COHOMOLOGY AS A LOCAL-TO-GLOBAL BRIDGE

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ABSTRACT. I discuss low dimensional incarnations of cohomology and illustrate how basic cohomological principles lead to a proof of Sperner's lemma.

CONTENTS

1. CHAINS AND COCHAINS WITH $\mathbb{Z}/2$-COEFFICIENTS

To keep the formalism at a minimum I will concentrate only on triangulated spaces of dimension ≤ 2. These are compact spaces equipped with a decomposition as a finite union of points *vertices* (or 0-*simplices*), *edges* (or 1-*simplices*) and triangles (or 2-*simplices*). Two edges can have in common at most one end-point, triangles can have in common only an edge or only a single vertex. An edge and a triangle can have in common either a single vertex, or the edge could be an entire edge of the triangle; see Figure 1.

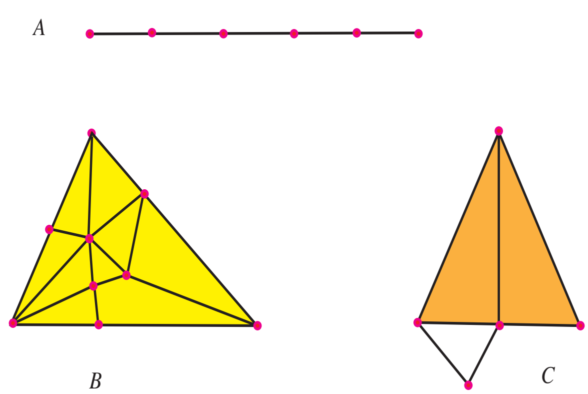

FIGURE 1. *Triangulated spaces.*

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To such a triangulated space X we can associate three finite sets, the set of 0-simplices $S_0(X)$, the set of 1-simplices $S_1(X)$ and the set of 2-simplices. The triangulation is then defined by the sets $S_k(X)$ togheter with a collection of gluing instructions that describe how these simplices are put together. I will use the letter Δ to indicate a triangulation of a space.

For the space A in Figure 1 we see that $S_0(A)$ consists six vertices while $S_1(A)$ consists of five segments. For the space B in Figure 1 we see that $S_0(B)$ consists of nine vertices, $S_1(B)$ consists of 17 edges and $S_2(B)$ consists of nine triangles.

If X is a triangulated space of dimension ≤ 2, and $k = 0, 1, 2$, then a chain of dimension k in X with $\mathbb{Z}/2$ coefficients is a formal sum of the form

$$c = \sum_{\sigma \in S_k(X)} c_\sigma[\sigma], \;\; c_\sigma \in \mathbb{Z}/2.$$

In the above sum it is convenient to think of c as the union of all the simplices σ such that $c_\sigma \neq 0$. The "number" c_σ is called the multiplicity of the simplex σ in the chain c. Thus, a 0-chain could be visualized as a union of vertices, a 1-chain as a union of edges etc.

We can add chains of the same dimension,

$$c = \sum_{\sigma \in S_k(X)} c_\sigma[\sigma], \;\; c' = \sum_{\sigma \in S_k(X)} c'_\sigma[\sigma] \Longrightarrow c + c' = \sum_{\sigma \in S_k(X)} (c_\sigma + c'_\sigma)[\sigma].$$

We also can multiply a chain c by a scalar $\lambda \in \mathbb{Z}/2$,

$$\lambda \cdot \sum_{\sigma \in S_k(X)} c_\sigma[\sigma] = \sum_{\sigma \in S_k(X)} \lambda c_\sigma[\sigma].$$

This shows that the set of k-dimensional chains is a vector space over $\mathbb{Z}/2$. We denote it by $C_k(X)$. Observe that the collection

$$\big\{ [\sigma]; \;\; \sigma \in S_k(X) \big\}$$

is a basis of $C_k(X)$. We will refer to the chains $[\sigma]$, $\sigma \in S_k(X)$, as *basic* chains. For later use we define $C_{-1}(X) = 0$.

There are some remarkable linear operators

$$\partial_k : C_k(X) \to C_{k-1}(X), \;\; k = 0, 1, 2,$$

defined by their action on the bases of $C_k(X)$ as follows.

- $\partial_0 = 0$.
- For any edge $\sigma \in S_1(X)$ we define $\partial[\sigma] \in C_0(X)$ to be the formal sum of vertices of σ
- For any triangle $\sigma \in S_2(X)$ we define $\partial[\sigma]$ to be the formal sum of edges of σ.

These operators are called the *boundary* operators of the triangulated space X. When the various dimensions are clear from the context I will drop the subscript from the notation of the boundary operators, so I will write ∂ instead of ∂_k.

For example, in Figure 2(a) we have

$$\partial\big([\boldsymbol{s}_1] + \cdots + [\boldsymbol{s}_5] \big) = [\boldsymbol{v}_0] + [\boldsymbol{v}_1],$$

while in Figure 2(b) we have

$$\partial\big([\boldsymbol{s}_1] + \cdots + [\boldsymbol{s}_6] \big) = 0.$$

A dual notion is that of *cochain*. If X is a triangulated space of dimension ≤ 2, then a cochain of degree k is a function

$$\alpha : S_k(X) \to \mathbb{Z}/2, \;\; \sigma \mapsto \alpha(\sigma)$$

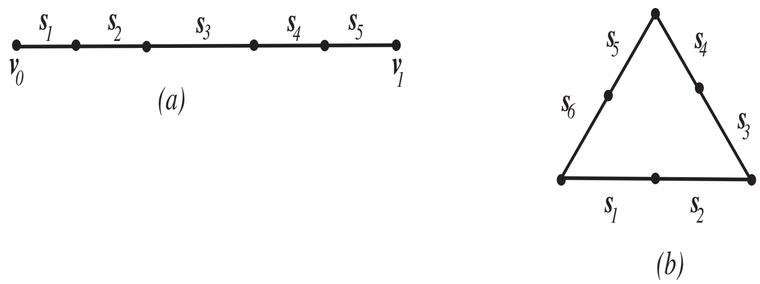

FIGURE 2. *Computing boundaries of chains.*

Thus, a degree 0-cochain is a gadget that associates a "number" in $\mathbb{Z}/2$ to every vertex. A degree 1-cochain is a gadget that associates a number in $\mathbb{Z}/2$ to every edge etc. We define the *support* of a cochain of degree k to be the set

$$\operatorname{supp}\alpha := \big\{\sigma \in S_k(X);\;\; \alpha(\sigma) \neq 0\,\big\}.$$

Thus, the support of a 1-cochain consists of those edges of the triangulation that are assigned nonzero numbers by α.

It is clear that the cochains of degree k form a vector space over $\mathbb{Z}/2$ that we denote by $C^k(X)$. For later usage we set $C^3(X) = 0$.

Let us observe that we have a bilinear map

$$\langle -, - \rangle : C^k(X) \times C_k(X) \to \mathbb{Z}/2,\;\; C^k(X) \times C_k(X) \ni (\alpha, c) \mapsto \langle \alpha, c\rangle \in \mathbb{Z}/2$$

defined as follows. If $\alpha \in C^k(X)$ and $c = \sum_{\sigma\in S_k(X)} c_\sigma[\sigma] \in C_k(X)$, then

$$\langle \alpha, c\rangle = \sum_{\sigma\in S_k(X)} c_\sigma \alpha(\sigma) \in \mathbb{Z}/2.$$

This pairing is called the *Kronecker pairing* and the "number" $\langle \alpha, c\rangle$ can be viewed as the integral of α over c. For this reason we will sometime use the notation

$$\int_c \alpha := \langle \alpha, c\rangle. \tag{I}$$

Just like in the case of chains, there are some remarkable linear operator

$$d_k : C^k(X) \to C^{k+1}(X),\;\; k = 0, 1, 2$$

defined as follows.

- $d_2 = 0$.
- If α is a cochain of degree 1, then $d_1\alpha$ is the degree 2 cochain that associates to each triangle σ the sum of the numbers associated by α to the boundary edges of σ, $d\alpha(\sigma) := \langle \alpha, \partial\sigma\rangle$
- If α is a cochain of degree 0 then $d_0\alpha$ is the degree 1 cochain that associates to each segment σ the sum of the numbers associated by α to the the boundary vertices of σ, $d\alpha(\sigma) := \langle \alpha, \partial\sigma\rangle$.

The operators d_k are called the *coboundary* operators of the triangulated space X. The boundary and coboundary operators are related by an important equality called the *discrete Stokes' formula*

$$\forall k = 0, 1, 2,\;\; c \in C_k(X),\;\; \alpha \in C_{k-1}(X):\;\; \langle d\alpha, c\rangle = \langle \alpha, \partial c\rangle. \tag{1.1}$$

Using the integral notation (I) we can rewrite (1.1) as

$$\int_c d\alpha = \int_{\partial c} \alpha. \tag{1.2}$$

This resembles the classical Stokes formula in multivariable calculus.

The proof of (1.1) is very simple. Invoking the bilinearity of the Kronecker pairing it suffices to prove (1.1) in the special case when c is a basic chain. In this case the equality follows immediately from the definition of d.

2. Application to Sperner Lemma

I want to explain how the above elementary arguments yield a cute proof of the very beautiful Sperner Lemma. I will begin with the 1-dimensional version of Sperner's lemma. This is rather trivial, but I want to give an argument that involves the above tricks and has the added advantage that it extends to higher dimensions.

Suppose the interval $[0, 1]$ is partitioned into n-subintervals

$$0 = x_0 < x_1 < \cdots < x_n = 1, \quad \sigma_i := [x_{i-1}, x_i],$$

and the vertices of these intervals are colored with two colors, 0 and 1. We denote by α_i the color of the vertex x_i. A subinterval σ_i is called *perfect* if its vertices have different colors, $\alpha_{i-1} \neq \alpha_i$. The one dimensional Sperner lemma states that if $\alpha_0 = 0$ and $\alpha_1 = 1$, then the number of perfect subintervals is odd.

Here is a cohomological proof of this fact. I will regard the coloring of the vertices as a degree 0 cochain that associaties to the vertex x_i the number $\alpha_i \bmod 2$. Then $d\alpha$ is the 1-cochain

$$d\alpha(\sigma_i) = \begin{cases} 1, & \text{if } \sigma_i \text{ is perfect} \\ 0, & \text{otherwise.} \end{cases}$$

Denote by c the 1-dimensional cochain defined as the sum of all the segments in the partition, i.e.,

$$c = \sum_{i=1}^{n} [\sigma_i].$$

Then

$$\int_c d\alpha = \sum_{i=1}^{n} d\alpha(\sigma_i) = \text{the number of perfect segments} \bmod 2. \tag{2.1}$$

On the other hand, we have $\partial c = [x_0] + [x_1]$ and thus

$$\int_{\partial c} \alpha = \alpha_0 + \alpha_1 = 1. \tag{2.2}$$

Using Stokes formula (1.1) we deduce

$$\text{the number of perfect segments} \bmod 2 = \alpha_0 + \alpha_1 = 1$$

which proves the 1-dimensional Sperner lemma.

The 2-dimensional version of Sperner's lemma is more complicated. Start with a triangle T whose vertices are labeled $0, 1, 2$. Fix an arbitrary triangulation of T. An *admissible labeling* of the triangulation is a labeling of the vertices of the triangulation with one of the labels $0, 1, 2$ subject to the following rule.

(R) *If a vertex v of the triangulation lies on an edge $\boldsymbol{e}$ of T, then the label of v must be equal to one of the labels of the vertices of $\boldsymbol{e}$; see Figure 3.*

If σ is an edge or a triangle of the triangulation then we define $L(\sigma)$ to be the set consisting of all the labels of its vertices. A triangle τ of the triangulation is called *perfect* if $L(\tau) = \{0, 1, 2\}$.

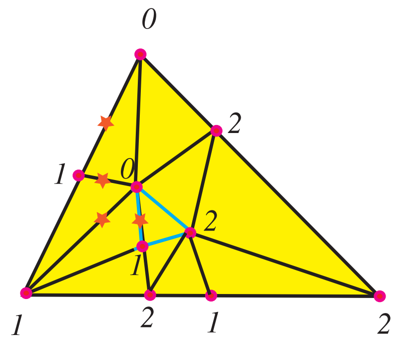

FIGURE 3. *A triangulation of a triangle.*

Theorem 2.1 ($2D$ Sperner Lemma)**.** *For any triangulation Δ_T of a triangle T and any admissible labeling of its vertices the number of perfect triangles is odd. In particular, there exists at least one perfect triangle.*

Proof. The following proof is a cohomological rendition of the elegant approach of D. Cohen [2] which in turn is also a variation of Sperner's original argument [6].

I will denote by $S_0(\Delta_T)$ (respectively $S_1(\Delta_T)$, $S_2(\Delta_T)$) the set of vertices (respectively edges, triangles) of the traingulation Δ_T.

The labeling of the vertices of the triangulation is a map $\ell : S_0(\Delta_T) \to \{0, 1, 2\}$. Using the map ℓ we construct a 1-cochain α such that for any edge $\boldsymbol{s}$ of the triangulation we have

$$\alpha(\boldsymbol{s}) = \begin{cases} 1, & L(\boldsymbol{s}) = \{0, 1\} \\ 0, & \text{otherwise.} \end{cases}$$

We say that an edge $\boldsymbol{s}$ is *special* if it lies in the support of α, i.e., $L\boldsymbol{s} = \{0, 1\} = 1$. For the triangulation depicted in Figure 3 we have indicated the special edges by a star. We can now give a simple description to the coboundary $d\alpha$. For any triangle τ of the triangulation we have

$$d\alpha(\tau) = (\text{the number of special edges of } \tau) \bmod 2.$$

Let us observe that if τ is a perfect triangle then exactly one of the edges of τ is special, while if τ is inperfect then the number of special edges of τ is even. We can rewrite this as follows

$$d\alpha(\tau) = \begin{cases} 1, & \tau \text{ is perfect} \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Let $[\Delta_T] \in C_2(\Delta_T)$ denote the 2-chain defined as the sum of all the triangles of the triangulation

$$[\Delta_T] = \sum_{\tau \in S_2(\Delta_T)} [\tau].$$

The equality (2.3) implies that

$$\int_{[\Delta_T]} d\alpha = (\text{the number of perfect triangles of } \Delta_T) \bmod 2. \tag{2.4}$$

We now want to apply Stokes formula. We can write

$$\partial[\Delta_T] = \sum_{\boldsymbol{s} \in S_1(\Delta_T)} \mu_{\boldsymbol{s}}[\boldsymbol{s}].$$

Form the definition of ∂ we deduce that

$$\mu_{\boldsymbol{s}} = (\text{the number of triangles of } \Delta_T \text{ that contain } \boldsymbol{s} \text{ as an edge}) \bmod 2.$$

This implies that

$$\mu_{\boldsymbol{s}} = \begin{cases} 1, & \boldsymbol{s} \text{ is contained in an edge of } T \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\partial[\Delta_T] = b_T = \text{the sum of the edges of } \Delta_T \text{ contained in an edge of } T. \tag{2.5}$$

Now observe that

$$\int_{\partial[\Delta_T]} \alpha = (\text{the number of special edges of } \Delta_T \text{ contained in an edge of } T) \bmod 2. \tag{2.6}$$

The special edges of Δ_T that lie on the boundary of T must be contained in the edge $[0, 1]$ of T. From the 1-dimensional case of Sperner's lemma we deduce that this number is even. The 2-dimensional case now follows from (2.4), (2.6) and Stokes formula. □

Remark 2.2. Theorem 2.1 was first proved by E. Sperner [6] in 1928 when he was 23. His proof was also based on a parity argument. Moreover, he showed that Theorem 2.1 implies Brower's fixed point theorem which states that any continuous function $T \to T$ has at least one fixed point. We refer to [5, §2.3] for more details and higher dimensional generalizations. □

3. WHERE IS THE COHOMOLOGY?

If X is triangulated space of dimension ≤ 2 then we can form the $\mathbb{Z}/2$-vector spaces $C^k(X)$ and the linear operators

$$d_k : C^k(X) \to C^{k-1}(X).$$

Let us first remark that the above terminology is a bit sloppy. The above vector spaces depend on a choice of an additional structure on X, namely a triangulation Δ. To emphasize the dependence on the triangulation Δ we ought to denote these spaces by $C^k(X, \Delta)$.

Next let us observe the definition of the boundary operators implies immediately that for any k the composition

$$C^{k-1}(X, \Delta) \xrightarrow{d_{k-1}} C^k(X, \Delta) \xrightarrow{d_k} C^{k-1}(X, \Delta)$$

is trivial, i.e., $d_k \circ d_{k-1} = 0$. We set

$$Z^k(X, \Delta) = \ker d_k \subset C^k(X, \Delta), \quad B^k = \operatorname{range} d_{k-1} \subset C^k(X, \Delta).$$

Since $d_k \circ d_{k+1} = 0$ we deduce that

$$B^k(X, \Delta) \subset Z_k(X, \Delta).$$

The elements of Z^k are called *cocycles*, while the elements of B^k are called *coboundaries*. We can then form the quotient

$$H^k(X, \Delta) := \frac{Z^k(X, \Delta)}{B^k(X, \Delta)}.$$

This is a finite dimensional vector space over $\mathbb{Z}/2$. Any cocycle $\alpha \in Z^k(X, \Delta)$ determines an element in $H^1(X, \Delta)$ called the *cohomology class* of α and denoted by $[\alpha]$. Note that $[\alpha] = 0$ if and only if there exists $\beta \in C^{k-1}(X, \Delta)$ such that $d\beta = \alpha$.

A priori, the group $H^k(X, \Delta)$ may seem to depend on the choice of the triangulation Δ. A rather deep theorem [4, Thm. 2.27, Cor. 3.4] shows that this is not the case. In other words, the isomorphism class of the group $H^k(X, \Delta)$ is independent of the triangulation Δ. It is called the k-th *cohomology* *group* of X with $\mathbb{Z}/2$-coefficients, and it is denoted by $H^k(X, \mathbb{Z}/2)$. The dimension of this vector space is a topological invariant called the k-th *Betti number* of X with $\mathbb{Z}/2$-coefficients and it is denoted by $b_k(X)$.

It is very easy to compute the cohomology of a triangle T using the obvious triangulation consisting of a single triangle, T itself, the three edges of T and the three vertices of T. In this case we deduce

$$H^k(T, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & k = 0 \\ 0, & k > 0. \end{cases} \tag{3.1}$$

Suppose now that T is a triangle with vertices labeled $0, 1, 2$. Fix a triangulation Δ_T of T and an admissible labeling.

We want to explain why the equality (3.1) is responsible for the existence of perfect triangles. We argue by contradiction and we assume that there exist no perfect triangles.

We denote by C the boundary of T. The triangulation of T induces a triangulation Δ_C of C. Observe that $S_k(\Delta_C) \subset S_k(\Delta_T)$ and thus we have natural restriction maps

$$r_k : C^k(\Delta_T) \to C^k(\Delta_C).$$

Consider again the cochain defined in the proof of Theorem 2.1. In Figure 4 we have indicated by stars the special edges $\boldsymbol{s}$.

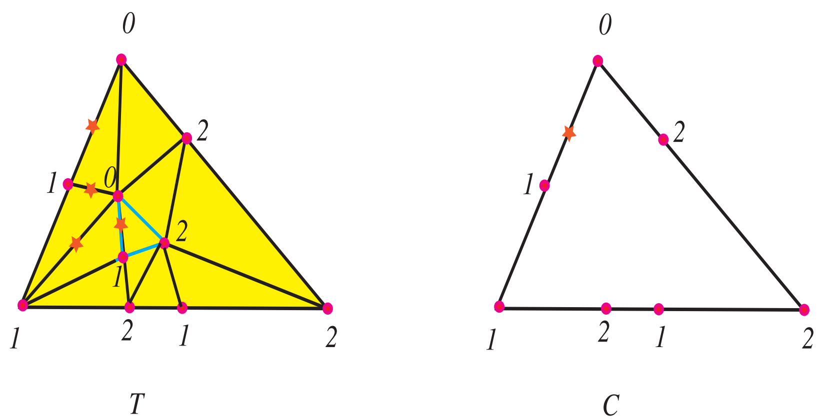

FIGURE 4. *A triangulation of a triangle, and the induced triangulation on the boundary.*

The equality (2.3) implies that if there exist no perfect triangles then $d\alpha = 0$. *At this point we want to invoke the equality* $H^1(T) = 0$ which implies that there exists a 0-cycle $\beta \in C^0(\Delta_T)$ such that

$$d\beta = \alpha.$$

If we denote by α_C and β_C the restrictions of α and respectively β to the triangulation Δ_C we deduce that

$$d\beta_C = \alpha_C \;\text{ in }\; C^1(\Delta_C).$$

This shows that the cohomology class $[\alpha_C] \in H^2(C)$ is trivial.

Now let us observe that the 1-dimensional Sperner lemma implies that support of α_C consists of an odd number of edges. This implies that

$$\int_{b_T} \alpha_C = 1,$$

where $b_T \in C_1(\Delta_C)$ is the 1-chain defined in (2.5). Note that $\partial b_T = 0$. Invoking Stokes formula we deduce
$$1 = \int_{b_T} \alpha_C = \int_{b_T} d\beta_C = \int_{\partial b_T} \beta_C = 0.$$
We have reached a contradiction!

Note that the above proof shows another interesting fact. If we let $\alpha \in C^1(\Delta_C)$ be the 1-cochain such that for any edge $\boldsymbol{s}$ we have
$$\alpha(\boldsymbol{s}) = \begin{cases} 1, & L(\boldsymbol{s}) = \{0,1\} \\ 0, & \text{otherwise}, \end{cases}$$
then α is a coycle, $d\alpha = 0$ but it is not a coboundary since $\int_{b_T} \alpha \neq 0$. This proves that $H^1(C) \neq 0$. More precisely, $H^1(C) \cong \mathbb{Z}/2$ and the cohomology class of α is a generator of this group.

4. WHAT NEXT?

The above arguments may seem like a bit of accidental magic. In fact, they are very special manifestations of a very general technology called *cohomology theory*. This has varied incarnations in topology, geometry, number theory, algebra, but all are governed by a core set of principles. In one form or another, the cohomology theory attempts to collect local data and produce meaningful global information.

Where can you learn more about these things? Everyone has his/her own favorites. I have two. The first is the classical book by Bott and Tu [1]. It takes you from humble beginnings to glorious heights while making sure that the geometric intuition is as in your face as possible.

The other one is the survey [3]. This is not for beginners and assumes a good familiarity with at least one incarnation of cohomology. It takes a panoramic view of the subject and provides a very illuminating look at the structure cohomology theory that is present in all its incarnations.

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