Chern classes of singular algebraic varieties

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Abstract

I am trying to understand the work MacPherson.

Contents

Notations

• $\mathbf{i} := \sqrt{-1}$.
• $\mathbb{N} := \mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z};\ \ n \geq 0\}$.
• For any set S we denote by $\mathbb{I}_S$ the identity map $S \to S$. For any subset $A \subset S$ we denote by $\mathbb{1}_A$ the characteristic function of A.

*Notes for myself and whoever else is reading this footnote.

1 Whitney stratifications of analytic spaces

§1.1 Regularity conditions The Whitney stratified spaces are topological spaces together with a partition $\mathcal{S} = \{S_i;\quad i \in I\}$ into locally closed subsets called *strata* which are (open) smooth manifolds and "interact" in a special way. The exact meaning of this interaction is specified by the *Whitney regularity conditions*

Definition 1.1. *Suppose X, Y are disjoint smooth submanifolds in the Euclidean space and $x \in \bar{Y} \cap X$. The triple (Y, X, x) is said to be* Whitney regular *(or that Y is W-regular over X at x) if given a sequence $(x_n, y_n) \in X \times Y$ such that $(x_n, y_n) \to (x, x)$ and the unit radial vector $\vec{v}_n = \frac{1}{|y_n - x_n|}(y_n - x_n)$ pointing from x_n to y_n converges to $\vec{v}$ and $T_{y_n}Y$ converges to T then*

$$\vec{v} \in T.$$

In particular

$$\lim \measuredangle(x_n y_n, T_{y_n}Y) = 0 \mod \pi.$$

Let (Y, X, x) as in the above definition and denote by π_X-the nearest point projection onto X, and by $\mathfrak{h}(y)$ the line $y\pi_X(y)$. We present an equivalent characterization of the W-regularity.

Proposition 1.2. *The manifold Y is W-regular over X at x if an only if the following two conditions hold.*

A. Given a sequence of points y_n in Y such that $y_n \to x$ and $T_{y_n}Y \to T$ then $T_xX \subset T$.

B. If y_n is a sequence in Y such that $y_n \to x$, $T_{y_n}Y \to T$ and the line $\mathfrak{h}(y_n) \to \mathfrak{h}$ then $\mathfrak{h} \subset T$.

For a proof of this proposition we refer to [6]. As explained in [6] we can avoid sequences altogether in (A) and (B). To get a feeling of the meaning of these regularity conditions we include below a few geometric consequences.

Proposition 1.3. *Suppose X, Y are open submanifolds of the Euclidean space E and $x \in X \cap \bar{Y}$. Let ρ_X denote the distance-from-X function.*

(i) If (Y, X, x) satisfies the condition A in Proposition 1.2 then there exists a neighborhood U of $x \in E$ such that $\pi_X\, |_{Y \cap U}$ is a submersion.

(ii) $B \Longrightarrow A$, i.e. if (Y, X, x) is B-regular then it is also A-regular.

(iii) If (Y, X, x) satisfies the regularity condition W then x has a neighborhood U in E such that the map

$$(\pi_X \times \rho_X)\, |_{U \cap Y} \to X \times (0, \infty)$$

is a submersion.

Example 1.4. Let us give some examples of situations when the regularity conditions are satisfied. Consider first the example depicted in Figure 1(a). Denote by G the open *grey* rectangle, $B_1, \cdots, B_4$ denote the open *black* edges, and $R_1, \cdots, R_4$ the *red* vertices. Observe that (G, B_i, p) is W-regular for any $p \in B_i$. Also (G, R_i, R_i) is W-regular.

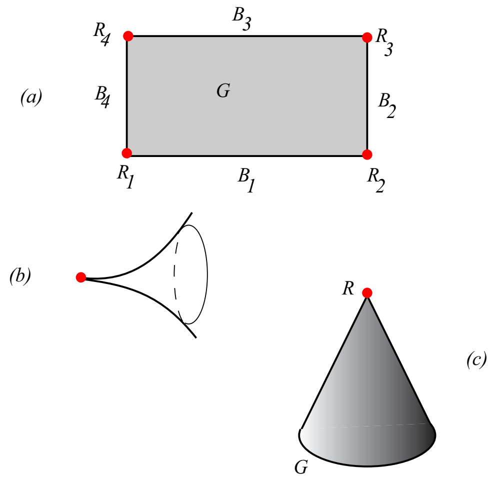

Figure 1: *A manifold with corners, a cusp $y^2 = x^3$ and an Euclidean cone.*

In Figure 1(b) we have depicted the real part of the cubic cusp, i.e the complex plane curve described by the equation

$$C := \{y^2 = x^3;\ \ (x, y) \in \mathbb{C}^2\}.$$

Topologically C is a cone over a circle, more precisely the trefoil knot in $S^3 \subset \mathbb{C}^2$. If We denote by C^0 the smooth part of this curve. The origin O is a singular point of the curve. We want to show that $C^0, O, O)$ is B-regular which in this case means that if p_n is a sequence of points on C^0 converging to O such that $T_{p_n}C \to T$ and $Op_n \to \ell$ then $\ell \subset T$. Set $P(x, y) = x^3 - y^2$ so that $\nabla P = (3x^2, 2y)$ If we write $p_n = (t_n^2, t_n^3) = t_n^2(1, t_n)$ then

$$\nabla P(p_n) = (3t_n^4, -2t_n^3) = t_n^3(3t_n, -2).$$

As $t_n \to 0$ the line Op_n converges to the line Op_0, $p_0 = (1, 0)$ which is the x-axis. Moreover $T_{p_n}C$ converges to the line given by the equation $-2y = 0$ which is the axis so that $\ell = T$.

In Figure 1(c) we have an Euclidean cone over a circle. We denote by G the complement of the vertex R. It is easy to see that (G, R, R) is W-regular. There is however a major difference between the cone in (c) and the topological cone in (b). Although both spaces are homeomorphic, they are not diffeomeorphic[1]. Note that in the case (b) the tangent spaces $T_{p_n}C$ converged to a unique position, the x-axis. For the Euclidean cone in (c) tangent spaces could converge to several limiting positions. It is easy to see that in this case the set of limiting positions can be identified with the circle the cone is based on.

Example 1.5. It is perhaps instructive to give examples when some of these regularity conditions fail.

① In Figure 2 we have depicted the real part of the *Whitney umbrella*, that is the singular complex hypersurface W in $\mathbb{C}^3$ defined by the equation

$$w(x, y, z) = x^2 - zy^2 = 0.$$

[1] whatever that means.

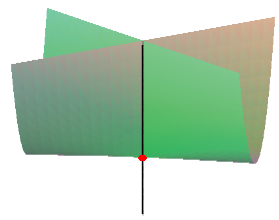

Figure 2: *Whitney umbrella* $x^2 = zy^2$.

This surface contains the origin O (marked in red on Figure 2), and two lines, the y-axis and the z-axis. W is singular along the z-axis (depicted in black). Let X denote the z-axis and Y the complement of X in W. We claim that Y is not A-regular over X at O.

Along the y-axis we have line we have

$$\nabla w = (2x, -2zy, -y^2) = (0, 0, -y^2).$$

If we choose a sequence of points $p_n \to 0$ along the y-axis then we see $T_{p_n}Y$ converges to the plane

$$T = \{z = 0\} \not\supseteq T_O X.$$

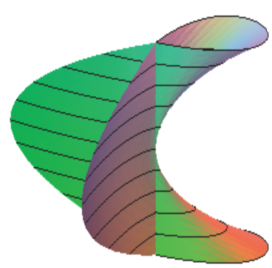

Figure 3: *Whitney cusp* $y^2 + x^3 - z^2x^2 = 0$.

② Consider the *Whitney cusp*, that is the hypersurface U of $\mathbb{C}^3$ described by the equation

$$f(x, y, z) = y^2 + x^3 - z^2x^2 = 0.$$

It is easy to construct a normalization of this surface. It is given by the map[2]

$$u : \mathbb{C}^2 \to U, \ \ (s,t) \mapsto (s^2 - t^2, s^2 t - t^3, s) \in U.$$

Using this normalization we can use the `plot3d` procedure in $MAPLE$ to generate the image of the real part of this surface depicted in Figure 3.

The vertical line visible in Figure 3 is the z-axis. Clearly the Whitney cusp is singular along this line. The surface has a "saddle" at the origin. Denote by X the z-axis, and by Y its complement. We claim that (Y, X, O) is A-regular, but not B-regular. We have to show that

$$\frac{|\partial_z f(p)|}{|\nabla f(p)|} \to 0 \quad \text{if } p = (x,y,z) \to O \text{ along } U. \tag{1.1}$$

Observe that

$$\nabla f = (3x^2 - 2xz^2, 2y, 2zx^2)$$

Obviously (1.1) holds for all sequences $p_n = (x_n, y_n, z_n) \in U$ such that $z_n = 0$, $\forall n \gg 1$. If $(x,y,z) \to 0$ along $U \cap \{z \neq 0\}$ then $y^2 = x^2(z^2 - x)$

$$|\nabla f|^2 = 4|x^2 z|^2 + 4|y|^2 + |3x^2 - 2xz^2|^2 = 4|x^2 z|^2 + 4|x|^2|z^2 - x|^2 + |x|^2|3x - 2z^2|^2$$

Then

$$\frac{|\nabla f|^2}{|\partial_z f|^2} = 1 + \frac{4|x|^2|z^2 - x|^2}{4|x^2 z|^2} + \frac{|x|^2|3x - 2z^2|^2}{4|x^2 z|^2}$$

$$= 1 + \frac{1}{4}\left|\frac{z^2 - x}{xz}\right|^2 + \frac{|x|^2|3x - 2z^2|^2}{4|x^2 z|^2} = 1 + \frac{1}{4}\left|\frac{z}{x} - \frac{1}{z}\right|^2 + \left|\frac{z}{x} - \frac{3}{2z}\right|^2 \stackrel{(x,z)\to 0}{\longrightarrow} \infty.$$

To show that condition B is violated at O we need to find a sequence

$$U \ni p_n = (x_n, y_n, z_n) \to 0$$

such that

$$T_{p_n} U \to T, \ \lim_{n\to\infty} \mathfrak{h}(p_n) = \mathfrak{h}, \ \text{ and } \ \mathfrak{h} \not\subset T. \tag{1.2}$$

The line is the line spanned by the vector $(x_n, y_n, 0)$. Thus we need to find a sequence p_n such that $\frac{1}{|\nabla f(p_n)|}\nabla f(p_n)$ is convergent and

$$\lim_{n\to\infty} \frac{x_n \partial_x f(p_n) + y_n \partial_y(p_n)}{|\nabla f(p_n)| \cdot \sqrt{|x_n|^2 + |y_n|^2}} \neq 0.$$

We will seek such sequences along paths in U which end up at O. Look at the parabola

$$C = \{y = 0\} \cap U = \{x = z^2; \ \ y = 0, \ \ y \neq 0\} = \{(z^2, 0, z); \ \ z \neq 0\} \subset U.$$

Along C line $\mathfrak{h}(z^2, 0, 0)$ is the line generated by the vector $\vec{e}_1 = (1,0,0)$ and we have

$$\nabla f = (z^4, 0, z^5) \Longrightarrow |\nabla f| = |z|^4(1 + O(|z|))$$

We conclude that along this parabola the tangent plane $T_p U$ converges to the plane perpendicular to $\vec{e}_1$ which shows that the B-conditions is violated by the sequence converging to zero along C.

$\square$

[2]We found this using the procedure `normal` of $SINGULAR$, [2].

§1.2 Whitney stratifications: existence.

Definition 1.6 (J. Mather, [5]). *A* prestratification *of a topological space X is a partition $\mathcal{P}$ into subsets called* strata *satisfying the following three conditions.*
(a) Each stratum U is a locally enclosed set, i.e. it is the intersection of an open set and a closed set. (Equivalently, U is locally closed if each point $u \in U$ has a neighborhood N such that $U \cap N$ is closed in N.

(b) $\mathcal{P}$ is locally finite*, i.e. every point $x \in X$ has a neighborhood which intersects only finitely many strata.*
(c) (Frontier axiom) If U and V are strata and $\bar{U} \cap V \neq \emptyset$ then $V \subset \bar{U}$. We denote by $V < U$ the relation defined by $V \subsetneq \bar{U}$

Definition 1.7 (J. Mather, [5]). *A* stratification *of X is a rule which assigns to each $x \in X$ a germ $\mathcal{S}_x$ at x of a closed subset of X with the following property:*

For each $x \in X$, there exists a neighborhood N of x and a prestratification $\mathcal{P}$ of N such that for any $y \in N$, $\mathcal{S}_y$ is the germ of y of the stratum of $\mathcal{P}$ containing y.

A prestratification $\mathcal{P}$ defines a stratification by associating to each $x \in X$ the germ at x of the stratum of $\mathcal{P}$ containing x. Two prestratifications $\mathcal{P}$, $\mathcal{P}'$ are called *equivalent* if they define the same stratification. We denote this equivalence relation by $\mathcal{P} \sim \mathcal{P}'$.

Any prestratification $\mathcal{P}$ of X defines a function

$$\mathrm{depth}_{\mathcal{P}} : X \to \mathbb{N}$$

where for each $x \in X$ $\mathrm{depth}_{\mathcal{P}}(x)$ is the largest integer $k \geq 0$ such that there exist strata $U_0, U_1, \cdots, U_k$ with the property

$$x \in U_0 < U_1 < \cdots < U_k.$$

Let us observe that

$$\mathcal{P} \sim \mathcal{P}' \Longrightarrow \mathrm{depth}_{\mathcal{P}} = \mathrm{depth}_{\mathcal{P}'}.$$

Using this fact we can associate a depth function to every stratification $\mathcal{S}$ as follows. For any $x \in X$ we define

$$\mathrm{depth}_{\mathcal{S}}(x) := \mathrm{depth}_{\mathcal{P}}(x),$$

where $\mathcal{P}$ is a prestratification in a neighborhood N of x as in Definition 1.7.

Any stratification $\mathcal{S}$ is associated to a natural prestratification of X defined by

$$\mathcal{P} = (U_k)_{k \geq 0}, \;\; U_k = \{x; \;\; \mathrm{depth}_{\mathcal{S}}(x) = k\}.$$

Given two stratified spaces $(X_i, \mathcal{S}_i)_{i=0,1}$ we can form in a natural way a stratification $\mathcal{S}_0 \times g S_1$ on the product $X_0 \times X_1$ and we write

$$(X_0 \times X_1, \mathcal{S}_0 \times \mathcal{S}_1) = (X_0, \mathcal{S}_1) \times (X_1, \mathcal{S}_1).$$

Given a stratified space $(X, \mathcal{S})$, any continuous map $f : Y \to X$ induces a natural stratification $f^*\mathcal{S}$ on Y defined by

$$(f^*\mathcal{S})_y = \text{the germ at } y \text{ of } f^{-1}(\mathcal{S}_{f(y)}).$$

Given two stratifications $\mathcal{S}_0, \mathcal{S}_1$ on X we denote by $\mathcal{S}_0 \cap \mathcal{S}_1$ the pullback of $\mathcal{S}_0 \times \mathcal{S}_1$ via the diagonal map $X \to X \times X$.

For every topological space X we denote by $\mathcal{T}_X$ the tautological stratification consisting of single stratum, X.

Definition 1.8. *Suppose X is a smooth manifold, and $Y \subset X$.*

(a) A prestratification $\mathcal{P}$ of Y is called Whitney *(or* regular*) if each stratum is a smooth submanifold and for any two strata $V < U$ the pair (U, V) is W-regular i.e. (U, V, v) is W-regular for every $v \in V$.*

(b) A stratification $\mathcal{S}$ of Y is called Whitney if it is the stratification associated to a Whitney pre-stratification.

To proceed further we need to introduce some special categories of subsets of analytic varieties.

Definition 1.9. *If X is a nonsingular complex analytic (or algebraic) variety. A subset of X is called* constructible *if it can be written as the set theoretic difference of two closed complex analytic (resp. algebraic) subsets. We denote by $\mathcal{C}(X)$ the collection of constructible subsets.*

Remark 1.10. The collection $\mathcal{C}(X)$ is the Boolean subalgebra $\mathcal{C}(X) \subset 2^X$ generated by the closed complex algebraic subsets of X. □

Definition 1.11. *Suppose U and V are two constructible subsets of the complex analytic manifold. For $\epsilon \in \{A, B, W\}$ we set*

$$S_\epsilon(U, V) := \Big\{ v \in V \cap \bar{U};\ \ (U, V, v) \quad \textit{is not } \epsilon\textit{-regular} \Big\}.$$

We have the following fundamental result.

Theorem 1.12 (Whitney, [7]). *Suppose U and V are two constructible smooth subsets of the complex manifold X. Assume $\dim V < \dim U$ and $V \subset \bar{U}$. The for every $\epsilon \in \{A, B, W\}$ the set $S_\epsilon(U, V)$ is*

(i) constructible

(ii) of dimension $< \dim V$.

Sketch of proof. We follow closely the approach in [6]. Set $m = \dim V$, $n = \dim U$, $N = \dim X$. Denote by $\mathbb{B}_\Delta X$ the blowup of $X \times X$ along the diagonal. We have a natural projection $\pi : \mathbb{B}_\Delta X \to X \times X$, and by E_Δ the exceptional divisor $\pi^{-1}(\Delta)$. We have a diagram

$$\begin{tikzcd}
 & E_\Delta \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{B}_\Delta X \arrow[d, "\pi"] & & \\
 & \Delta \arrow[r, hook] & X \times X \arrow[dl, "l"] \arrow[dr, "r"] & & \\
U \arrow[r, hook] & X & & X & V \arrow[l, hook']
\end{tikzcd}.$$

We set $\mathbb{T}_l := (l \circ \pi)^* TX \to \mathbb{B}_\Delta X$. We denote by $G_n(\mathbb{T}_l) \to \mathbb{B}_\Delta X$ the Grassmanian of n-dimensional subspaces in $\mathbb{T}_l$. For each $x \in X$ we denote by $G_n(\mathbb{T}_l)_x$ the total space of the restriction of $G_n(\mathbb{T}_l)$ over $\pi^{-1}(x,x) \cong \mathbb{P}(T_x X)$. Note that

$$G_n(\mathbb{T}_l)_x \cong \mathbb{P}(T_x X) \times G_n(T_x X).$$

Consider the Gauss map

$$\gamma : U \times V \setminus \Delta \to G_n(\mathbb{T}_l), \ \ (u,v) \mapsto (u,v,T_u U) \in G_n(\mathbb{T}_l).$$

The triple (U,V,v) is W regular iff the intersection of the closure of $\Gamma(U \times V \setminus \Delta)$ with $G_n(\mathbb{T}_l)_v$ lies inside the locus

$$\mathcal{R}_v := \Big\{ (\ell, T) \in \mathbb{P}(T_v X) \times G_n(T_v X), \ \ \ell \subset T \Big\}.$$

The locus

$$\mathcal{R} = \bigcup_{v \in V} \mathcal{R}_v$$

is constructible since V is so. The closure of $\gamma(U \times V \setminus \Delta)$ is constructible and in particular

$$\mathcal{Z} := \mathcal{R} \setminus \overline{\gamma(U \times V \setminus \Delta)}$$

is constructible. The projection $G_n(\mathbb{T}_l)|_{E_\Delta} \to \Delta$ has projective fibers so that the projection of $\mathcal{Z}$ onto $V \times V \cap \Delta$ is constructible by a theorem of Łojasewicz, [3]. The "bad" locus $S_W(U,V)$ coincides with this projection of $\mathcal{Z}$. This proves (i).

The proof of (ii) is considerably more complicated and it is based on the following generalization of the curve selection lemma. For a proof we refer to [6, 7].

Theorem 1.13 (Wing Lemma, H. Whitney[7]). *Suppose M is a complex submanifold of the complex variety V in a complex vector space E such that $\dim_{\mathbb{C}} M < \dim_{\mathbb{C}} V$. Fix a complex subvariety $V' \subset V$ such that $\dim_{\mathbb{C}} V' < \dim_{\mathbb{C}} V$. Then every $p \in M \setminus V'$ has an open neighborhood U in M such that there exists a real analytic embedding*

$$w : U \times [0,1) \to E$$

such that

$$w|_{U \times 0} = \mathbb{I}_U, \ \ w(U \times (0,1) \subset V - (M \cup V')$$

Moreover w is holomorphic in the U-directions.

Remark 1.14. When M is a point the Wing Lemma specializes to the well known curve selection lemma.

□

From Theorem 1.12 one can deduce easily the following important result.

Theorem 1.15 (Whitney, [7]). *If X is a complex subvariety of a smooth variety M then M admits a Whitney stratification such that X is a finite union of strata.*

§1.3 Whitney stratifications: local structure. It is time to answer one fundamental question. *What is the main point of all the above constructions?* Why did we have to go through all this trouble to construct partitions of complex varieties into complex *manifolds* satisfying Whitney regularity conditions when we could have achieved this by paying a less costly technical price. What makes Whitney stratifications better than other stratifications?

Loosely speaking, the Whitney stratification have a much nicer structure in the directions transversal to strata: the points in a connected component of a Whitney stratum "all look the same". To explain what we meant that two points look the same we consider again the Whitney umbrella W in Example 1.5, Figure 2. This surface has one obvious stratification.

- A 1-dimensional stratum Z consisting of the z-axis. To describe the local structure of a Whitney stratification we need to describe with great care the concept of tubular neighborhood.
- A (disconnected) 2-dimensional stratum W^0 defined as the complement of Z in W.

For every $z \in Z$, and $0 < \varepsilon \ll 1$ we denote by $NW_\varepsilon(z)$ the intersection of W with the closed ball of radius ε centered at z. The topological type of $W_\varepsilon(z)$ depends on the position of z on the z axis. As z varies for $-\infty$ to ∞ this topological type changes as z crosses the origin. That was precisely the point where the Whitney regularity conditions were violated. A similar phenomenon takes place with the Whytney cusp.

In this subsection we will show that Whitney stratifications have the desired local homogeneity condition: two nearby points on the same stratum have the same local structure.

Definition 1.16 ([4, 5]). *Suppose U is a complex submanifold of the complex manifold X. A* tubular neighborhood *of $U \hookrightarrow X$ is a quadruple $T = (\pi, E, \epsilon, \phi)$ where $E \to U$ is a hermitian vector bundle, $\epsilon : U \to (0, \infty)$ is a smooth function, and if we set*

$$B_\epsilon := \big\{ (v, x) \in E;\ \ \|v\|_x < \epsilon(x) \big\}$$

then ϕ is a diffeomorphism $B_\epsilon \to X$ onto an open subset of X such that the diagram below is commutative.

$$\begin{tikzcd} B_\epsilon \arrow[dr, "\phi"] \\ U \arrow[u, shift left, "\zeta"] \arrow[r, hook] & X \end{tikzcd}, \quad \zeta = \textit{zero section.}$$

We set $|T| := \phi(B_\epsilon)$.

Given a tubular neighborhood $T = (\pi, E, \epsilon, \phi)$ we get a natural projection

$$\pi_T : |T| \to U.$$

Moreover the function $\rho(v, x) = \|v\|_x^2$ induces a smooth function $\rho_T : |T| \to U$. We say that ρ_T is *projection* and ρ_T is the *tubular function* associated to the tubular neighborhood T. We get a submersion

$$(\pi_T, \rho_T) : |T| \setminus U \to U \times \mathbb{R}.$$

The restriction of a tubular neighborhood of U to an *open* subset of U is defined in an obvious fashion.

Definition 1.17. *Suppose that T is a tubular neighborhood of $U \hookrightarrow X$ and $f : X \to Y$ is a map. We say that f is compatible with T if the restriction of f to $|T|$ is constant along the fibers of π_T i.e. the diagram below is commutative*

$$\begin{tikzcd} |T| \arrow[d, two heads, "\pi_T"'] \arrow[dr, "f"] & \\ U \arrow[r, "f"] & Y \end{tikzcd}$$

Theorem 1.18 (Tubular Neighborhood Theorem). *Suppose $f : X \to Y$ is a smooth map between smooth manifolds and $U \hookrightarrow X$ a smooth submanifold of X such that $f|_U$ is a submersion. Suppose*

$$W \hookrightarrow V \hookrightarrow U$$

are open subsets such that the closure of W in U lies inside V, and T_0 is a smooth tubular neighborhood of $V \hookrightarrow X$ which is compatible with f.

Then there exists a tubular neighborhood T of $U \hookrightarrow X$ which is

(i) compatible with f and
(ii) $T\,|_W \subset T_0\,|_W$.

Suppose now that X is a smooth manifold and $\mathcal{P}$ is a Whitney prestratification of a subset in X. Assume that for every stratum $U \in \mathcal{P}$ we are given a tubular neighborhood T_U of $U \hookrightarrow X$. We denote by π_U (resp. ρ_U) the projection (resp. the tubular function) associated to T_U. For any stratum $V < U$ we distinguish two commutativity relations.

$$\pi_V \circ \pi_U(x) = \pi_V(x), \;\; \forall x \in |T_U| \cap |T_V| \cap \pi_U^{-1}(|T_V| \cap U). \tag{C_π}$$

$$\rho_V \circ \pi_U(x) = \rho_V(x), \;\; \forall x \in |T_U| \cap |T_V| \cap \pi_U^{-1}(|T_V| \cap U). \tag{C_ρ}$$

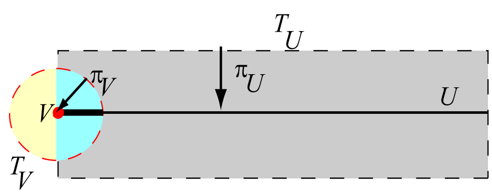

Figure 4: *Non-compatible tubular neighborhoods*

Example 1.19. In Figure 4 the thick black segment is $|T_V| \cap U$ while the blue area is $|T_U| \cap |T_V| \cap \pi_U^{-1}(|T_V| \cap U)$ and we see that (C_π) is satisfied. The condition (C_ρ) signifies that for any point p in the blue area the distance to the point V is equal to the distance from V to the projection of the point p to the half-line U. Clearly the condition (C_ρ) is violated. The tubular neighborhoods in Figure 5 are compatible, i.e. both commutativity relations are satisfied.

□

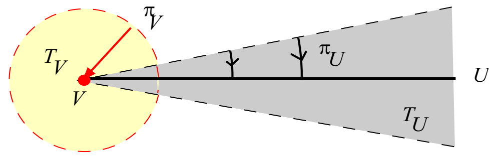

Figure 5: *Compatible tubular neighborhoods*

Definition 1.20 (Controlled stratifications). *Suppose $\mathcal{P}$ is a Whitney prestratification of subset X in an Euclidean space E. A collection of tubular neighborhoods (T_U), one tubular neighborhood for each stratum, is called* controlled *if for any strata $V < U$ the corresponding projections and tubular functions satisfy the commutativity conditions (C_π) and (C_ρ).*

A controlled Whitney prestratification *of X is a pair $(\mathcal{P}, \mathcal{T})$ consisting of a Whitney prestratification of X and a controlled system of tubular neighborhoods.*

Example 1.21. Consider the quadrant $Q = \{(x, y, 0) \in \mathbb{R}^3;\ \ x, y \geq 0\}$ depicted in Figure 6. It has a stratification consisting of 4 strata: the interior of the quadrant, the two half axes

$$V_1 = \{(x, 0, 0);\ \ x > 0\},\ \ V_2 = \{(0, y, 0);\ \ y > 0\}$$

and the origin W. This is a Whitney stratification and we have

$$W < V_1 < U,\ \ W < V_2 < U$$

We seek smooth functions $\rho_W, \rho_{V_1}, \rho_{V_2}, \rho_U$ defined in an open neighborhood of the quadrant in $\mathbb{R}^3$ such that

$$\bar{S} = \{\rho_S = 0\} \cap Q,\ \ \forall S = \{W, V_1, V_2, U\} = \mathcal{P}.$$

For each stratum S we set $|T_S| = \{\rho_S < r_S \ll 1\}$. If the level sets of these functions intersect transversally in the overlaps of these tubular neighborhoods then it is easy to construct controlled tubular neighborhoods.

The projection π_W has only one possible definition, the constant map. The overlap $|T_W| \cap |T_{V_1}|$ will be a tubular neighborhood of a portion of V_1 and here we define

$$\pi_{V_1}(p) := \text{the intersection of } V_1 \text{ with the level set of } \rho_W \text{ passing through } p.$$

In the triple overlap $|T_U| \cap |T_{V_1}| \cap |T_W|$ we define

$$\pi_U(p) = \text{the intersection of } U \text{ with the level sets of } \rho_{V_1} \text{ and } \rho_W \text{ passing throught } p.$$

We choose $\rho_W = x^2 + y^2 + z^2$. Next we define ρ_{V_1} so that its level sets are cones of axis V_1. More precisely

$$\rho_{V_1} = \frac{y^2 + z^2}{x^2}$$

Finally, we define

$$\rho_U = \frac{z^2}{y^2}$$

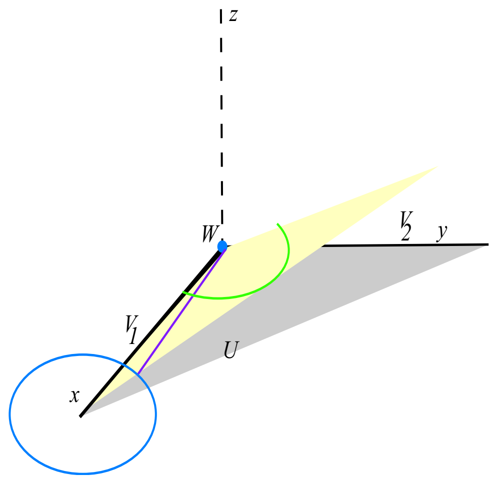

Figure 6: *Constructing a controlled tubular system.*

whose level sets are planar wedges containing the axis V_1. These three functions satisfy the transversality conditions and will provide the desired controlled systems of neighborhoods. $\square$

The existence of a controlled system of tubular neighborhoods follows from a more general result. To state it we need to introduce *Thom's A_f condition.*

Definition 1.22 (The A_f condition). *Suppose $f : X \to Y$ is a smooth map between two smooth manifolds. Assume U, V are submanifolds such that df has constant rank along U and along V. Let $v \in V \cap \bar{U}$. We say that the triple (U, V, v) satisfies the A_f* condition *if for any sequence $u_n \in U$ such that*

- $\lim_{n\to\infty} u_n = v$.
- $\lim_{n\to\infty} \ker df\,|_U = T$

we have

$$\ker df\,|_V \subset T.$$

We say that the pair (U, V) satisfies the condition A_f if (U, V, v) satisfies the condition A_f for any $v \in V \cap \bar{U}$.

Definition 1.23 (Stratifications of maps). *Suppose $f : X \to X'$ is a smooth map and let $A \subset X$ such that $f(A) \subset A'$. A* stratification *of f is a pair $(\mathcal{P}, \mathcal{P}')$ of Whitney stratifications of A and A' satisfying the following two conditions.*

- *f maps strata into strata (not necessarily onto).*
- *If $(U, U') \in \mathcal{P} \times \mathcal{P}'$ are strata such that $f(U) \subset U'$ then $f\,|_U$ is a submersion.*

A stratification $(\mathcal{P}, \mathcal{P}')$ is called a Thom stratification *if any two strata $V < U$ in $\mathcal{P}$ satisfy the condition A_f.*

We have the following important result. The proof is an iterated application of the Tubular Neighborhood Theorem. For details we refer to [1, 4].

Theorem 1.24 (Controlled Thom stratifications). *Suppose $f : X \to X'$ is a smooth map between two smooth manifolds and $A \subset X$, $A' \subset X$ are such that $f(A) \subset A'$. Suppose $(\mathcal{P}, \mathcal{P}')$ is a Thom stratification for $f : A \to A'$. Suppose we are given a collection of tubular neighborhoods $T_{U'}$ of the strata in $\mathcal{P}'$ satisfying the commutativity conditions (C_π). Then there exists a controlled system of tubular neighborhoods T_U of the strata in $\mathcal{P}$ which is compatible with f, i.e. the following holds.*

If $U \in \mathcal{P}$ and U' is the stratum of $\mathcal{P}'$ which contains $f(U)$ then

$$f(|T_U|) \subset |T_{U'}|$$

and the diagram below is commutative.

$$\begin{tikzcd} |T_U| \arrow[r, two heads, "\pi_U"] \arrow[d, "f"'] & U \arrow[d, "f"] \\ |T_{U'}| \arrow[r, two heads, "\pi_{U'}"] & U' \end{tikzcd} . \qquad (C_{f,\pi})$$

Definition 1.25. *A* controlled Thom stratification *of the map $f : A \to A'$ is a quadruple $(\mathcal{P}, \mathcal{T}; \mathcal{P}', \mathcal{T}')$ with the following properties.*

- $(\mathcal{P}, \mathcal{T})$ *is a controlled Whitney prestratification of A.*
- $(\mathcal{P}', \mathcal{T}')$ *is a controlled Whitney prestratification of A'.*
- $(\mathcal{P}, \mathcal{P}')$ *is a Thom stratification of $f : A \to A'$.*
- $(\mathcal{T}, \mathcal{T}')$ *satisfies the compatibility conditions $(C_{f,\pi})$.*

By setting $X' = \{point\}$ in the above theorem we deduce the following important consequence.

Corollary 1.26. *Every complex analytic space admits a controlled Whitney prestratification.*

To prove the local homogeneity of Whitney stratifications we need to have a way of constructing plenty of strata preserving homeomorphisms. We require an additional feature of these homeomorphisms, namely that their restrictions to any given stratum are smooth maps. We will produce such homeomorphisms by integrating certain vector fields on the variety.

If M is a smooth manifold and A is a subset of M then vector field on A is a (possibly discontinuous) section V of the restriction to A of the tangent bundle TM. Such a vector field is called *locally integrable* if the following conditions hold.

- For each $a \in A$ there exists $\varepsilon > 0$ and a C^1-curve

$$\Gamma = \Gamma_{V,a,\varepsilon} : (-\varepsilon, \varepsilon) \to A$$

such that

$$\Gamma(0) = a, \;\; \frac{d}{dt}\gamma = V(\Gamma(t)), \;\; \forall |t| < \varepsilon.$$

Such a curve is called an *integral curve* of V.

• Two integral curves with the same initial value coincide on their common domain.
For each $a \in A$ there exists a neighborhood N of a in A and an $\varepsilon > 0$ such that the mapping

$$N \times (-\varepsilon, \varepsilon) \to A, \quad (a,t) \mapsto \Gamma_{V,a,\varepsilon}(t)$$

is defined and continuous.

The vector field is called globally integrable if the conditions in the above definition are satisfied for $\varepsilon = \infty$.

Example 1.27. (a) In the plane $\mathbb{R}^2$ with polar coordinates (r,θ) the vector field $\frac{\partial}{\partial\theta}$ extended by 0 at the origin defines a globally integrable yet discontinuous vector field. The integral curves of this vector field are the circles centered at the origin.
(b) In the Euclidean space $\mathbb{R}^3$ consider the vector field described in cylindrical coordinates (r,θ,z) by

$$V = \begin{cases} \frac{\partial}{\partial z} + \frac{\partial}{\partial \theta} & \text{if} \quad z \neq 0 \\ \\ \frac{\partial}{\partial z} & \text{if} \quad z = 0 \end{cases}.$$

The integral curves of this vector field are helices around the z-axis. □

Definition 1.28. *Suppose $(\mathcal{P}, \mathcal{T})$ is a controlled Whitney (pre)stratification of a closed subset of a smooth manifold X. A* weakly controlled vector field *on A is a vector field Ξ on A such that for every stratum $U \in \mathcal{P}$ it satisfies the commutativity condition*

$$\begin{tikzcd} {|T_U|} \arrow[r, "\Xi"] \arrow[d, "\pi_U"'] & TX \arrow[d, "D\pi_U"] \\ U \arrow[r, "\Xi"] & TX \end{tikzcd} \qquad (C_{v,\pi})$$

A weakly controlled vector field Ξ is called controlled *if it is tangent to the level sets of the tubular functions ρ_U, i.e.*

$$D\rho_U(\Xi) = 0 \qquad (C_{v,\rho})$$

Theorem 1.29. *Suppose $f : X \to X'$ is a smooth map between two smooth manifolds, and $A \subset X$, $A' \subset X'$ are two locally closed subsets such that $f(A) \subset A'$ Suppose $(\mathcal{P}, \mathcal{T}; \mathcal{P}', \mathcal{T}')$ is a controlled Thom stratification of $f : A \to A'$. Then for every weakly controlled vector field Ξ' on A' there exists a controlled vector field Ξ over A such that*

$$Df(\Xi) = \Xi'.$$

Moreover, if Ξ' is locally integrable, then we can choose Ξ to be locally integrable as well. This vector field Ξ is globally integrable if the restriction of f to the closure of each stratum is proper.

By taking $X' = \{point\}$ we obtain the following important result.

Corollary 1.30. *Every controlled Whitney stratification $(\mathcal{P}, \mathcal{T})$ of a closed subset A of a smooth manifold X admits a controlled, locally integrable vector field. Moreover, if A is compact, there exist globally integrable controlled vector fields.*

Corollary 1.31 (Thom's First Isotopy Lemma). *Suppose $f : X \to X'$ is a smooth map and $(\mathcal{P}, \mathcal{T})$ a controlled Whitney stratification of a closed subset $A \subset X$. Suppose that the following hold.*

- *The restriction of f to every stratum of $\mathcal{P}$ is submersive.*
- *The restriction of f to the closure of any stratum of $\mathcal{P}$ is proper.*

Then the map $f : A \to X'$ is topologically *a locally trivial fibration.*

Corollary 1.32 (Local triviality of Whitney stratifications). *Suppose $\mathcal{P}$ is a Whitney stratification of a closed subset A of a smooth manifold X. Then $\mathcal{P}$ is locally trivial in the following sense.*

For every stratum U of A, there exists a sphere bundle $\phi : \Sigma \to U$, a closed Whitney stratified subset $(S, \mathcal{S}) \subset \Sigma$ such that

- *$\phi|_S : S \to U$ is a topological locally trivial fibration.*
- *There exists a neighborhood Z of U in N and a homeomorphism of the pair $(Z, A \cap Z)$ onto the pair of mapping cylinders $(\mathbf{Cyl}\,\phi, \mathbf{Cyl}\,\phi\,|_S)$ which is the identity on X.*

2 The Euler characteristic

3 The Euler obstruction

4 The Chern-MacPherson classes

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