measure γ on $\boldsymbol{U}$ is completely determined by its covariance form $C = C_\gamma$ which is the symmetric, nonnegative definite bilinear form

$$C : \boldsymbol{U}^* \times \boldsymbol{U}^* \to \mathbb{R}, \;\; C(\xi_1, \xi_2) = \boldsymbol{E}_\gamma(\xi_1 \cdot \xi_2),$$

where $\xi_1, \xi_2 \in \boldsymbol{U}^*$ are viewed as random variables on $(\boldsymbol{U}, \gamma)$. The Gaussian measure γ is called *nondegenerate* if its covariance form is nondegenerate. If this is the case, the bilinear form defines an Euclidean inner product on $\boldsymbol{U}^*$ and, by duality, an inner product on $\boldsymbol{U}$.

Conversely, given an inner product $\boldsymbol{\sigma}$ on $\boldsymbol{U}$ with norm $|-|_{\boldsymbol{\sigma}}$, we have a Gaussian measure

$$\gamma_{\boldsymbol{\sigma}} = (2\pi)^{-\frac{\dim \boldsymbol{U}}{2}} e^{-\frac{|\boldsymbol{u}|^2_{\boldsymbol{\sigma}}}{2}} |d\boldsymbol{u}|_{\boldsymbol{\sigma}}, \tag{3.1}$$

and $\boldsymbol{\sigma}$ coincides with the inner product determined by $\gamma_{\boldsymbol{\sigma}}$.

The inner product $\boldsymbol{\sigma}$ identifies $\boldsymbol{U}$ with $\boldsymbol{U}^*$ and the covariance form of an arbitrary Gaussian measure γ on $\boldsymbol{U}$ can be identified with a symmetric nonnegative operator $T_\gamma : \boldsymbol{U} \to \boldsymbol{U}$. The measure γ is nondegenerate iff T_γ is invertible. In this case

$$\gamma = \frac{1}{\sqrt{\det 2\pi T_\gamma}} e^{-\frac{1}{2}\boldsymbol{\sigma}(T_\gamma^{-1}\boldsymbol{u}, \boldsymbol{u})} |d\boldsymbol{u}|_{\boldsymbol{\sigma}} = \left(T_\gamma^{\frac{1}{2}}\right)_\# \gamma_{\boldsymbol{\sigma}}. \tag{3.2}$$

Note that if γ is a centered Gaussian measure on $\boldsymbol{U}$ with covariance form C_γ and $L : \boldsymbol{U} \to \boldsymbol{V}$ is a linear map to another finite dimensional vector space $\boldsymbol{V}$ then the pushforward $L_\# \gamma$ is a Gaussian measure on $\boldsymbol{V}$ with covariance form $C_{L_\# \gamma} = L^* C_\gamma$. In particular, if γ is as in (3.2), then

$$\gamma = \left(T_\gamma^{\frac{1}{2}}\right)_\# \gamma_{\boldsymbol{\sigma}}.$$

3.2. **Probabilistic descriptions of special metrics and connection.** Suppose that we are given a smooth real vector bundle $E \to M$ of rank r, an ample finite dimensional subspace $\boldsymbol{U} \subset C^\infty(E)$ and an inner product $(-,-)_{\boldsymbol{U}}$ on $\boldsymbol{U}$. The metric $(-,-)_{\boldsymbol{U}}$ determines a Gaussian measure $\gamma_{\boldsymbol{U}}$ on $\boldsymbol{U}$.

As we have seen, the metric $(-,-)_{\boldsymbol{U}}$ on $\boldsymbol{U}$ induces a metric $(-,-)_E$ on the bundle E and by duality, a metric on E^*. We want to give a probabilistic description of the induced metric $(-,-)_{E^*}$ in a fiber $E^*_{\boldsymbol{x}}$ of E^*.

To simplify the presentation we introduce some notations and conventions.

(i) We will use the $\bullet$-notation to denote the inner product in $\boldsymbol{U}$ or $\boldsymbol{U}^*$.
(ii) We will use the Latin letters i, j, k, ℓ to denote indices in the range $1, \dots, m = \dim M$.
(iii) We will use the Greek letters α, β, γ to denote indices in the range $1, \dots, r = \mathrm{rank}\,(E)$.

Let

$$\langle -,- \rangle : E^*_{\boldsymbol{x}} \times E_{\boldsymbol{x}} \to \mathbb{R}$$

denote the natural pairing. Fix an orthonormal basis $\Psi_1, \dots, \Psi_N$ of $\boldsymbol{U}$ and denote by (Ψ_n^*) the dual orthonormal basis of $\boldsymbol{U}^*$. Then

$$\mathbf{ev}^*_{\boldsymbol{x}} : E^*_{\boldsymbol{x}} \to \boldsymbol{U}^*$$

is given by

$$\mathbf{ev}_{\boldsymbol{x}}(\boldsymbol{u}^*) = \sum_{n=1}^{N} \big\langle\, \boldsymbol{u}^*, \Psi_n(x)\,\big\rangle \Psi_n^*,$$

and

$$(\boldsymbol{u}_1^*, \boldsymbol{u}_2^*)_{E^*} = (\mathbf{ev}_{\boldsymbol{x}}^*\, \boldsymbol{u}_1^*) \bullet (\mathbf{ev}_{\boldsymbol{x}}^*\, \boldsymbol{u}_2^*) = \sum_{n=1}^{N} \big\langle\, \boldsymbol{u}_1^*, \Psi_n(x)\,\big\rangle \big\langle\, \boldsymbol{u}_2^*, \Psi_n(x)\,\big\rangle.$$

Thus the metric $(-,-)_{E^*}$ is described by the bilinear form $\boldsymbol{C}(\boldsymbol{x})$ on $E_{\boldsymbol{x}}^*$ given by

$$\boldsymbol{C}_{\boldsymbol{x}} = \sum_{n=1}^{N} \Psi_n(\boldsymbol{x}) \otimes \Psi_n(\boldsymbol{x}) \in E_{\boldsymbol{x}} \otimes E_{\boldsymbol{x}} \cong \operatorname{Hom}(E_{\boldsymbol{x}}^* \otimes E_{\boldsymbol{x}}^*, \mathbb{R}).$$

The bilinear form $\boldsymbol{C}_{\boldsymbol{x}}$ has a probabilistic interpretation: it is the covariance form of the Gaussian measure $(\mathbf{ev}_{\boldsymbol{x}})_\# \gamma_{\boldsymbol{U}}$ on $E_{\boldsymbol{x}}$.

We have a metric duality isomorphism

$$\boldsymbol{D} = \boldsymbol{D}_{\boldsymbol{x}} : E_{\boldsymbol{x}} \to E_{\boldsymbol{x}}^*, \;\; (\boldsymbol{v}^*, \boldsymbol{D}\boldsymbol{u})_{E^*} := \langle \boldsymbol{v}^*, \boldsymbol{u} \rangle.$$

Fix a point $\boldsymbol{x}_0$ and a small coordinate neighborhood $\mathcal{O}$ of $\boldsymbol{x}_0$ with coordinates (x^i) such that $x^i(\boldsymbol{x}_0) = 0$. Suppose that $(\boldsymbol{e}^\alpha(x))$ is a local frame of E^* defined on $\mathcal{O}$. Denote by $(\boldsymbol{e}_\alpha(x))$ the dual moving frame. We set

$$C^{\alpha\beta}(x) := \boldsymbol{C}_x\big(\, \boldsymbol{e}^\alpha(x), \boldsymbol{e}^\beta(x)\,\big).$$

The matrix $C(x) = (C^{\alpha\beta}(x))$ is symmetric and positive definite. We denote by $(C_{\alpha\beta}(x))$ the inverse matrix. If we write

$$\boldsymbol{D}\boldsymbol{e}_\alpha =: \sum_\beta D_{\beta\alpha} \boldsymbol{e}^\beta,$$

then we deduce

$$\delta_\alpha^\gamma = \langle \boldsymbol{e}^\gamma, \boldsymbol{e}_\alpha \rangle = \Big(\, \boldsymbol{e}^\gamma, \sum_\beta D_{\beta\alpha} \boldsymbol{e}^\beta \,\Big)_{E^*} = \sum_\beta C^{\gamma\beta} D_{\beta\alpha}$$

which shows that the duality isomorphism $\boldsymbol{D}$ is represented in these bases by the inverse of the matrix C, $D_{\beta\alpha}(x) = C_{\beta\alpha}(x)$.

We want to compute the covariant derivatives

$$\nabla_i^{E^*} \boldsymbol{e}^\alpha(0) := \nabla_{\partial_{x^i}}^{E^*} \boldsymbol{e}^\alpha(0).$$

We set

$$\Psi_n^\alpha(x) := \Big\langle\, \boldsymbol{e}^\alpha(x), \Psi_n(x)\,\Big\rangle \in \mathbb{R}, \;\; \forall n = 1, \dots, N,$$

and we deduce

$$\mathbf{ev}_x^*\, \boldsymbol{e}^\alpha(x) = \sum_{n=1}^{N} \Psi_n^\alpha(x) \Psi_n^*, \;\; \partial_{x^i}\Big(\mathbf{ev}_x^*\, \boldsymbol{e}^\alpha(x)\Big) = \sum_{n=1}^{N} \partial_{x^i} \Psi_n^\alpha(x) \Psi_n^*.$$

We denote by P_x the orthogonal projection $\boldsymbol{U}^* \to E_x^*$. Then

$$\nabla_i^{E^*} \boldsymbol{e}^\alpha(x) = P_x \partial_i \Big(\mathbf{ev}_x^*\, \boldsymbol{e}^\alpha(x)\Big) = \boldsymbol{D}_x \left(\sum_n \partial_i \Psi_n^\alpha(x) \Psi_n(x) \right)$$

$$= \boldsymbol{D}_x \left(\sum_{n,\beta} \partial_i \Psi_n^\alpha(x) \Psi_n^\beta(x) \boldsymbol{e}_\beta(x) \right) = \sum_{n,\beta,\gamma} \partial_i \Psi_n^\alpha(x) \Psi_n^\beta(x) C_{\gamma\beta}(x) \boldsymbol{e}^\gamma(x)$$

$$= \sum_{\gamma} \underbrace{\left(\sum_{n} \sum_{\beta} \partial_i \Psi_n^{\alpha}(x) \Psi_n^{\beta}(x) C_{\gamma\beta}(x) \right)}_{=:\Gamma^{\alpha}_{\gamma|i}(x)} \boldsymbol{e}^{\gamma}(x).$$

For every $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{O}$, we denote by (x^i) the coordinates of $\boldsymbol{x}$, by (y^i) the coordinates of $\boldsymbol{y}$, and we set

$$\begin{gathered} C_{\boldsymbol{x},\boldsymbol{y}} := \sum_{n=1}^{N} \Psi_n(\boldsymbol{x}) \otimes \Psi_n(\boldsymbol{y}) \in E_{\boldsymbol{x}} \otimes E_{\boldsymbol{y}}, \\ C^{\alpha\beta}(x,y) := \sum_{n=1}^{N} \big\langle\, \boldsymbol{e}^{\alpha}(\boldsymbol{x}), \Psi_n(\boldsymbol{x}) \,\big\rangle \big\langle\, \boldsymbol{e}^{\beta}(\boldsymbol{y}), \Psi_n(\boldsymbol{y}) \,\big\rangle. \end{gathered} \tag{3.3}$$

One should think of $C_{\boldsymbol{x},\boldsymbol{y}}$ as a *covariance kernel* defined by the random section $\boldsymbol{u} \in \boldsymbol{U}$ because it captures the correlations between the values of $\boldsymbol{u}$ at $\boldsymbol{x}$ and $\boldsymbol{y}$. We deduce that

$$\sum_{n} \partial_i \Psi_n^{\alpha}(x) \Psi_n^{\beta}(x) = \partial_{x^i} C^{\alpha\beta}(x,y)|_{x=y}.$$

Hence

$$\nabla_i^{E^*} \boldsymbol{e}^{\alpha}(x) = \sum_{\gamma} \Gamma^{\alpha}_{\gamma|i}(x) e^{\gamma}(x), \;\; \Gamma^{\alpha}_{\gamma|i}(x) = \sum_{\beta} \partial_{x^i} C^{\alpha\beta}(x,y)|_{x=y} C_{\gamma\beta}(x). \tag{3.4}$$

By duality we deduce

$$\nabla_i^{E} \boldsymbol{e}_{\alpha}(x) = -\sum_{\beta} \Gamma^{\beta}_{\alpha|i}(x) \boldsymbol{e}_{\beta}(x). \tag{3.5}$$

We denote by $\Gamma_i(x)$ the endomorphism of $E_{\boldsymbol{x}}$ given by

$$\boldsymbol{e}_{\alpha}(x) \mapsto \sum_{\beta} \Gamma^{\beta}_{\alpha|i}(x) \boldsymbol{e}_{\beta}(x).$$

From (3.4) and the symmetry of the bilinear form $C(x)$ we deduce that

$$\Gamma_i(x) = \partial_{x^i} C(x,y)|_{x=y} \cdot (C(x)^T)^{-1} = \partial_{x^i} C(x,y)|_{x=y} \cdot C(x)^{-1}. \tag{3.6}$$

We set

$$\Gamma = \sum_i dx^i \Gamma_i = d_x C(x,y)|_{x=y} C(x)^{-1}$$

The operator valued 1-form $-\Gamma$ describes the connection ∇^E in the local frame $(\boldsymbol{e}_{\alpha}(x))$,

$$\nabla^E = d - \Gamma.$$

The curvature is then

$$F^E = -d\Gamma + \Gamma \wedge \Gamma = -\sum_{i,j} (\partial_{x^i} \Gamma_j - \partial_{x^j} \Gamma_i) dx^i \wedge dx^j + \sum_{i<j} [\Gamma_i, \Gamma_j] dx^i \wedge dx^i. \tag{3.7}$$

Concretely

$$\partial_{x^i} \Gamma^{\alpha}_{\gamma|j} = \partial_{x^i} \sum_n \sum_{\beta} \partial_{x^j} \Psi_n^{\alpha}(x) \Psi_n^{\beta}(x) C_{\gamma\beta}(x)$$

$$= \sum_n \sum_{\beta} \partial^2_{x^i x^j} \Psi_n^{\alpha}(x) \Psi_n^{\beta}(x) C_{\gamma\beta}(x) + \sum_n \sum_{\beta} \partial_{x^j} \Psi_n^{\alpha}(x) \partial_{x^i} \Psi_n^{\beta}(x) C_{\gamma\beta}(x)$$

$$\sum_n \sum_{\beta} \partial_{x^j} \Psi_n^{\alpha}(x) \Psi_n^{\beta}(x) \partial_{x^i} C_{\gamma\beta}(x).$$

We deduce

$$\begin{aligned}\partial_{x^i}\Gamma_j &= \partial^2_{x^ix^j}C(x,y)|_{x=y}C(x)^{-1} + \partial^2_{x^jy^i}C(x,y)|_{x=y}C(x)^{-1}\\ &\quad + \Big(\partial_{x^j}C(x,y)|_{x=y}\Big)\cdot\partial_{x^i}\big(C(x)^{-1}\big).\end{aligned} \tag{3.8}$$

Suppose that E came equipped with another metric $\boldsymbol{\sigma}_0(-,-)$ and connection ∇^0 compatible with this metric. Then

$$\nabla^E = \nabla^0 + A = \nabla^0 + \sum dx^i\wedge A_i,$$

where A is a *globally defined* operator valued 1-form, $A\in\Omega^1\big(\operatorname{End}(E)\big)$.

If we choose the local frame frame $(\boldsymbol{e}^\alpha(x))$ on $\mathcal{O}$ to be orthonormal with respect to the metric $\boldsymbol{\sigma}_0$, and $\nabla^0\boldsymbol{e}^\alpha|_{x=0}=0$, then

$$\partial_i\,\mathbf{ev}^*_{\boldsymbol{x}}\,\boldsymbol{e}^\alpha(x)|_{x=0} = \sum_{n=1}^N \partial_i\boldsymbol{\sigma}\big(\Psi_n(0),\boldsymbol{e}_\alpha(0)\big)\Psi_n = \sum_{n=1}^N \boldsymbol{\sigma}\big(\nabla^0_i\Psi_n(0),\boldsymbol{e}_\alpha(0)\big)\Psi_n.$$

It follows that

$$\nabla^E_i\boldsymbol{e}^\alpha(0) = \sum_\gamma \underbrace{\left(\sum_n\sum_\beta(\nabla^0_i\Psi_n)^\alpha(0)\Psi^\beta_n(0)C_{\gamma\beta}(0,0)\right)}_{=:A^\alpha_{\gamma|i}(0)}\boldsymbol{e}^\gamma(0), \tag{3.9}$$

where

$$(\nabla^0\Psi_n)^\alpha(x) := \langle\boldsymbol{e}^\alpha(x),\nabla^0_i\Psi_n(x)\rangle.$$

We deduce

$$\nabla^E_i\boldsymbol{e}_\gamma(0) = -\sum_\alpha A^\alpha_{\gamma|i}(0)dx. \tag{3.10}$$

We denote by $(A_i(x))$ the endomorphism of E_x given by the matrix $(-A^\alpha_{\gamma|i})_{1\le\alpha,\gamma\le r}$.

We can rewrite this in an invariant way as follows. Consider the natural projections

$$M\overset{p_+}{\leftarrow}M\times M\overset{p_-}{\rightarrow}M,\ \ p_\pm(\boldsymbol{x}_+,\boldsymbol{x}_-)=\boldsymbol{x}_\pm,$$

and the bundle

$$E\boxtimes E := p_+^*E\otimes p_-^*E.$$

Then $C(\boldsymbol{x}_+,\boldsymbol{x}_-)$ is a global section of $E\boxtimes E$. Its restriction to the diagonal can be identified with the section $C(\boldsymbol{x})$ of the bundle $E\otimes E$ over M. We deduce

$$A(\boldsymbol{x}) = \sum_i A_i(x)dx^i = -\sum_i \nabla^0_{x^i}C(x,y)_{x=y}\,\cdot\,C(x)^{-1}. \tag{3.11}$$

Indeed, both sides of the above equality are globally defined $\operatorname{End}(E)$-valued 1-forms on M. It therefore suffices to verify (3.11) at an arbitrary point $\boldsymbol{x}_0$ in some local coordinates near $\boldsymbol{x}_0$ and some local trivialization of E. We have done this already in (3.10).

We denote by F^0 the curvature of ∇^0 and by F^E the curvature of ∇^E. Then

$$F^0 = \sum_{i<j}F^0_{ij}dx^i\wedge dx^j,\ \ F^E = \sum_{i<j}F^E_{ij}dx^i\wedge dx^j,$$

and

$$F^E_{ij} = F^0_{ij} + \nabla^0_{x^i}A_j - \nabla^0_{x^j}A_i + [A_i,A_j]. \tag{3.12}$$

Observe that

$$\begin{aligned}\nabla^0_{x^i} A_j = &-\underbrace{\Big(\nabla^0_{x^i}\nabla^0_{x^j} C(x,y)|_{x=y} + \nabla^0_{y^i}\nabla^0_{x^j} C(x,y)|_{x=y}\Big)}_{=:T_{ij}(x)} \cdot C(x)^{-1} \\ &- \nabla^0_{x^j} C(x,y)|_{x=y} \cdot \nabla^0_{x^i}\big(\,C(x)^{-1}\,\big),\end{aligned} \tag{3.13a}$$

$$\nabla^0_{x^i}\big(\,C(x)^{-1}\,\big) = -C_x^{-1}\big(\,\nabla^0_{x^i} C(x)\,\big)C_x^{-1}, \tag{3.13b}$$

$$\nabla^0_{x^i} C(x) = \nabla^0_{x^i} C(x,y)|_{x=y} + \nabla^0_{y^i} C(x,y)|_{x=y}. \tag{3.13c}$$

3.3. **Probabilistic reconstruction of the geometry of a vector bundle.** Suppose that we are given a smooth rank r real vector bundle $E \to M$ over the smooth compact manifold M. We fix a metric $\boldsymbol{\sigma}_0$ on E and a connection ∇^0 on E compatible with $\boldsymbol{\sigma}_0$. We want to construct a family of sample spaces $(\boldsymbol{U}_\varepsilon, \boldsymbol{\gamma}_\varepsilon) \subset C^\infty(E)$ with associated special (metric, connection)-pair $(\boldsymbol{\sigma}_\varepsilon, \nabla^\varepsilon)$ satisfying the conditions (1.10a,1.10b,1.10c). We use a spectral geometry approach.

We fix a Riemann metric g on M with volume density $|dV_g|$. We can form the covariant Laplacian

$$\Delta_0 = \big(\,\nabla^0\,\big)^* \nabla^0 : C^\infty(E) \to C^\infty(E).$$

This is a symmetric, nonnegative definite second order elliptic operator whose principal symbol is scalar

$$\sigma(\Delta_0)(\boldsymbol{x}, \xi) = |\xi|_g^2 \mathbb{1}_{E_{\boldsymbol{x}}}, \;\; \forall \boldsymbol{x} \in M, \;\; \xi \in T^*_{\boldsymbol{x}} M.$$

Let

$$\operatorname{spec}(\Delta_0) = \lambda_1 \leq \lambda_2 \leq \cdots,$$

where in the above sequence each eigenvalue appears as many times as its multiplicity. We fix an orthonormal eigenbasis $(\Psi_n)_{n\geq 1}$ of $L^2(E)$

$$\Delta_0 \Psi_n = \lambda_n \Psi_n, \;\; \forall n.$$

Now fix an even, smooth, compactly supported function $w : \mathbb{R} \to [0, \infty)$. Assume that $w(0) \neq 0$.

For each $\varepsilon > 0$ we have a smoothing selfadjoint operator

$$W_\varepsilon : w\big(\,\varepsilon\sqrt{\Delta_0}\,\big) : L^2(E) \to L^2(E).$$

Define

$$\boldsymbol{U}_\varepsilon := \operatorname{Range}(T_\varepsilon) = \operatorname{span}\big\{\,\Psi_n; \;\; w(\varepsilon\sqrt{\lambda_n}) \neq 0\,\big\} \subset C^\infty(E).$$

Note that $\boldsymbol{U}_\varepsilon$ is a finite dimensional invariant subspace of W_ε. The restriction of W_ε to $\boldsymbol{U}_\varepsilon$ is invertible and selfadjoint with respect to the L^2-inner product on $\boldsymbol{U}_\varepsilon$. As such, it defines a nondegenerate Gaussian measure γ_ε on $\boldsymbol{U}_\varepsilon$ following the prescription (3.2)

$$\gamma_\varepsilon(d\boldsymbol{u}) = \frac{1}{\sqrt{\det 2\pi W_\varepsilon}} e^{-\frac{1}{2}(W_\varepsilon^{-1}\boldsymbol{u}, \boldsymbol{u})_{L^2}} |d\boldsymbol{u}|_{L^2},$$

where $(-,-)_{L^2}$ denotes the L^2-inner product on $\boldsymbol{U}_\varepsilon$ and $|d\boldsymbol{u}|_{L^2}$ denotes the associated Lebesgue measure on $\boldsymbol{U}_\varepsilon$.

We denote generically by $L^{1,p}$ the Sobolev spaces norms of L^p-functions with first order derivatives in L^p.

Theorem 3.1. *Denote by $(\boldsymbol{\sigma}_\varepsilon, \nabla^\varepsilon)$ the special (metric, connection)-pair determined on E by the sample space $(\boldsymbol{U}_\varepsilon, \gamma_\varepsilon)$ constructed as above. For each $\varepsilon \geq 0$ we denote by F^ε the curvature of ∇^ε. Then for each $p \in (1,\infty)$ there exists a positive constant $\kappa = \kappa(p)$ such that the following hold*

$$\|\varepsilon^m \boldsymbol{\sigma}_\varepsilon - \kappa \boldsymbol{\sigma}_0\|_{C^0} + \|\nabla^\varepsilon - \nabla^0\|_{L^{1,p}} + \|F^\varepsilon - F^0\|_{C^0} \leq \kappa\varepsilon \ \text{ as } \varepsilon \searrow 0.$$

Proof. Consider the covariance form $C_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) \in C^\infty(E \boxtimes E)$ determined as in Subsection 3.2 by the inner product on $\boldsymbol{U}_\varepsilon$ defined by the Gaussian measure γ_ε. If we identify E with E^* using the metric $\boldsymbol{\sigma}_0$ we can view C_ε as a section of $E \boxtimes E^*$. As such, it coincides with the Schwartz kernel of W_ε.

The next result contains the key estimates responsible for the conclusions in Theorem 3.1. We defer its very technical proof to the next subsection.

Lemma 3.2. *Let ρ denote the injectivity radius of (M,g), Fix a point $\boldsymbol{x}_0 \in M$ and normal coordinates (x^i) on the open geodesic ball $B_\rho(\boldsymbol{x}_0)$ centered at $\boldsymbol{x}_0$. Fix a trivialization of E over $B_\rho(\boldsymbol{x}_0)$ obtained by ∇^0-parallel transport along the geodesic rays starting at $\boldsymbol{x}_0$. Then the following hold.*

(a) There exist constants $\kappa, K, \varepsilon_0 > 0$ such that

$$|C_\varepsilon(x,x) - \kappa\varepsilon^{-m}\mathbb{1}_{E_x}| \leq K\varepsilon^{2-m}, \ \ \forall \varepsilon < \varepsilon_0, \quad \forall x \in B_{\rho/2}(\boldsymbol{x}_0). \tag{3.14}$$

(b) For $1 \leq i \leq m$ the limits

$$\lim_{\varepsilon\to 0} \varepsilon^m \nabla^0_{x^i} C_\varepsilon(x,y)_{x=y}, \ \ \lim_{\varepsilon\to} \varepsilon^m \nabla^0_{y^i} C_\varepsilon(x,y)_{x=y} \tag{3.15}$$

exist uniformly in $x \in B_{\rho/2}(\boldsymbol{x}_0)$ and the rate of convergence in $C^0\big(B_{\rho/2}(\boldsymbol{x}_0)\big)$ is $O(\varepsilon)$. Moreover

$$\lim_{\varepsilon\to 0} \varepsilon^m \nabla^0_{x^i} C_\varepsilon(x,y)_{x=y=\boldsymbol{x}_0} = 0. \tag{3.16}$$

(c) For $1 \leq i \neq j \leq m$ the limits

$$\lim_{\varepsilon\to 0} \varepsilon^m \nabla^0_{x^i x^j} C_\varepsilon(x,y)_{x=y}, \ \ \lim_{\varepsilon\to 0} \varepsilon^m \nabla^0_{y^i} \nabla^0_{y^j} C_\varepsilon(x,y)_{x=y} \quad \lim_{\varepsilon\to 0} \varepsilon^m \nabla^0_{x^i} \nabla^0_{y^j} C_\varepsilon(x,y)_{x=y} \tag{3.17}$$

exist uniformly in $x \in B_{\rho/2}(\boldsymbol{x}_0)$ and the rate of convergence in $C^0\big(B_{\rho/2}(\boldsymbol{x}_0)\big)$ is $O(\varepsilon)$.

(d) For $1 \leq i \leq m$ the limit

$$\lim_{\varepsilon\to 0} \varepsilon^m \Big(\nabla^0_{x^i}\nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} + \nabla^0_{y^i}\nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} \Big) \tag{3.18}$$

exists uniformly in $x \in B_{\rho/2}(\boldsymbol{x}_0)$ and the rate of convergence in $C^0\big(B_{\rho/2}(\boldsymbol{x}_0)\big)$ is $O(\varepsilon)$. □

Assuming the validity of Lemma 3.2 we proceed as follows. Fix $\boldsymbol{x}_0 \in M$ and normal coordinates in $B_\rho(\boldsymbol{x}_0)$ centered at $\boldsymbol{x}_0$ deduce from (3.14) that

$$\|\varepsilon^m \boldsymbol{\sigma}_\varepsilon - \kappa\boldsymbol{\sigma}_0\|_{C_0} = O(\varepsilon^2) \ \text{ as } \varepsilon \to 0.$$

In the sequel the Landau symbol O refers to the C^0-norm on $B_{\rho/2}(\boldsymbol{x}_0)$. Note also that (3.14) implies that

$$C_\varepsilon(\boldsymbol{x})^{-1} = \varepsilon^m \Big(\kappa^{-1}\mathbb{1}_{E_{\boldsymbol{x}}} + O(\varepsilon^2) \Big). \tag{3.19}$$

If we write $A^\varepsilon := \nabla^\varepsilon - \nabla^0$, then we deduce from (3.11) and (3.15) that

$$A^\varepsilon_i(x) = -\nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} \cdot C_\varepsilon(x)^{-1} = -\varepsilon^m \nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} \Big(\kappa^{-1}\mathbb{1}_{E_{\boldsymbol{x}}} + O(\varepsilon^2) \Big)$$

has a limit as $\varepsilon \to 0$ uniform in $x \in B_{\rho/2}(\boldsymbol{x}_0)$. We set

$$\bar{A}_i(x) := \lim_{\varepsilon\to 0} A_i^\varepsilon(x). \tag{3.20}$$

Moreover (3.16) implies

$$\bar{A}_i(\boldsymbol{x}_0) = 0. \tag{3.21}$$

We have

$$\big\| \bar{A}_i - A_i^\varepsilon \big\|_{C^0(B_{\rho/2}(\boldsymbol{x}_0))} = O(\varepsilon). \tag{3.22}$$

Using (3.12) we deduce that along $B_\rho(\boldsymbol{x}_0)$ and for $i \neq j$ we have

$$F_{ij}^\varepsilon - F_{ij}^0 = \nabla^0_{x^i} A_j^\varepsilon - \nabla^0_{x^j} A_i^\varepsilon + [A_i^\varepsilon, A_j^\varepsilon].$$

From (3.22) we deduce

$$\big\| \, [A_i^\varepsilon, A_j^\varepsilon] - [\bar{A}_i, \bar{A}_j] \, \big\|_{C^0(B_{\rho/2}(\boldsymbol{x}_0))} = O(\varepsilon). \tag{3.23}$$

To estimate $\nabla^0_{x^i} A_j^\varepsilon(x)$ we use (3.13a) and we have

$$\nabla^0_{x^i} A_j^\varepsilon(x) = -T_{ij}^\varepsilon(x) C_\varepsilon(x)^{-1} - \nabla^0_{x^j} C_\varepsilon(x,y)_{x=y} \cdot \nabla^0_{x^i}\big(C_\varepsilon(x)^{-1} \big),$$

$$T_{ij}^\varepsilon(x) = \nabla^0_{x^i}\nabla^0_{x^j} C_\varepsilon(x,y)_{x=y} + \nabla^0_{y^i}\nabla^0_{x^j} C_\varepsilon(x,y)_{x=y}.$$

The estimate (3.20) and Lemma 3.2(b) imply that

$$\lim_{\varepsilon\to 0} T_{ij}^\varepsilon(x) C_\varepsilon(x)^{-1}$$

exists uniformly in $x \in B_{\rho/2}(\boldsymbol{x}_0)$ and the rate of convergence in $C^0\big(B_{\rho/2}(\boldsymbol{x}_0) \big)$ is $O(\varepsilon)$. Using (3.13b), (3.13c) and (3.20) we deduce that

$$\lim_{\varepsilon\to 0} \nabla^0_{x^j} C_\varepsilon(x,y)_{x=y} \cdot \nabla^0_{x^i}\big(C_\varepsilon(x)^{-1} \big) \ \text{ exists uniformly in } x \in B_{\rho/2}(\boldsymbol{x}_0),$$

and the rate of convergence in $C^0\big(B_{\rho/2}(\boldsymbol{x}_0) \big)$ is $O(\varepsilon)$. We conclude that

$$\bar{F}_{ij}(x) := \lim_{\varepsilon\to 0} F_{ij}^\varepsilon(x) \ \text{ exists uniformly in } x \in B_{\rho/2}(\boldsymbol{x}_0), \tag{3.24}$$

and

$$\big\| \bar{F}_{ij} - F_{ij}^\varepsilon \big\|_{C^0(B_{\rho/2}(\boldsymbol{x}_0))} = O(\varepsilon). \tag{3.25}$$

Observe now that

$$\begin{aligned} \nabla^0_{x_i} A_i^\varepsilon(x) = &-\big(\nabla^0_{x_i}\nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} + \nabla^0_{x_i}\nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} \big) \cdot C(x)^{-1} \\ &- \nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} \cdot \nabla^0_{x^i}\big(C(x)^{-1} \big). \end{aligned}$$

Lemma 3.2(c) together with (3.20) imply that the limit

$$\lim_{\varepsilon\to 0} \big(\nabla^0_{x_i}\nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} + \nabla^0_{x_i}\nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} \big) \cdot C(x)^{-1}$$

exists uniformly for $x \in B_{\rho/2}(\boldsymbol{x}_0)$ and the rate of convergence in $C^0\big(B_{\rho/2}(\boldsymbol{x}_0) \big)$ is $O(\varepsilon)$. Finally (3.15) and (3.22) imply that

$$\big\| \nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} \cdot \nabla^0_{x^i}\big(C(x)^{-1} \big) \big\|_{C^0(B_{\rho/2}(\boldsymbol{x}_0))} = O(\varepsilon).$$

Hence

$$\lim_{\varepsilon\to 0} \nabla^0_{x_i} A_i^\varepsilon(x) \ \text{ exists uniformly in } x \in B_{\rho/2}(\boldsymbol{x}_0), \tag{3.26}$$

and the rate of convergence in $C^0\big(B_{\rho/2}(\boldsymbol{x}_0) \big)$ is $O(\varepsilon)$.

The connection ∇^0 defines a first order elliptic (Hodge) operator

$$\mathcal{H} : \Omega^\bullet\big(\operatorname{End}(E)\big) \to \Omega^\bullet\big(\operatorname{End}(E)\big),\;\; \mathcal{H} = d^{\nabla^0} + \Big(d^{\nabla^0}\Big)^*.$$

Since $A^\varepsilon(x)$ converges uniformly on $B_{\rho/2}(x)$ as $\varepsilon \to 0$, we deduce from (3.24) and (3.26) that $\mathcal{H}A^\varepsilon(x)$ converges uniformly on $B_{\rho/2}(x)$ as $\varepsilon \to 0$.

Invoking elliptic L^p-estimates we deduce that for any $p \in (1,\infty)$ there exists a constant $C > 0$ such that for any $\varepsilon_1, \varepsilon_2 > 0$ we have

$$\|A^{\varepsilon_1} - A^{\varepsilon_2}\|_{L^{1,p}(\,B_{\rho/4}(\boldsymbol{x}_0)\,)} \leq C\Big(\|A^{\varepsilon_1} - A^{\varepsilon_2}\|_{L^p(\,B_{\rho/2}(\boldsymbol{x}_0)\,)} + \|\mathcal{H}A^{\varepsilon_1} - \mathcal{H}A^{\varepsilon_2}\|_{L^p(\,B_{\rho/2}(\boldsymbol{x}_0)\,)}\Big).$$

The right-hand side of the above inequality goes to 0 as $\varepsilon_1, \varepsilon_2 \to 0$ so

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \|A^{\varepsilon_1} - A^{\varepsilon_2}\|_{L^{1,p}(B_{\rho/4}(\boldsymbol{x}_0))} = 0.$$

This proves that as $\varepsilon \to 0$ the 1-forms $A^\varepsilon(x)$ converge in the $L^{1,p}$-norm on $B_{\rho/4}(\boldsymbol{x}_0)$. Since these forms converge uniformly to $\bar{A}$ on this ball we deduce that

$$\lim_{\varepsilon\to 0} \|A^\varepsilon - \bar{A}\|_{L^{1,p}(\,B_{\rho/4}(\boldsymbol{x}_0)\,)} = 0.$$

Since M is compact we conclude that exists a globally defined $\operatorname{End}(E)$-valued 1-form

$$\bar{A} \in L^{1,p}\big(\,T^*M \otimes \operatorname{End}(E)\,\big)$$

such that

$$\lim_{\varepsilon\to 0} \|A^\varepsilon - \bar{A}\|_{L^{1,p}(M)} = 0,\;\; \forall p \in (1,\infty).$$

Moreover the equality (3.16) shows that $\bar{A}(\boldsymbol{x}_0) = 0$. Since the point $\boldsymbol{x}_0$ was arbitrary we deduce $\bar{A} = 0$. In turn, this implies that $F^\varepsilon = F^0 + d^{\nabla^0}A^\varepsilon$ converges in $L^p(M)$ to F^0. From (3.24) we deduce that this convergence is in fact uniform. This proves Theorem 3.1 assuming the validity of Lemma 3.2. □

3.4. **Proof of Lemma 3.2.** We rely on the techniques pioneered by L. Hörmander [11] to describe asymptotic estimates for the Schwartz kernel of W_ε as $\varepsilon \to 0$. We follow closely the presentation in [21, XII.2]. We allow w to be an *arbitrary even Schwartz function* $w \in \mathcal{S}(\mathbb{R})$. We denote by C^w_ε the Schwartz kernel of $w(\varepsilon\sqrt{\Delta_0})$.

Fix a point $\boldsymbol{x}_0 \in M$ and normal coordinates (x^i) on $B_\rho(\boldsymbol{x}_0)$. We fix a local orthonormal frame $(\boldsymbol{e}_\alpha)$ of E over this ball which is ∇^0-synchronous of $\boldsymbol{x}_0$, i.e.,

$$\nabla^0 \boldsymbol{e}_\alpha(\boldsymbol{x}_0) = 0,\;\; \forall\alpha. \tag{3.27}$$

We will describe another integral kernel $\mathcal{K}^w_\varepsilon(x,y) \in \operatorname{Hom}(E_y \otimes \mathbb{C}, E_x \otimes \mathbb{C})$, defined for $x, y \in B_\rho(\boldsymbol{x}_0)$, $|x-y|$ sufficiently small, such that

$$C^w(x,y) = \mathcal{K}^w_\varepsilon(x,y) + O(\varepsilon^\infty)$$

i.e.,

$$\|C^w_\varepsilon(x,y) - \mathcal{K}^w_\varepsilon(x,y)\|_{C^k} = O(\varepsilon^N) \;\text{ as } \varepsilon \to 0,\;\; \forall k, N \in \mathbb{Z}_{>0},$$

where the C^k-norm above refers to the C^k-norms of functions defined in a neighborhood of the diagonal in $M \times M$.

Fix a smooth $a : \mathbb{R} \to \mathbb{R}$ such that

$$a(t) = \begin{cases} 0, & |t| < 1,\\ 1, & |t| > 2.\end{cases}$$

For $x \in B_\rho(\boldsymbol{x}_0)$ and $\xi \in \mathbb{R}^m$ we denote by $|\xi|_x$ the length of ξ as an element of T_x^*M. The approximate kernel $\mathcal{K}_\varepsilon^w(x,y)$ has the form [21, Chap. XII, (2.2)]

$$\mathcal{K}_\varepsilon^w(x,y) = \int_{\mathbb{R}^m} q_\varepsilon(x,\xi) e^{\boldsymbol{i}(x-y,\xi)} d\xi, \tag{3.28}$$

where for any positive integer ν we have

$$q_\varepsilon(x,\xi) = a(|\xi|_x) w(|\xi|_x) c_0(x,\xi) + a(|\xi|_x) \sum_{j=1}^{2\nu} \varepsilon^j w^{(j)}\big(\varepsilon|\xi|_x\big) c_j(x,\xi) + R_\nu^\varepsilon(\varepsilon,x,\xi), \tag{3.29}$$

where for every $\varepsilon > 0$ the remainder $R_\nu^\varepsilon(x,\xi)$ is a classical symbol of order $\leq -\nu-1$ and the family $(R_\nu^\varepsilon(x,\xi))_{\varepsilon\in(0,1)}$ is bounded in the space of such symbols.

Moreover, $c_0(x,\xi) = \mathbb{1}_{E_x}$, each of the terms $c_j(x,\xi)$ is independent of w and it has an asymptotic expansion as $\xi \to \infty$

$$c_j(x,\xi) \sim \sum_{k \leq \lfloor j/2 \rfloor} c_{jk}(x,\xi),$$

where $c_{jk}(x,\xi)$ is homogeneous of order k in ξ.

Sublemma 3.3. *Suppose that $\phi \in \mathcal{S}(\mathbb{R})$ and*

$$c : B_\rho(\boldsymbol{x}_0) \times (\mathbb{R}^m \setminus 0) \to \operatorname{End}(E_0 \otimes \mathbb{C}), \ \ (x,\xi) \mapsto c(x,\xi),$$

is a smooth function homogeneous of order $k \in \mathbb{Z}$. We set

$$L_\varepsilon[\phi, c(x)] := \int_{\mathbb{R}^m} a(|\xi|_x)\phi(\varepsilon|\xi|_x) c(x,\xi) d\xi, \tag{3.30}$$

$$\hat{c}(x) = \int_{|\xi|_x=1} c(x,\xi) d\xi.$$

Then the following hold.

(i) *If $k \leq -m-1$, then*

$$\big|L_\varepsilon[\phi, c(x)]\big| = O\big(\|\phi\|_{C^0}\big).$$

(ii) *If $k = -m$, then there exist temperate distributions*

$$T_{j,m} : \mathcal{S}(\mathbb{R}) \to \mathbb{R}, \ \ j = -1, 0, 2, \dots,$$

such that as $\varepsilon \to 0$ we have the asymptotic expansion

$$L_\varepsilon[\phi, c(x)] \sim \hat{c}(x) \left((\log\varepsilon) T_{-1,m}(\phi) + \sum_{j=0}^{\infty} \varepsilon^j T_{j,m}(\phi) \right).$$

Moreover,

$$T_{-1,m}(\phi) = \phi(0).$$

(iii) *If $k > -m$, then there exist temperate distributions*

$$T_{j,k} : \mathcal{S}(\mathbb{R}) \to \mathbb{R}, \ \ j = 0, 1, \dots,$$

such that as $\varepsilon \to 0$ we have an asymptotic expansion

$$L_\varepsilon[\phi, c(x)] \sim \varepsilon^{-m-k} \hat{c}(x) \sum_{j=0}^{\infty} \varepsilon^j T_{j,m}(\phi).$$

Moreover

$$T_{0,k}(\phi) = \left(\int_0^\infty \phi(s) s^{k+m-1} ds \right).$$

Proof. Part (i) is obvious because $a(|\xi|_x)c(x,\xi)$ in integrable in ξ over $\mathbb{R}^m$ if the order k of c is $< -m$. Assume that $k \geq -m$. We set

$$\hat{c}(x) := \int_{|\xi|_x=1} c(x,\xi) d\xi.$$

We have

$$L_\varepsilon[\phi, c(x)] = \int_0^\infty \left(\int_{|\xi_x|=1} c(x, t\xi_x) d\xi_x \right) a_0(t) \phi\varepsilon(t) t^{m-1} dt.$$

$$= \left(\int_0^\infty a_0(t)\phi(\varepsilon t) t^{k+m-1} dt \right) \hat{c}(x) = \varepsilon^{-k-m} \left(\int_0^\infty a_0(s/\varepsilon)\phi(s) s^{k+m-1} ds \right) \hat{c}(x).$$

The last 1-dimensional integral has a complete asymptotic expansion as $\varepsilon \to 0$ described explicitly in [3, Eq.(4.4.22)]. Sublemma 3.3 follows by unraveling the details of this asymptotic expansion. □

Fix two multi-indices $\alpha, \beta \in \mathbb{Z}^m_{\geq 0}$ such that $|\alpha| + |\beta| \leq 2$. Using (3.28) we deduce that

$$\partial_x^\alpha \partial_y^\beta \mathcal{K}_\varepsilon^w(x,y)|_{x=y} = (-1)^{|\beta|} \boldsymbol{i}^{|\alpha|+|\beta|} \xi^\alpha \xi^\beta \int_{\mathbb{R}^m} q(x,\xi) + \int_{\mathbb{R}^m} q_1(x,\xi) d\xi$$

where

$$q_1(x,\xi) = \partial_x^\alpha \partial_y^\beta \Big(q(x,\xi) e^{\boldsymbol{i}(x-y,\xi)} \Big)_{x=y} - q(x,\xi) \Big(\partial_x^\alpha \partial_y^\beta e^{\boldsymbol{i}(x-y,\xi)} \Big)_{x=y}.$$

$$= \sum_{0 \leq \gamma < \alpha} Z_{\alpha,\beta,\gamma} \xi^\gamma \xi^\beta \partial_x^{\alpha-\gamma} q_\varepsilon(x,y,\xi) d\xi,$$

and $Z_{\alpha,\beta,\gamma}$ are certain universal complex constants. Using (3.29) with $\nu = m+2$ and Sublemma 3.3 we deduce that there exist universal temperate distributions

$$S^j_{\alpha,\beta} : \mathcal{S}(\mathbb{R}) \to \mathbb{C}, \;\; j = 0,1,2,$$

and endomorphisms

$$\boldsymbol{K}^j_{\alpha,\beta}(x) : E_x \to E_x, \;\; j = 0,1,2,$$

depending smoothly on x but independent of w such that

$$\varepsilon^m \partial_x^\alpha \partial_y^\beta \mathcal{K}_\varepsilon^w(x,y)|_{x=y} = \varepsilon^{-|\alpha|-|\beta|} \left(\sum_{j=0}^2 \varepsilon^j S^j_{\alpha,\beta}(w) \boldsymbol{K}^j_{\alpha,\beta}(x) + O(\varepsilon^3) \right). \tag{3.31}$$

Moreover, since $c_0(x,\xi) = \mathbb{1}_{E_x}$ we deduce

$$\begin{aligned} S^0_{\alpha,\beta}(w) &= \int_0^\infty w(t) t^{m+|\alpha|+|\beta|-1} dt, \\ \boldsymbol{K}^0_{\alpha,\beta}(x) &= (-1)^{|\beta|} \boldsymbol{i}^{|\alpha|+|\beta|} \left(\int_{|\xi|=1} \xi^\alpha \xi^\beta \right) \mathbb{1}_{E_x}. \end{aligned} \tag{3.32}$$

For any Schwartz function $w \in \mathcal{S}(\mathbb{R})$ and any $\lambda > 0$ we set

$$w_\lambda(x) = w(\lambda x).$$

Observe that $w_\lambda(\varepsilon\sqrt{\Delta_0}) = w(\lambda\varepsilon\sqrt{\Delta_0})$ so that, for fixed $\lambda > 0$, we have

$$\mathcal{K}_\varepsilon^{w_\lambda} = \mathcal{K}_{\lambda\varepsilon}^{w} + O(\varepsilon^\infty).$$

Using this in (3.31) we deduce that for $|\alpha| + |\beta| \leq 2$ and $j = 0, 1, 2$ we have

$$S^j_{\alpha,\beta}(w_\lambda) = \lambda^{-m-|\alpha|-|\beta|+j}\mathcal{S}^j_{\alpha,\beta}(w). \tag{3.33}$$

Sublemma 3.4. *(a) Let $|\alpha| + |\beta| \in \{0, 2\}$. If $\phi \in \mathcal{S}(\mathbb{R})$ is even, then*

$$S^1_{\alpha,\beta}(\phi)\boldsymbol{K}^1_{\alpha,\beta}(x) = 0, \;\; \forall x \in B_{\rho/2}(\boldsymbol{x}_0). \tag{3.34}$$

(b) If $\phi \in \mathcal{S}(\mathbb{R})$ is even, then

$$\lim_{\varepsilon\to} \varepsilon^m \nabla^0_{x^i}\mathcal{K}^\phi_\varepsilon(x, y)|_{x=y=\boldsymbol{x}_0} = 0. \tag{3.35}$$

Proof. Denote by $\mathcal{S}_+(\mathbb{R})$ the space of even Schwartz functions on $\mathbb{R}$ and by $\mathcal{X}_{\alpha,\beta}$ the subspace of of $\mathcal{S}_+(\mathbb{R})$ consisting of functions ϕ satisfying (3.34). Clearly $\mathcal{X}_{\alpha,\beta}$ is a closed subspace of $\mathcal{S}_+$ so it suffices to prove that $\mathcal{X}_{\alpha,\beta}$ is dense in $\mathcal{S}_+(\mathbb{R})$ with respect to the natural locally convex topology of $\mathcal{S}(\mathbb{R})$. The family $\gamma_\lambda(s) = e^{-\lambda^2 s^2}$ spans a vector space dense in $\mathcal{S}_+(\mathbb{R})$; see [22, Chap. 8, Lemma 2.3]. Thus, it suffices to show that $\gamma_\lambda \in \mathcal{X}_{\alpha,\beta}$ for any $\lambda > 0$. In view of the homogeneity condition (3.33) we see that

$$\gamma_1 \in \mathcal{X}_{\alpha,\beta} \iff \gamma_\lambda \in \mathcal{X}_{\alpha,\beta}, \;\; \forall \lambda > 0.$$

For $t > 0$ we denote by H_t the heat kernel, i.e., the Schwartz kernel of $e^{-t\Delta_0}$. Note that H_{ε^2} is the the Schwartz kernel of $\gamma_1(\varepsilon\sqrt{\Delta_0})$.

The heat kernel $H_t(x, y)$ has a rather well understood structure. We denote by $d(x, y)$ the geodesic distance between $x, y \in B_{\rho/2}(\boldsymbol{x}_0)$ with respect to the metric g on M. For x, y in a neighborhood of the diagonal we have an asymptotic expansion as $t \searrow 0$ (see [17, Thm. 7.15])

$$H_t(x, y) = h_t(x, y)\underbrace{\sum_{\nu=0}^\infty t^\nu \Theta_\nu(x, y)}_{=:\Theta_t(x,y)}, \;\; \nu \in \mathbb{Z}_{\geq 0}, \tag{3.36}$$

where $\Theta_k(x, y) \in \operatorname{Hom}(E_y, E_x)$ and

$$h_t(x, y) = t^{-\frac{m}{2}} e^{-\frac{d(x,y)^2}{4t}}.$$

The asymptotic expansion (3.36) is differentiable with respect to all the variables t, x, y. Hence

$$\varepsilon^m H_{\varepsilon^2}(x, y) \sim e^{-u_\varepsilon}\sum_{\nu=0}^\infty \varepsilon^{2\nu}\Theta_\nu(x, y), \tag{3.37}$$

where $u_\varepsilon := \frac{d(x,y)^2}{4\varepsilon^2}$. When $x = y$ we have $u_\varepsilon = 0$ and thus

$$\varepsilon^m H_{\varepsilon^2}(x, x) \sim \sum_{\nu=0}^\infty \varepsilon^{2\nu}\Theta_\nu(x, x).$$

This proves (3.34) in the case $\alpha = \beta = 0$ for the test function γ_1 since the expansion in the right-hand side above involves only even powers of ε.

Differentiating (3.37) we deduce

$$\varepsilon^m \nabla^0_{x^i} H_{\varepsilon^2}(x,y) \sim -(\partial_{x^i} u_\varepsilon) e^{-u_\varepsilon} \sum_{\nu=0}^{\infty} \varepsilon^{2\nu} \Theta_\nu(x,y) + e^{-u_\varepsilon} \sum_{\nu=0}^{\infty} \varepsilon^{2\nu} \nabla^0_{x^i} \Theta_\nu(x,y). \tag{3.38}$$

To compute $\varepsilon^m \nabla^0_{x^j} \nabla^0_{x^i} H_{\varepsilon^2}(x,y)$ when $x=y$ we will take into account that $\partial_{x^i} u_\varepsilon = 0$ when $x=y$. We deduce

$$\begin{aligned} \varepsilon^m \nabla^0_{x^j} \nabla^0_{x^i} H_{\varepsilon^2}(x,y)_{x=y} &\sim \frac{1}{4\varepsilon^2} \partial^2_{x^j x^i} d(x,y)^2|_{x=y} \sum_{\nu=0}^{\infty} \varepsilon^{2\nu} \Theta_\nu(x,x) \\ &+ \sum_{\nu=0}^{\infty} \varepsilon^{2\nu} \nabla^0_{x^j} \nabla^0_{x^i} \Theta_\nu(x,y)_{x=y}. \end{aligned} \tag{3.39}$$

This proves that $\varepsilon^{m+2} \nabla^0_{x^j} \nabla^0_{x^i} H_{\varepsilon^2}(x,y)_{x=y}$ has an asymptotic expansion in *even, nonnegative powers of* ε. Arguing in a similar fashion we deduce that the kernels

$$\varepsilon^{m+2} \nabla^0_{y^j} \nabla^0_{y^i} H_{\varepsilon^2}(x,y)_{x=y}, \;\; \varepsilon^{m+2} \nabla^0_{y^j} \nabla^0_{x^i} H_{\varepsilon^2}(x,y)_{x=y}$$

also have asymptotic expansions in *even, nonnegative powers of* ε. We conclude that $\gamma_1 \in \mathcal{X}_{\alpha,\beta}$ if $|\alpha| + |\beta| = 2$.

Let us observe that (3.38) implies

$$\varepsilon^m \nabla^0_{x^i} H_{\varepsilon^2}(x,y)|_{x=y} \sim \sum_{\nu=0}^{\infty} \varepsilon^{2\nu} \nabla^0_{x^i} \Theta_\nu(x,y)_{x=y}.$$

We deduce that

$$\lim_{\varepsilon \to 0} \varepsilon^m \nabla^0_{x^i} H_{\varepsilon^2}(x,y)|_{x=y} = \nabla^0_{x^i} \Theta_0(x,y)|_{x=y}.$$

From the transport equations [17, Eq.(7.17)] we deduce that *in normal coordinates at* $\boldsymbol{x}_0$ and under the synchronicity condition (3.27) we have

$$\nabla^0_{x^i} \Theta_0(x,y)|_{x=y=\boldsymbol{x}_0} = 0.$$

This proves (3.35) for $\phi = \gamma_1$ and thus for any even Schwartz function ϕ. □

We can now complete the proof of Lemma 3.2. Using (3.31) and (3.32) with $\alpha = \beta = 0$ and Sublemma 3.4(a) we deduce that

$$\varepsilon^m C_\varepsilon(x,x) = \kappa \mathbb{1}_{E_x} + O(\varepsilon^2),$$

where

$$\kappa = \left(\int_0^\infty w(t) t^{m-1} dt \right) \operatorname{vol}(S^{m-1}).$$

For $1 \leq i \leq m$ we set

$$\alpha_i = (\delta_{i1}, \dots, \delta_{im}) \in \mathbb{Z}^m_{\geq 0},$$

where δ_{ij} is Kronecker's delta. From (3.32) we deduce that

$$\boldsymbol{K}^0_{\alpha_j,0} = -\boldsymbol{K}^0_{0,\alpha_j} = \boldsymbol{i} \left(\int_{|\xi|=1} \xi_j \right) \mathbb{1}_{E_x} = 0.$$

Thus

$$\begin{aligned} \varepsilon^m \nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} &= S^1_{\alpha_i,0}(w) \boldsymbol{K}^1_{\alpha_i,0} + O(\varepsilon), \\ \varepsilon^m \nabla^0_{y^i} C_\varepsilon(x,y)_{x=y} &= S^1_{\alpha_i,0}(w) \boldsymbol{K}^1_{0,\alpha_i} + O(\varepsilon). \end{aligned}$$

These estimates prove (3.15). The equality (3.16) follows from (3.35).

From (3.32) we deduce that for $1 \leq i \neq j \leq m$

$$\boldsymbol{K}^0_{\alpha_i+\alpha_j,0}(x) = -\boldsymbol{K}^0_{\alpha_i,\alpha_j}(x) = \boldsymbol{i}\left(\int_{|\xi|=1} \xi_i\xi_j\right)\mathbb{1}_{E_x} = 0,$$

and invoking (3.34) we conclude that

$$\varepsilon^m \nabla^0_{x^i}\nabla^0_{x^j} C_\varepsilon(x,y)_{x=y} = S^2_{\alpha_i+\alpha_j,0}(w)\boldsymbol{K}^2_{\alpha_i+\alpha_j,0}(x) + O(\varepsilon),$$

$$\varepsilon^m \nabla^0_{x^i}\nabla^0_{y^j} C_\varepsilon(x,y)_{x=y} = S^2_{\alpha_i,\alpha_j}(w)\boldsymbol{K}^2_{\alpha_i,\alpha_j}(x) + O(\varepsilon),$$

$$\varepsilon^m \nabla^0_{y^i}\nabla^0_{y^j} C_\varepsilon(x,y)_{x=y} = S^2_{0,\alpha_i+\alpha_j}(w)\boldsymbol{K}^2_{0,\alpha_i+\alpha_j}(x) + O(\varepsilon).$$

These estimates prove (3.17). Note that Sublemma 3.4 implies that

$$\varepsilon^m\left(\nabla^0_{x^i}\nabla^0_{x^i} C_\varepsilon(x,y)_{x=y} + \nabla^0_{y^i}\nabla^0_{x^i} C_\varepsilon(x,y)_{x=y}\right)$$

$$= \varepsilon^{-2}\Big(S^0_{2\alpha_i,0}(w)\boldsymbol{K}^0_{2\alpha_i,0}(x) + S^0_{\alpha_i,\alpha_i}(w)\boldsymbol{K}^0_{0,2\alpha_i}(x)\Big)$$

$$+\Big(S^2_{2\alpha_i,0}(w)\boldsymbol{K}^2_{2\alpha_i,0}(x) + S^2_{\alpha_i,\alpha_i}(w)\boldsymbol{K}^2_{\alpha_i,\alpha_i}(x)\Big) + O(\varepsilon).$$

The equalities (3.32) imply that

$$S^0_{2\alpha_i,0}(w)\boldsymbol{K}^0_{2\alpha_i,0}(x) + S^0_{\alpha_i,\alpha_i}(w)\boldsymbol{K}^0_{\alpha_i,\alpha_i}(x) = 0.$$

This proves (3.18) and completes the proof of Lemma 3.2. □

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