The many faces of the Gauss-Bonnet theorem

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Abstract

The Gauss-Bonnet theorem, like few others in geometry, is the source of many fundamental discoveries which are now part of the everyday language of the modern geometer. This is an informal survey of some of the most fertile ideas which grew out of the attempts to better understand the meaning of this remarkable theorem.

1 A classical result in spherical geometry

Denote by Σ_R the round sphere of radius R centered at the origin of the Euclidean space $\mathbb{R}^3$. Consider three points A, B, C on Σ. Join these three points by arcs of great circles[1] to obtain a *spherical triangle*. We denote by α, β, γ the angles of this triangle at A, B and respectively C. We have the following classical result.

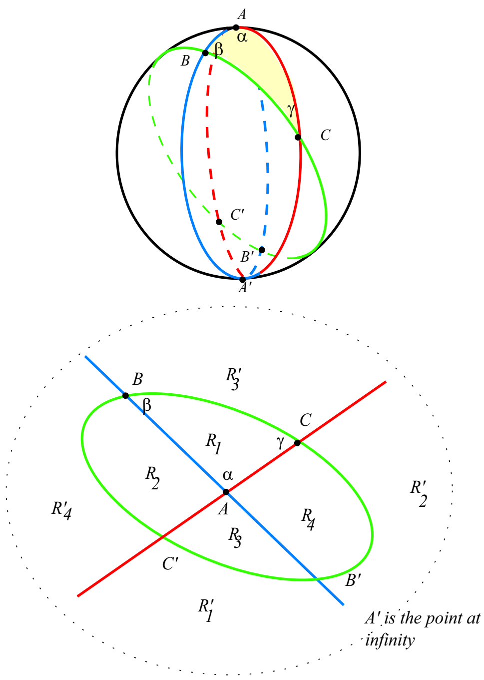

Figure 1: *A spherical triangle and its stereographic projection.*

[*]Talk at the Graduate Student Seminar Fall 2003.

[1]The great circles are obtained by intersecting the sphere with an arbitrary plane through its center.

Theorem 1.1 (Legendre).

$$\text{Area}\,(\Delta ABC) = (\alpha + \beta + \gamma - \pi)R^2. \tag{$*$}$$

In particular the sum of angles of a spherical triangle is bigger than the sum of angles of an Euclidean triangle.

Proof We could attempt to use calculus to determine this area, but due to the remarkable symmetry of the sphere an elementary solution is not only possible, but it is also faster. First some terminology.

For every set of points $S \subset \Sigma$ we denote by S' the set consisting of the antipodal points. A *lune* is one of the regions bounded by two meridians joining two antipodal points. We will indicate a lune by indicating the angle it is determined by. We thus obtain the lunes $L_\alpha, L_\beta, L_\gamma$ at A, B and respectively C. The area of a lune depends linearly on the angle between the two meridians. When the angle is zero the area is zero, while when the angle is 2π the lune coincides with the whole sphere so the area is $4\pi R^2$. Thus the area of a lune of angle θ is $2\theta R^2$.

By eventually relabelling the points we can assume $A' \neq B, C$. Consider the stereographic projection from A' of the great circles determining the triangle ABC. We Obtain the diagram depicted at the bottom of Figure 1. Upon inspecting this figure we deduce

$$L_\alpha = R_1 \cup R_3',\ \ L_\beta = R_1 \cup R_4,\ \ L_\gamma = R_1 \cup R_2.$$

We denote by a_i the area of R_i. Since $\text{Area}\,(R_i') = \text{Area}\,(R_i)$ we deduce

$$a_1 + a_2 = 2\gamma R^2,\ \ a_1 + a_3 = 2\alpha R^2,\ \ a_1 + a_4 = 2\beta R^2.$$

Now observe that the regions $R_1, \cdots, R_4$ cover the hemisphere containing the point A and bounded by the great circle $BCB'C'$ so that

$$a_1 + a_2 + a_3 + a_4 = 2\pi R^2.$$

Thus

$$2(\alpha + \beta + \gamma)R^2 = 3a_1 + a_2 + a_3 + a + 4 = 2a_1 + (a_1 + a_2 + a_3 + a_4) = 2a_1 + 2\pi$$

so that

$$a_1 = (\alpha + \beta + \gamma)R^2 - \pi R^2.$$

□

Remark 1.2. The condition that the edges of the triangle ABC are arcs of great circles is obviously essential. Indeed, it easy to deform these arcs while keeping the angles between them fixed while changing the area they surround. One may ask what is so special about these great circles. The answer is simple: on the round sphere the great circles play the same role the straight lines play in the plane. More precisely, the shortest path between two nearby points on the sphere is the unique arc of great circle determined by these two points. To rephrase this in modern language, the great circles are precisely the *geodesics* of the round sphere.

□

2 Enter the curvature

To put Legendre's theorem in some perspective we need to give a different interpretation of the right-hand-side of ($*$). Let us perform the following ideal experiment.

Consider a vector $\vec{v}$ tangent to the sphere at the point A. Denote by θ the angle between $\vec{v}$ and the arc AB. We measure the angles in counterclockwise fashion. Let us now *parallel transport* this vector along the perimeter of this triangle. This means that we move the vector along the perimeter of the *geodesic* triangle ABC, in counterclockwise fashion, such that during the motion the angle between the vector and the corresponding edge of the triangle stays constant. At the end of this experiment the vector $\vec{v}$ will not return to the initial position. In fact is has turned in counterclockwise fashion exactly by the angle $(\alpha + \beta + \gamma) - \pi$. The rotation performed by $\vec{v}$ during this transport along the perimeter is called the *holonomy* along the triangle ABC.

Consider more generally a *geodesic polygon* $A_1 \cdots A_n$ on Σ, i.e a polygon whose edges are geodesics. We denote its angles by $\alpha_1, \cdots, \alpha_n$. We define the *defect* of this polygon to be the amount by which the sum of the angles of this polygon differs from the sum of angles of an Euclidean polygon with the same number of edges,

$$\delta(A_1 \cdots A_n) = (\alpha_1 + \cdots + \alpha_n) - (n-2)\pi.$$

Arguing as above we see that the defect measures the holonomy around the polygon, i.e. the size of the rotation suffered by a vector during a (counterclockwise) parallel transport around the polygon.

Let us observe that the defect is *additive* with respect to polygonal decompositions (see Figure 2)

$$\delta(A_0 A_1, \cdots A_n) = \delta(A_1 \cdots A_n) + \delta(A_0 A_1 A_n).$$

Thus the defects of very small geodesic polygons completely determine the defects of

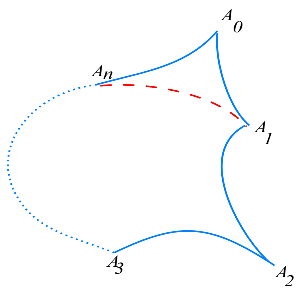

Figure 2: *Decomposing a geodesic polygon into smaller geodesic triangles.*

any geodesic polygon in the following simple fashion: partition the large polygon into smaller ones and then add up their defects. In other words, the defect behaves very much like a measure and thus we expect it to have the (Radon-Nicodym) form

$$\text{defect} = \text{function} \times \text{area}.$$

Given this fact it is natural to define the *(gaussian) curvature function*

$$\mathcal{K} : \Sigma_R \to \mathbb{R}, \;\; \kappa(x) = \lim_{\mathcal{P} \searrow x} \frac{\delta(\mathcal{P})}{\text{Area}\,(\mathcal{P})}, \tag{2.1}$$

where the above limit is taken over smaller and smaller geodesic polygons $\mathcal{P}$ which converge to the point $x \in \Sigma_R$. To compute the defect of a geodesic polygon it suffices to integrate the curvature function

$$\delta(\mathcal{P}) = \int_{\mathcal{P}} \mathcal{K}(x) dA(x),$$

where $dA(x)$ denotes the area element on the sphere Σ_R. We can reformulate Legendre's theorem as a statement about the curvature of Σ_R, more precisely

$$\mathcal{K}_{\Sigma_R} \equiv \frac{1}{R^2}. \tag{**}$$

This formulation has one clear advantage over $(*)$: it relies entirely on quantities *intrinsic* to the surface Σ_R, i.e. quantities which can be measured by an observer living on this surface and has no idea that this surface lies in the bigger three dimensional space $\mathbb{R}^3$.

As Gauss noticed, the definition (2.1) extends word for word to Riemann surfaces. These are two dimensional manifolds along which the smooth curves have well defined lengths. It is convenient to think of such a surface as made of canvas: it is flexible but inelastic so that the "fibers" of this canvas do not change their length as it is deformed in one way or another.

We will denote the Riemann surfaces by pairs (M, g) where M is a 2-manifold and g is a Riemann metric, i.e. an assignment of an inner product in each tangent space $T_m M$ of M which depends smoothly on $m \in M$. We denote by $\mathcal{K}_M : M \to \mathbb{R}$ the curvature of such a surface defined by the equality (2.1). The defect of a region $R \subset M$ is by definition the quantity

$$\delta(R) = \int_R \mathcal{K}_M dA_g,$$

where dA_g denotes the area element of (M, g). Here is a nice consequence of these simple observations.

Corollary 2.1. *We cannot wrap a planar piece of canvas around a round sphere of radius R.*

Proof Indeed the defect of any planar region is zero. If we could wrap a planar piece of canvas C around a region of Σ_R then we would conclude from $(**)$ that the defect of this piece, viewed as a region of Σ_R is equal to $\mathrm{Area}\,(C)/R^2 > 0$.

□

Theorem 2.2 (Bonnet). *Suppose (M, g) is a closed, oriented Riemann surface. Then*

$$\delta(M) = 2\pi\chi(M), \tag{2.2}$$

where $\chi(M) = 2 - 2\mathrm{genus}\,(M)$ denotes the Euler characteristic of M.

Proof Consider a triangulation of M consisting of geodesic triangles $\Delta_1, \cdots, \Delta_T$. Denote by V the number of vertices and by E the number of edges of this triangulation. Then

$$\chi(M) = V - E + T.$$

Now observe that since each edge separates exactly two triangles we have and since each triangle consists of three edges we deduce that

$$3T = 2E \Longleftrightarrow \chi(M) = V - \frac{T}{2}.$$

If $\alpha_i, \beta_i, \gamma_i$ denote the angles of the geodesic triangle Δ_i we deduce that

$$\delta(M) = \sum_i \delta(\Delta_i) = \sum_i (\alpha_i + \beta_i + \gamma_i) - T\pi.$$

The angles of all the triangles Δ_i fill up the angles formed at the vertices of the triangulation. The sum of the angles formed at one vertex is 2π so that

$$\delta(M) = 2\pi V - T\pi = 2\pi\big(V - \frac{T}{2}\big) = 2\pi\chi(M).$$

$\square$

Corollary 2.3 (Gauss). *If (M, g) is as above then*

$$\frac{1}{2\pi}\int_M \mathcal{K}_M dA_g = \chi(M). \tag{2.3}$$

Definition 2.4. *For any oriented Riemann surface (M, g) we denote by $\boldsymbol{e}(M, g)$ the 2-form on M*

$$\boldsymbol{e}(M, g) = \frac{1}{2\pi}\mathcal{K}_M dA_g.$$

This 2-form is called the ***Euler form*** *of the Riemann surface (M, g).*

For example, Legendre theorem implies that

$$\boldsymbol{e}(\Sigma_R) = \frac{1}{2\pi R^2} dA_{\Sigma_R}.$$

3 The Gauss map and Theorema Egregium

The above considerations are very intuitive but computationally unfriendly. To see how one can get numbers out of these heuristic considerations we will follow the path opened by C.F. Gauss in his famous memoir *"Disquisitiones generales circa superficies curvas"* (see [7, vol. II, Chap.3] for an English translation and commentaries on the fundamental ideas of contained in this very important work).

Suppose $M \hookrightarrow \mathbb{R}^3$ is a closed oriented surface smoothly embedded in the 3-dimensional Euclidean space. Since we can measure the lengths of curves in $\mathbb{R}^3$ we can measure the lengths of smooth curves on M, i.e. M is equipped with a natural Riemann metric called the *induced metric*. We denote it by g_{ind}.

Since M is oriented every tangent space $T_m M$ is equipped with a natural orientation. For every $m \in M$ we denote by $\vec{\mathfrak{n}}(m) \in \mathbb{R}^3$ the unit vector uniquely determined by the properties

$$\vec{\mathfrak{n}}(m) \perp T_m M, \;\; \vec{\mathfrak{n}}(m) \wedge \text{orientation}\,(T_m M) = \text{orientation}\,(\mathbb{R}^3).$$

The ensuing smooth map

$$\mathcal{G}_M : M \to \Sigma_1, \;\; m \mapsto \vec{\mathfrak{n}}(m)$$

is called the *Gauss map* of the embedding $M \hookrightarrow \mathbb{R}^3$. The following result explains the importance of the Gauss map.

Proposition 3.1.

$$\deg \mathcal{G}_M = \frac{1}{2}\chi(M).$$

Sketch of proof. We sketch below an intuitive argument explaining why such an equality could be true. This argument could be arranged into a completely argument proof. For the technical details we refer to [4, §6, Thm.1].

Loosely speaking the degree of the map $\mathcal{G}_M$ is the number of points in a fiber $\mathcal{G}_M^{-1}(\vec{v}_0)$, $\vec{v}_0 \in \Sigma_1$, i.e. the number of solutions of the equation

$$\vec{\mathfrak{n}}(m) = \vec{v}_0, \;\; m \in M,$$

where $\vec{v}_0$ is a fixed *generic* point in Σ_1. For every $m \in M$ we denote by X_m the orthogonal projection of $\vec{v}_0$ onto the tangent plane $T_m M \subset \mathbb{R}^3$. X_m is a vector field on

M which vanishes exactly when $\vec{n}(m) = \pm\vec{v}_0$. Thus the number of zeroes of the vector field X_m should be twice the degree of the Gauss map. According to the Poincaré-Hopf theorem the number of zeroes of a vector field is equal to the Euler characteristic of M. $\square$

In the memoir quoted above Gauss proved the following fundamental result.

Theorem 3.2 (Theorema Egregium). *The pullback of the Euler form on Σ_1 via the Gauss map is the Euler form of (M, g_{ind}), i.e.*

$$\mathcal{G}_M^* \boldsymbol{e}(\Sigma_1) = \boldsymbol{e}(M, g_{ind}). \tag{†}$$

We refer to [6, §4.2.4] for a proof of this result closer to the spirit of our discussion.

Here we only want to comment on the meaning of this result which Gauss himself thought to be fundamental. First the name, Theorema Egregium (Golden Theorem) was chosen by Gauss to emphasize how fundamental he considers this result. Why is this result surprising?

The right hand side of (†) is a quantity *intrinsic* to the Riemann surface (M, g_{ind}), i.e. it can be determined by doing measurements only inside M. The left-hand-side $\mathcal{G}_M^* \boldsymbol{e}(\Sigma_1)$ is by definition an *extrinsic* invariant of M since its determination requires a good understanding of how the unit normal field $\vec{\mathfrak{n}}$ changes along M.

Roughly speaking the equality (†) states that the Gauss map takes a small neighborhood of a point $m_0 \in M$ and wraps along a small region of the unit sphere containing the point $\vec{\mathfrak{n}}(m_0)$. During this process the area of this neighborhood of m_0 is distorted by a factor equal to the curvature of M at m_0.

☞ *Why should we be amazed by Theorema Egregium?* The surface of a sphere in $\mathbb{R}^3$ perceived to be "curved" because the human eye detects the changes in the position of the outer normal from one point to another. Theorema Egregium states that even if we live on the surface and we cannot "see" this variation in the position of the normal we can still compute the "rate of change" per unit of area by performing only two-dimensional measurements. Here is what you have to do. Pick a small geodesic triangle, measure its angles and then its defect and area. The ratio defect/area is measures the rate of change in the position of the normal per unit area.

The meaning of this rate of change can however be deceiving. The surface of a cylinder looks curved to the human eye, yet the "rate of change" in position per unit area is zero for the following simple reason. We can roll a planar sheet of paper over a cylinder. This rolling over process does not modify the lengths of curves on the paper and thus the cylinder has the same curvature as a plane which is zero.

The Gauss-Bonnet identity (2.3) follows immediately from (†). Indeed we have

$$\begin{aligned} \frac{1}{2\pi}\int_M \mathcal{K}_M dA_g &= \int_M \boldsymbol{e}(M, g_{ind}) = \int_M \mathcal{G}^* \boldsymbol{e}(\Sigma_1) \\ &= \deg \mathcal{G} \cdot \int_{\Sigma_1} \boldsymbol{e}(\Sigma_1) = 2\deg \mathcal{G} = \chi(M). \end{aligned}$$

The Gauss-Bonnet formula is a beautiful example of a local-to-global result: it assembles in an ingenious fashion *local* information (gaussian curvature) to reach a *global* conclusion (the value of the Euler characteristic). It is perhaps less surprising that the operation of integration is involved in such a process.

4 Higher dimensional generalizations

It is natural to enquire if any of the above results has a higher dimensional counterpart. More precisely we are seeking a *universal* procedure which would associate to each closed oriented n-dimensional Riemann manifold (M,g) an *Euler form* $\boldsymbol{e}(M,g)$. This should be a differential n-form with the following properties.

(i) The coefficients of $\boldsymbol{e}(M,g)$ depend only on the *local geometry of* (M,g). In other words, to compute the form $\boldsymbol{e}$ in a neighborhood U of a point we only need to perform computations inside this neighborhood, an not very far away from this point. In particular, if two points m_1, m_2 have isometric neighborhoods U_1, U_2 then along U_1 the differential form $\boldsymbol{e}$ should look exactly as on U_2.

(ii) The integral of the Euler form should be the Euler characteristic of the manifold

$$\int_M \boldsymbol{e}(M,g) = \chi(M).$$

Denote by $S^{n-1}(R)$ the sphere of radius R centered at the origin of $\mathbb{R}^n$. We denote by $A_{n-1}(R)$ its n-dimensional "area"

$$A_{n-1}(R) = A_{n-1}R^{n-1}, \;\; A_{n-1} = A_{n-1}(1) = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

where

$$\Gamma(1) = 1, \;\; \Gamma(1/2) = \pi^{1/2}, \;\; \Gamma(x+1) = x\Gamma(x), \;\; \forall x.$$

Using the general principles ***(i)*** and ***(ii)*** above it is easy to figure out what $\boldsymbol{e}(S^n)$ should be. Since any two points in S^n have isometric neighborhoods we deduce that

$$\boldsymbol{e}(S^n) = const \cdot dV_{S^n},$$

where dV_{S^n} is the Euclidean volume from on S^n. Since we must have the equality

$$const \int_{S^n} dV_{S^n} = \chi(S^n)$$

we guess that

$$\boldsymbol{e}(S^n) := \frac{\chi(S^n)}{A_n} dV_{S^n}. \tag{4.1}$$

We take (4.1) as the definition of the Euler form of the round n-dimensional sphere of radius 1.

To seek an expression for the Euler form of an arbitrary Riemann n-manifold we try to guess its shape in some more special cases. The two-dimensional Theorema Egregium will be our guide.

Suppose $M^n \hookrightarrow \mathbb{R}^n$ is a smooth, closed, oriented, hypersurface in $\mathbb{R}^n$. As in the case $n=2$ we see that it is equipped with an induced metric g_{ind}. Using the orientation we obtain an unit normal vector field $\vec{\mathfrak{n}}$ along M which defines a Gauss map

$$\mathcal{G}_M : M \to S^n.$$

Arguing as in the proof of Proposition 3.1 we deduce the following.

Proposition 4.1. *Suppose M is even dimensional. Then*

$$\deg \mathcal{G}_M = \frac{1}{2}\chi(M).$$

We could then attempt to set[2]

$$\boldsymbol{e}(M, g_{ind}) := \mathcal{G}_M^* \boldsymbol{e}(S^n) = \frac{\chi(S^n)}{A_n} \mathcal{G}_M^* dV_{S^n}. \tag{4.2}$$

This would be a good definition provided $\mathcal{G}_M^* dV_{S^n}$ is a *local quantity intrinsic to* M, i.e. it is an algebraic expression involving only of the Riemann metric tensor g_{ind} and its partial derivatives. Fortunately this is the case.

Theorem 4.2 (Higher dimensional Theorema Egregium). *$\mathcal{G}_M^* dV_{S^n}$ is a local quantity intrinsic to (M, g_{ind}).*

For a proof of this fact we refer to [6, §2.2.4]. In the case $n = 2$ this local intrinsic quantity coincides up to a multiplicative universal constant with the curvature function. In higher dimensions this quantity can be expressed as a polynomial in the *curvature of* M. This last statement needs a bit of explaining.

The key to understanding the curvature is the notion of *parallel transport*. In dimension 2 it was easy to visualize this process. To obtain a higher dimensional description of this process we first need to come up with an alternate 2-dimensional description, one which is better suited for generalization.

Suppose $M^n \hookrightarrow \mathbb{R}^{n+1}$ is an oriented hypersurface with unit (oriented) normal vector field $\vec{\mathfrak{n}}$. Denote by $\langle \bullet, \bullet \rangle$ the Euclidean inner product in $\mathbb{R}^{n+1}$. Choose local coordinates $(u^1, \cdots, u^n)$ in a neighborhood of a point m_0 such that $u^i(m_0) = 0$. In other words we are given a one-to-one smooth map $\vec{r}(u^1, u^2)$ from a neighborhood of the origin in the $(u^1, \cdots, u^n)$-space onto a neighborhood of m_0 in M

$$\vec{r} : \mathbb{R}^n \supset (U, 0) \to (M, m_0).$$

We set $\partial_i = \frac{\partial}{\partial u^i}$ and $g_{ij} := \langle \partial_i \vec{r}, \partial_j \vec{r} \rangle$. Consider a smooth path $\gamma : [0, 1] \to M$, $\gamma(t) = \vec{r}(u^1(t), \cdots, u^n(t))$ such that $\gamma(0) = m_0$. For any tangent vector field $Y(t) = Y^1 \partial_1 \vec{r} + \cdots + Y^n \partial_2 \vec{r}$ along $\gamma(t)$ the derivative $\dot{Y}$ has an orthogonal decomposition into a normal component $\dot{X}^\nu$ and a tangential component. We have the following fundamental result due to Gauss.

Theorem 4.3 (Gauss).

$$\dot{Y}^\tau = \sum_{i=1}^n \dot{Y}^i \partial_i \vec{r} + \sum_{i,j,k=1}^n \Gamma^i_{jk} Y^j \dot{u}^k \partial_i \vec{r}, \tag{4.3}$$

where the quantities Γ^i_{jk} are algebraic expressions in the coefficients g_{ij} and their first order partial derivatives,

$$\Gamma^\ell_{ij} = \frac{1}{2} \sum_{k=1}^n g^{k\ell} \Big(\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik} \Big), \quad \forall i, j, k, \tag{Γ}$$

where $\big(g^{k\ell}\big)_{1 \leq k, \ell \leq n}$ denotes the matrix inverse to $\big(g_{ij}\big)_{1 \leq i,j \leq n}$. In other words, $\dot{Y}^\tau$ is an **intrinsic** *quantity, which can be determined by performing computations only inside the manifold M.*

For a proof we refer to [8, §. 3-2]. The formulæ (Γ) make sense on *any Riemann manifold* and following Levi-Civita's insight we can take (4.3) as the definition of the derivative of a tangent vector field Y along a smooth path γ in an arbitrary Riemann

[2]Note that this definition implies automatically that $\boldsymbol{e}(M) = 0$ if M is odd dimensional since in this case $\chi(S^{\dim M}) = 0$.

manifold. Traditionally one uses the notation $\nabla_{\dot\gamma} Y$, and this operation is referred to as *the Levi-Civita covariant derivative of the vector field* Y *along the path* γ.

Suppose now that $n = 2$ and $\gamma(t)$ is a *geodesic*[3]. Assume $X_0 \in T_{m_0}M$ is a tangent vector such that $X_0 \perp \dot\gamma(0)$ and $|X_0| = 1$. The parallel transport of X_0 along $\gamma(t)$ is the smooth vector field $X(t) \in T_{\gamma(t)}M$ such that

$$X(0) = X_0, \;\; |X(t)| = 1, \;\; X(t) \perp \dot\gamma(t), \;\; \forall t.$$

We have the following fact.

Proposition 4.4. *The vector field* $Y(t)$ *is the parallel transport of* X_0 *along the geodesic* $\gamma(t)$ *iff* $Y(0) = X_0$ *and* $\nabla_{\dot\gamma} Y = 0$.

Suppose now that we are on an arbitrary Riemann manifold (M, g). Choose local coordinates $(u^1, \cdots, u^n)$ near a point m_0 and set $\partial_i := \frac{\partial}{\partial u^i}$. Using the Levi-Civita covariant derivative we can now define an abstract notion of parallel transport along a path. Thus the vector field $Y(t)$ will be parallel along the path γ iff $\nabla_{\dot\gamma} Y = 0$. In this case we say that $Y(t)$ is the parallel transport of $Y(0)$ along γ. We denote by R_{ij} the parallel transport along the perimeter of an infinitesimal parallelogram spanned by the tangent vectors ∂_i, ∂_j at m_0. This is a linear operator

$$R_{ij} : T_{m_0}M \to T_{m_0}M, \;\; R_{ij}\partial_\ell = \sum_k R^k_{\ell ij}\partial_k.$$

This is known as the *Riemann curvature tensor.* The Euler form of a hypersurface provisionally defined by (4.2) can be expressed as an *universal polynomial* in the components of this tensor. This polynomial is called the *pfaffian* (of the curvature). In particular the pfaffian makes sense for *any* Riemann manifold and this the one we choose as our candidate of the Euler form (see [6, Chap.8]) or [7, vol.5, Chap.13] for more details on the pfaffian). In 1944 S.S. Chern succeeded in proving that this guess was right.

Theorem 4.5 (Chern). *The integral of the pfaffian of the Riemann tensor of a closed, oriented, even dimensional Riemann manifold is equal to the Euler characteristic of the manifold.*

For a proof we refer to the original paper [1].

5 The story is about to become even more interesting

It may seem that Chern's theorem closes the book on the Gauss-Bonnet theorem. There is no obvious way to generalize this formula any more. In Chern's formulation is a statement about all the Riemann manifolds. Fortunately, Chern had the remarkable insight that the Gauss-Bonnet formula is not just a statement about a Riemann manifold: it is a statement about an oriented vector bundle (the tangent bundle) together with a special connection on it (the Levi-Civita connection). The shift of emphasis from the manifold to the vector bundle is fundamental. He noticed that the Higher Dimensional Theorema Egregium is only the tip of a massive iceberg.

To understand Chern's revolutionary ideas consider an oriented submanifold $M^n \hookrightarrow \mathbb{R}^N$. Each tangent space T_mM is an oriented n-dimensional subspace of $\mathbb{R}^N$. The family of such oriented subspaces can be structured as a smooth manifold. It is called the *the grassmanian of oriented* n*-panes in* $\mathbb{R}^N$ and it is denoted by $G^+(n, N)$. We obtain in this fashion a map

$$\mathbb{G}_M : M \to G^+(n, N), \;\; m \mapsto T_mM.$$

[3]This is equivalent to the condition that the acceleration vector $\ddot\gamma$ is parallel to the normal vector $\vec{\mathfrak{n}}$.

For example, when M is a surface in $\mathbb{R}^3$, so that $n = 2$, $N = 3$, then any oriented plane in $\mathbb{R}^3$ is uniquely determined by the its unit normal uniquely determined by the condition

$$\text{orientation (normal)} \wedge \text{orientation (plane)} = \text{orientation} (\mathbb{R}^3).$$

Thus $G^+(2, 3)$ can be identified with the family of unit length vectors in $\mathbb{R}^3$, i.e. with the unit round sphere S^2. The map $\mathbb{G}_M$ in this case is an old acquaintance, namely the Gauss map $\mathcal{G}_M$.

The grassmanian $G^+(n, N)$ is equipped with a tautological rank n oriented vector bundle $E_{n,N}$ and by definition

$$\mathbb{G}_M^* E_{n,N} = TM.$$

Note that the embedding $\mathbb{R}^N \hookrightarrow \mathbb{R}^{N+1}$ induces a sequence of embeddings

$$G^+(n, N) \overset{i_N}{\hookrightarrow} G^+(n, N+1) \hookrightarrow G^+(n, N+2) \hookrightarrow \cdots$$

and we denote by $G^+(n, \infty)$ the inductive limit of these spaces. Note that

$$i_N^* E_{n,N+1} = E_{n,N},$$

so that the space $G^+(n, \infty)$ is equipped with a tautological oriented rank n real vector bundle $E_{n,\infty}$. Each of the manifolds $G^+(n, N)$ is equipped with a natural, symmetric Riemann metric. Moreover, the group $SO(N)$ acts in a tautological fashion on this space. This action is transitive and an old result of Elie Cartan states that the cohomology of $G^+(n, N)$ with real coefficients is isomorphic as a vector space with the space of differential forms invariant under this $SO(N)$ action. Fortunately, in his classic work [9], H. Weyl has explained how to compute invariants of groups and thus we can get a very explicit description of the cohomology of this manifold. In particular we get an explicit description of $H^*(G^+(2n, \infty), \mathbb{R})$. It is the quotient of the polynomial ring

$$\mathbb{R}[p_1, \cdots, p_n, \boldsymbol{e}], \ \ \deg \boldsymbol{e} = 2n, \ \ \deg p_k = 4k,$$

modulo the ideal generated by the polynomial $e^2 - p_n$. Gauss-Bonnet theorem for a submanifold in $M^{2n} \hookrightarrow \mathbb{R}^N$ can now be rephrased as a two part statement.

- **Theorema Egregium:** $\mathbb{G}_M^* \boldsymbol{e}$ is an intrinsic invariant of (M, g_{ind}).
- $$\int_M \mathbb{G}_M^* \boldsymbol{e} = \chi(M). \tag{5.1}$$

It is perhaps more illuminating to rephrase the second statement using the Poincaré-Hopf theorem: *the signed count of zeroes of a vector field on a closed oriented manifold is equal to the Euler characteristic of the manifold.*

If X is a vector field with nondegenerate zeroes then we can construct the 0-cycle on M

$$[X^{-1}(0)] = \sum_{X(p)=0} \operatorname{ind}(X, p) \cdot p \in H_0(M, \mathbb{R}),$$

so that if $\epsilon : H_0(M, \mathbb{R}) \to \mathbb{R}$ denotes the natural augmentation map then

$$\epsilon([X^{-1}(0)]) := \sum_{X(p)=0} \operatorname{ind}(X, p) = \chi(M).$$

The equality (5.1) can now be rephrased as follows: *the cohomology class* $\mathbb{G}_M^* \boldsymbol{e}$ *is the Poincaré dual of the homology class carried by the zero set of a generic section of the tangent bundle.*

Suppose now that M is a closed oriented manifold and E is an oriented real vector bundle over M of rank $2n$. One can show that there exists a continuous map $\mathbb{G}_E : M \to G^+(2n, \infty)$ such that

$$E = \mathbb{G}_E^* E_{2n,\infty}.$$

Moreover, any two such Gauss maps are homotopic so that we get a canonic morphism

$$\mathbb{G}_E^* : H^*(G^+(2n, \infty), \mathbb{R}) \to H^*(M, \mathbb{R}).$$

We set

$$\boldsymbol{e}(E) := \mathbb{G}_E^*(\boldsymbol{e}) \in H^{2n} M, \mathbb{R}).$$

This cohomology class is called the *Euler class of E*. The final generalization of the Gauss-Bonnet formula is the following.

Theorem 5.1. *(a) The pfaffian of the curvature of any connection on E is a closed differential form of degree $2n$ whose DeRham cohomology class is $\mathbf{e}(E)$.*

(b) The Euler class of E is the Poincaré dual of the homology class carried by the zero set of a generic section of E.

The Euler class is one very special instance of a more general construction called *characteristic class.*

6 What next?

The notion of characteristic class is key to most developments in geometry and topology during the past fifty years. The facts presented so far had only one modest goal in mind: to awaken your curiosity for a more in depth look at this subject. I strongly believe you cannot call yourself a geometer (or topologist) if you do not have some idea of what these objects can do for you. There are many places where you can read of these things. The classical reference [5] is always a good place to start with.

Chern's original approach, is not as efficient or polished as the one in [5] but contains a tremendous amount of geometrical jewels which are hard to find in any other source. I strongly recommend the original article [2] or the beautiful presentation in [3, §1.5,§3.3] which is closer to Chern's original line of thought. For a differential geometric approach to characteristic classes you can consult [6, Chap. 8].

Theorem 5.1 and the notion of characteristic class do not represent the final word in the Gauss-Bonnet saga. We can still try to seek an extension of this statement to more general spaces, namely spaces whose points are solutions of polynomial equations. These are known as algebraic varieties, and while they look almost everywhere as manifolds, the develop singularities which require special consideration. This direction is still under investigation.

A less obvious yet extremely useful generalization has its origin in an analytic interpretation of the Euler characteristic as the index of an elliptic partial differential operator. From this perspective the Atiyah-Singer index theorem can be regarded as a very broad generalization of the Gauss-Bonnet theorem. The characteristic classes play a key role in the formulation and the proof of the index theorem.

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