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Seiberg–Witten invariants and surface singularities: splicings and cyclic covers

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Abstract. We verify the conjecture formulated in [36] for suspension singularities of type $g(x,y,z) = f(x,y) + z^n$, where f is an irreducible plane curve singularity. More precisely, we prove that the modified Seiberg–Witten invariant of the link M of g, associated with the canonical spin^c structure, equals $-\sigma(F)/8$, where $\sigma(F)$ is the signature of the Milnor fiber of g. In order to do this, we prove general splicing formulae for the Casson–Walker invariant and for the sign-refined Reidemeister–Turaev torsion. These provide results for some cyclic covers as well. As a by-product, we compute all the relevant invariants of M in terms of the Newton pairs of f and the integer n.

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More than four decades ago D. Mumford proved a very surprising result: if a point on a normal complex surface looks *topologically* like a smooth point then it must be *analytically* so. Soon after this, Brieskorn showed that such a phenomenon can only take place in complex dimension two. Since then a large number of facts suggesting an unusual rigidity of certain families of surface singularities have been discovered. The present paper has a twofold goal. The first and broader goal is to advertise some of these exotic rigidity phenomena, and to illustrate how techniques which became available only during the last decade can be used to unify and explain a substantial number of apparently disparate results. The second and more focused goal is to establish new results as important steps of an extensive program concentrating on this rigidity property. Accordingly, the article has two "introductions". The first one, Section 1, states the main new result (which was

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the motivation of the article), lists some of the key points of its proof, and guides the reader through the sections of the article. Then Section 2, at a more introductory level, gives definitions and historical remarks. (We let the reader decide in which order to read them.)

1. Introduction

This article is a natural continuation of [36] and [37], and it is closely related to [30, 31, 32]. In [36], the authors formulated a very general conjecture which connects the topological and analytical invariants of a complex normal surface singularity whose link is a rational homology sphere.

Even if we restrict ourselves to the case of hypersurface singularities, the conjecture is still highly non-trivial. The "simplified" version for this case reads as follows.

1.1. Conjecture ([36]). *Let $g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a complex analytic germ which defines an isolated hypersurface singularity. Assume that its link M is a rational homology sphere. Denote by $\mathbf{sw}^0_M(\sigma_{\mathrm{can}})$ the modified Seiberg–Witten invariant of M associated with the canonical* spin^c *structure σ_{can} (cf.* 2.4*). Moreover, let $\sigma(F)$ be the signature of the Milnor fiber F of g (cf.* 2.1*). Then*

$$-\mathbf{sw}^0_M(\sigma_{\mathrm{can}}) = \sigma(F)/8. \tag{1}$$

The goal of the present paper is to verify this conjecture for suspension hypersurface singularities. More precisely, in 7.15 we prove the following.

1.2. Theorem. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible plane curve singularity. Fix an arbitrary positive integer n such that the link M of the suspension singularity $g(x, y, z) := f(x, y) + z^n$ is a rational homology sphere (cf.* 7.2(c)*). Then* 1.1(1) *holds.*

The numerical identity 1.1(1) covers a very deep qualitative analytic-rigidity phenomenon.

Above, a certain realization of the Seiberg–Witten invariant will be used, namely the linear combination $\mathbf{sw}^{\mathrm{TCW}}_M(\sigma_{\mathrm{can}}) := \mathcal{T}_{M,\sigma_{\mathrm{can}}}(1) - \lambda_{\mathrm{W}}(M)/2$ of the sign-defined Reidemeister–Turaev torsion (associated with a "canonical" spin^c structure σ_{can}) and the Casson–Walker invariant (see 2.4). In order to prove the theorem, we will establish different splicing formulas for these topological invariants, facts with strong independent interest.

1.3. A few words about the proof and the organization of the article. If M is the link of $g = f + z^n$ (as in 1.2), then M has a natural splice decomposition into Seifert varieties of type $\Sigma(p, a, m)$. Moreover, in [31, 3.2], the first author established an additivity formula for $\sigma(F)$ compatible with the geometry of this decomposition. On the other hand, for any Brieskorn singularity $(x, y, z) \mapsto x^p + y^a + z^m$ (whose link is $\Sigma(p, a, m)$) the conjectured identity is valid by [36, 37]. Hence it was natural to carry out the proof by proving an additivity result for $\mathbf{sw}_M^{\mathrm{TCW}}(\sigma_{\mathrm{can}})$ with respect to the splice decomposition of M into Seifert varieties.

This additivity result is proved in 7.14 (as an outcome of all the preparatory results of the previous sections), but its proof contains some *surprising steps*.

Our original plan was the following. First, we identify the splicing data of M. Then, for such splicing data, we establish splicing formulas for the Casson–Walker invariant and for the Reidemeister–Turaev sign-refined torsion with the hope that we can do this in a purely topological setup without any additional reference to our special analytic context. For the Casson–Walker invariant this program was straightforward, thanks to the results of Fujita [13] and Lescop [22] (see Section 4). But when dealing with the torsion we encountered some serious difficulties (and we finally had to return back to singularities for some additional information).

The torsion computations require the explicit description of the supports of all the relevant characters of $H_1(M, \mathbb{Z})$, and then the computation of some sophisticated Fourier–Dedekind sums. The computation turned out to be feasible because these sums are not arbitrary. They have two very subtle special features which follow from various properties of irreducible plane curve singularities. The first one is a numerical inequality (see 6.1(6)) measuring some special algebraicity property. The second (new) property is the alternating behavior of the coefficients of their Alexander polynomials (see 6.2).

In Section 5 we establish different splicing formulas for $\mathbf{sw}_M^{\mathrm{TCW}}(\sigma_{\mathrm{can}})$, and we show the *limits of a possible additivity*. We even introduce a new invariant $\mathcal{D}$ which measures the non-additivity property of $\mathbf{sw}_M^{\mathrm{TCW}}(\sigma_{\mathrm{can}})$ with respect to (some) splicing (see e.g. 5.8) or (some) cyclic covers (see 5.10). This invariant vanishes in the presence of the alternating property of the Alexander polynomial involved (but not in general; see e.g. Example 5.12).

This shows clearly (and rather surprisingly) that the behavior of $\mathbf{sw}_M^{\mathrm{TCW}}(\sigma_{\mathrm{can}})$ with respect to splicing and cyclic covers (constructions topological in nature) definitely prefers some special algebraic situations. For more comments, see 5.11, 6.4 and 7.8.

Section 6 contains the needed results about irreducible plane curve singularities and the Algebraic Lemma used in the summation of the Fourier–Dedekind sums mentioned above.

In Section 7 we provide a list of properties of the link of $f+z^n$. Here basically we use almost all the partial results proved in the previous sections. Most of the formulae are formulated as inductive identities with respect to the number of Newton pairs of f.

Appendix A illustrates some of the key formulas and invariant computations on a (rather) representative example. Appendix B contains an index of notations in order to help the reader.

1.3.1. The intention of the authors was to make a self-explanatory presentation, at least of the proofs of the main new results (although, in many places, even if the corresponding statement is deducible from the sequence of statements of the article, we also provide additional references). Nevertheless, we use a few important results from the literature including the following: Fujita's splicing formula (cf. 4.3), Walker–Lescop surgery formula (cf. 4.4), the authors' result about the presentation of the sign-refined Reidemeister–Turaev torsion from the plumbing graph (cf. 3.7), and the combinatorial algorithm which provides the plumbing graph of cyclic coverings (cf. 7.2(b) and 7.3). The interested reader can find the corresponding proofs at the indicated places. Some familiarity with Seifert 3-manifolds and plumbing and splicing diagrams (of algebraic links) may also help (cf. 3.4 and 3.5).

For more details and discussions about the invariants involved see [36]. For different properties of (hypersurface) singularities, the reader may consult [2].

1.4. Notation. All the homology groups with unspecified coefficients are defined over the integers.

2. Background and historical perspective

To help the reader place the results of this paper in the proper context we decided to devote an entire section to the notions and concepts which play a central role in the study of surface singularities, and to highlight some of the historical developments which best explain the significance of Conjecture 1.1.

Our guiding conjecture deals with complex analytic *normal surface singularities.* These are two-dimensional germs of normal analytic spaces: they can be represented in some smooth germ $(\mathbb{C}^N, 0)$ as the zero set of some local analytic germs $f_i = 0$ for a finite index set $i \in I$. If $|I| = N - 2$ then $(X, 0)$ is called a *complete intersection*; if $N = 3$ (hence $|I| = 1$) then $(X, 0)$ is a *hypersurface singularity.* In any case, they have an isolated singularity at the origin. If the analytic line bundle of analytic 2-forms above the regular part $X \setminus \{0\}$ (resp. one of its powers) is analytically trivial, we say that $(X, 0)$ is *Gorenstein* (respectively $\mathbb{Q}$-*Gorenstein*). Isolated complete intersections (in particular, isolated hypersurface singularities) are Gorenstein.

We prefer to separate the invariants of such a singularity in three categories (with rather strong relationships between them): topological, analytical and smoothing invariants.

2.1. Some invariants. In general the *analytic invariants* are provided by analytic or algebraic methods from the local ring of analytic functions of $(X, 0)$. In this way is defined, e.g., the multiplicity or the embedding dimension (the smallest N for which $(X, 0) \subset (\mathbb{C}^N, 0)$ is realizable) (these will not be used in the present paper; for more details see e.g. [33]).

Another way to obtain analytic invariants is by sheaf cohomology associated with some "standard" sheaves living in one of the resolutions of $(X, 0)$. More precisely, one fixes a regular map $\pi : \tilde{X} \to X$ with $\tilde{X}$ smooth and π isomorphic above $X \setminus \{0\}$ (where X is a small Stein representative of $(X,0)$). Then $\tilde{X}$ and its structure sheaf $\mathcal{O}_{\tilde{X}}$ are a rich source of invariants. For example, the geometric genus p_g of $(X,0)$ is defined as $\dim_{\mathbb{C}} H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$.

From a *topological* point of view, any normal two-dimensional analytic singularity $(X,0)$ is characterized by its *link* M defined as follows. If $(X,0) \subset (\mathbb{C}^N,0)$ is as above, then $M = X \cap S_\epsilon$, where S_ϵ is a sphere with a sufficiently small radius centered at the origin. The intersection is independent of all the choices, and it is an oriented 3-manifold. The germ $(X,0)$ is locally homeomorphic to the real cone over M.

We say that M is an *integral* (resp. *rational*) *homology sphere* if $H_1(M,\mathbb{Z}) = 0$ (resp. $H_1(M,\mathbb{Q}) = 0$).

In general, it is rather difficult to relate topological invariants (e.g. $\pi_1(M)$, or recently defined highly non-trivial invariants of M provided by algebraic topology, gauge theory or quantum field theory) to analytic invariants provided by sheaf theory or algebraic/analytic geometry. The most important bridge between them is the plumbing graph of M. (For definition, see 3.4; very briefly: a plumbing graph is a decorated graph such that each vertex v has two decorations g_v and e_v. From such a graph one can construct an oriented *plumbed* 3-manifold as follows: for each vertex one considers an S^1-bundle P_v with Euler number e_v over a Riemann surface E_v of genus g_v. Corresponding to each edge connecting v and w, one glues P_v and P_w by a simple surgery; see e.g. [41].)

Indeed, not every 3-manifold can be realized as the link of a singularity: by [14], analytic links are exactly those plumbed manifolds which are associated with negative definite plumbing graphs (cf. 3.4).

On the other hand, from any resolution π one can read the combinatorial data about its resolution graph [20], i.e. the data describing the topology and combinatorics of the exceptional divisor $E := \pi^{-1}(0)$ and its topological embedding in $\tilde{X}$; or equivalently, the data from which one can recover topologically the space $\tilde{X}$. In fact, this data is completely codified by the negative definite intersection matrix $(E_v, E_w)_{\tilde{X}}$, where $\{E_v\}_{v=1}^s$ are the irreducible components of E, and by the genera g_v of the components E_v. (M is a rational homology sphere if and only if each g_v is 0 and the graph is a tree.)

The point is that the graph of any "good" resolution (i.e. of a resolution whose exceptional divisor E is a normal crossing divisor) can be considered as a possible (negative definite) plumbing graph of M. (The irreducible components of E correspond to the vertices of the graph, the decoration g_v is clear, while e_v is the Euler number $(E_v, E_v)_{\tilde{X}}$ of the normal bundle of E_v in $\tilde{X}$. The vertices v and w are connected by $\#E_v \cap E_w = (E_v, E_w)_{\tilde{X}}$ edges.)

Moreover, by a result of Neumann [40], from M itself one can recover all the possible resolution graphs of $(X,0)$ (which are related by simple blow up/blow down modifications). In particular, any resolution (or plumbing) graph carries the same information as M. A property of $(X,0)$ will be called *topological* if it can be determined from M, or equivalently, from any of these graphs.

For example, the following numerical invariant will appear in the next discussion. Assume that the resolution is good, and consider the unique rational cycle (supported by E and associated with the "canonical divisor") $K = \sum_v r_v E_v$ which satisfies (the "adjunction relations") $(K, E_w)_{\tilde{X}} = -(E_w, E_w)_{\tilde{X}} - 2 + 2\mathrm{g}_w$ for any $1 \leq w \leq s$. Write $K^2 := (K, K)_{\tilde{X}}$. Then the rational number $K^2 + s$ is independent of the choice of the resolution, it is an invariant of M. (In fact, one of the goals of the next program is to express any topological invariant in terms of the graph, or in terms of the intersection matrix; more complicated examples will be considered in the body of the paper.)

Finally, we say a few words about *smoothing invariants*. By a *smoothing* of $(X, 0)$ we mean a proper flat analytic germ $f : (\mathcal{X}, 0) \to (\mathbb{C}, 0)$ with an isomorphism $(f^{-1}(0), 0) \to (X, 0)$ (where $(\mathcal{X}, 0)$ has at most an isolated singularity at 0). If $\mathcal{X}$ is a sufficiently small contractible Stein representative of $(\mathcal{X}, 0)$, then for sufficiently small η $(0 < |\eta| \ll 1)$ the fiber $F := f^{-1}(\eta) \cap \mathcal{X}$ is smooth, and its diffeomorphism type is independent of the choices. It is a connected oriented real 4-manifold with boundary ∂F which can be identified with the link M of $(X, 0)$. It is called the Milnor fiber associated with the smoothing. In general not every $(X, 0)$ has any smoothings, and even if it has some, the Milnor fiber F depends essentially on the choice of the smoothing. In the case of complete intersections (in particular, of hypersurfaces) there is only one (semi-universal) smoothing component, hence F depends only on $(X, 0)$.

There are some standard notations: $\mu(F) = \operatorname{rank} H_2(F, \mathbb{Z})$ (called the *Milnor number* of the smoothing); $(\, , \,)_F =$ the *intersection form* of F on $H_2(F, \mathbb{Z})$; (μ_0, μ_+, μ_-), the *Sylvester invariant* of $(\, , \,)_F$; and finally $\sigma(F) := \mu_+ - \mu_-$, the *signature* of F. (If $M = \partial F$ is a rational homology sphere then $\mu_0 = 0$.)

2.2. Question: Which invariants are topological? A very intriguing issue, which has generated intense research efforts, is the possibility of expressing the analytic invariants of $(X, 0)$ (like the geometric genus p_g, multiplicity, etc.) or the smoothing invariants (if they exist, like the signature $\sigma(F)$ of the Milnor fiber F) in terms of the topology of M.

2.3. A short historical survey. M. Artin proved in [3, 4] that the rational singularities (i.e. the vanishing of p_g) can be characterized completely from the plumbing graph of M. In [19], H. Laufer extended Artin's results to minimally elliptic singularities, showing that Gorenstein singularities with $p_g = 1$ can be characterized topologically. Additionally, he noticed that the program breaks down for more complicated singularities (see also the comments in [36] and [29]). On the other hand, the first author noticed in [29] that Laufer's counterexamples do not signal the end of the program. He conjectured that if we restrict ourselves to the case of those Gorenstein singularities whose links are rational homology spheres then p_g is topological. In fact, even before [29], the question on the topological nature of p_g was formulated in [42, 3.2] for Gorenstein singularities whose links are integral homology spheres.

The topological nature of p_g for elliptic singularities with rational homology sphere links was shown explicitly in [29] (partially based on some results of S. S.-T. Yau [51]).

For Gorenstein singularities which have a smoothing (with Milnor fiber F), the topological invariance of p_g can be reformulated in terms of $\sigma(F)$ and/or $\mu(F)$. Indeed, via some results of Laufer, Durfee, Seade, Wahl and Steenbrink, any of p_g, $\sigma(F)$ and $\mu(F)$ determines the remaining two modulo K^2+s (for the precise identities see e.g. [25] or [36]). For example, the identity which connects p_g and $\sigma(F)$ is

$$8p_g + \sigma(F) + K^2 + s = 0. \tag{2}$$

Via this identity the following facts about the signature $\sigma(F)$ can be transformed into non-trivial properties of p_g.

Fintushel and Stern proved in [12] that for a hypersurface Brieskorn singularity *whose link is an integral homology sphere*, the Casson invariant $\lambda(M)$ of the link M equals $-\#\mathcal{R}(M)/2$, where $\#\mathcal{R}(M)$ is the number of conjugacy classes of irreducible $SU(2)$-representations of $\pi_1(M)$. It turns out that this number equals $\sigma(F)/8$ as well (a fact conjectured by Neumann and Wahl). This result was generalized by Neumann and Wahl in [42]. They proved the same statement for all Brieskorn–Hamm complete intersections and suspensions of irreducible plane curve singularities (with the same assumption about the link). Moreover, they conjectured the identity $\lambda(M) = \sigma(F)/8$ for any isolated complete intersection singularity whose link is an *integral* homology sphere.

In [36] the authors extended the above conjecture for smoothing of Gorenstein singularities with *rational* homology sphere link, in such a way that the conjecture incorporates the previous conjecture and results about the geometric genus as well. Here the Casson invariant $\lambda(M)$ is replaced by a certain Seiberg–Witten invariant $\mathbf{sw}^0_M(\sigma_{\mathrm{can}})$ of the link associated with the *canonical* spin^c structure of M. (In the next subsection we provide more details about the definition and properties of the topological invariant $\mathbf{sw}^0_M(\sigma_{\mathrm{can}})$; right now we only mention that for integral homology sphere links it equals $-\lambda(M)$.)

In fact, the conjecture in [36] is more general. A part of it says that for any $\mathbb{Q}$-Gorenstein singularity whose link is a rational homology sphere, one has

$$8p_g - 8\cdot \mathbf{sw}^0_M(\sigma_{\mathrm{can}}) + K^2 + s = 0. \tag{3}$$

Notice that (in the presence of a smoothing and of the Gorenstein property, e.g. for any hypersurface singularity) (3) via (2) is exactly (1). We also notice that by extending the conjecture to the family of $\mathbb{Q}$-Gorenstein singularities (many of which are not smoothable) we also incorporate important classes like the rational singularities or the singularities which admit a good $\mathbb{C}^*$-action (and have rational homology sphere link).

The identity (3) was verified in [36] for cyclic quotient singularities, Brieskorn–Hamm complete intersections and some rational and minimally elliptic singularities. [37] contains the case when $(X,0)$ has a good $\mathbb{C}^*$-action.

To end this historical presentation, we mention that the theory of suspension hypersurface singularities also has its own long history. This class (together with the weighted-homogeneous singularities) serves as an important "testing and exemplifying" family for various properties and conjectures. For more information, the reader is invited to check [31, 32] and the survey paper [30], and the references listed in these articles. See also [28] for the topological behavior of some other analytic invariants (cf. 7.8(1)) of suspension singularities.

The recent survey [34] may help the reader to obtain a more global picture and deeper understanding of the main question 2.2 and also about more recent developments. See also [26].

2.4. The Seiberg–Witten invariant. If M is a rational homology 3-sphere with $H := H_1(M, \mathbb{Z})$, then the set $\mathrm{Spin}^c(M)$ of spin^c structures of M is an H-torsor. If M is the link of a normal surface singularity (or, equivalently, if M has a plumbing representation with a negative definite intersection matrix), then $\mathrm{Spin}^c(M)$ has a distinguished element σ_{can}, called the *canonical* spin^c structure (cf. [36]). Its definition is the following.

We fix a resolution $\tilde{X} \to X$. The spin^c structures on $\tilde{X}$ are classified by (the so-called characteristic) elements k of $H^2(\tilde{X}, \mathbb{Z})$ for which $k(E_j) + (E_j, E_j)_{\tilde{X}}$ is even for any j. The correspondence is given by the first Chern class: for any $\sigma \in \mathrm{Spin}^c(\tilde{X})$, we denote by $\mathbb{S}_\sigma$ the associated bundle of complex spinors, and we set $k := c_1(\mathbb{S}_\sigma) \in H^2(\tilde{X}, \mathbb{Z})$.

Notice that $-K$ (via Poincaré duality) can be identified with such an element, hence it determines a spin^c structure on $\tilde{X}$. This induces on $M = \partial\tilde{X}$ the spin^c structure σ_{can}. It can be verified that it is independent of the choice of $\tilde{X}$. Since K is determined combinatorially from the graph, it follows that the identification of $\sigma_{\mathrm{can}} \in \mathrm{Spin}^c(M)$ is a topological procedure.

To describe the Seiberg–Witten invariants one has to consider additional geometric data belonging to the space of parameters

$$\mathcal{P} = \{u = (g, \eta) : g = \text{Riemann metric}, \ \eta = \text{closed 2-form}\}.$$

Then for each spin^c structure σ on M one defines the (σ, g, η)-*Seiberg–Witten monopoles*, as solutions of some non-linear elliptic equations. For a generic parameter u, the Seiberg–Witten invariant $\mathbf{sw}_M(\sigma, u)$ is the signed monopole count. This integer depends on the choice of the parameter u and thus it is not a topological invariant. To obtain an invariant of M, one needs to alter this monopole count by the *Kreck–Stolz* invariant $\mathrm{KS}_M(\sigma, u)$ (associated with the data (σ, u)) (cf. [24] or see [18] for the original "spin version"). Then, by [8, 24, 27], the rational number

$$\frac{1}{8}\mathrm{KS}_M(\sigma, u) + \mathbf{sw}_M(\sigma, u)$$

is independent of u and thus it is a topological invariant of the pair (M, σ). We denote this *modified Seiberg–Witten invariant* by $\mathbf{sw}^0_M(\sigma)$.

2.4.1. In [36], for some very special singularity links (lens spaces and some Seifert 3-manifolds) the identity (3) was verified using this analytic definition of $\mathbf{sw}^0_M(\sigma)$. But, in general, it is very difficult to compute $\mathbf{sw}^0_M(\sigma)$ using this definition (and there are only sporadic results in this direction).

In the theory of Seiberg–Witten invariants there is an intense effort to replace the present construction/definition of the modified Seiberg–Witten invariants with a different one, which allows easier computations (and even is topological—or, in the case of plumbed manifolds, combinatorial—in nature). Presently, there exist a few candidates. One of them is $\mathbf{sw}^{\mathrm{TCW}}_M(\sigma)$ provided by the sign-refined Reidemeister–Turaev torsion (normalized by the Casson–Walker invariant), see below; another is provided by the Ozsváth-Szabó theory (for possible connections, see [44] and [35]). In particular, the above conjecture can be transformed into similar conjectures where $\mathbf{sw}^0_M(\sigma)$ is replaced by any candidate $\mathbf{sw}^*_M(\sigma)$ (even if at this moment the identity $\mathbf{sw}^0_M(\sigma) = \mathbf{sw}^*_M(\sigma)$ is not proven yet).

In the present paper we will consider $\mathbf{sw}^{\mathrm{TCW}}_M(\sigma)$.

2.4.2. Turaev's torsion function and the Casson–Walker invariant. For any spin^c structure σ on M, we denote by

$$\mathcal{T}_{M,\sigma} = \sum_{h\in H} \mathcal{T}_{M,\sigma}(h)\, h \in \mathbb{Q}[H]$$

the sign-refined *Reidemeister–Turaev torsion* associated with σ (for its detailed description, see [47]). We think of $\mathcal{T}_{M,\sigma}$ as a function $H \to \mathbb{Q}$ given by $h \mapsto \mathcal{T}_{M,\sigma}(h)$. The *augmentation map* $\mathfrak{aug} : \mathbb{Q}[H] \to \mathbb{Q}$ is defined by $\sum a_h\, h \mapsto \sum a_h$. It is known that $\mathfrak{aug}(\mathcal{T}_{M,\sigma}) = 0$.

Denote the Casson–Walker invariant of M by $\lambda_{\mathrm{W}}(M)$ [49]. It is related to Lescop's normalization $\lambda(M)$ [22, 4.7] by $\lambda_{\mathrm{W}}(M) = 2\lambda(M)/|H|$. One defines

$$\mathbf{sw}^{\mathrm{TCW}}_M(\sigma) := \mathcal{T}_{M,\sigma}(1) - \lambda_{\mathrm{W}}(M)/2. \tag{4}$$

This will be the Seiberg–Witten invariant considered in this article. More precisely, what we really prove is the following:

2.4.3. Theorem. *If g is a suspension singularity as in* 1.2 *then*

$$-\mathbf{sw}^{\mathrm{TCW}}_M(\sigma_{\mathrm{can}}) = \sigma(F)/8. \tag{5}$$

We mention that the identity (5) and its proof are completely independent of the (conjectural) identification $\mathbf{sw}^0_M(\sigma) = \mathbf{sw}^{\mathrm{TCW}}_M(\sigma)$ (and of the "classical" Seiberg–Witten theory).

In 3.7 we present a combinatorial formula for $\mathcal{T}_{M,\sigma}(1)$ in terms of the plumbing graph of M (proved in [36]). Those readers who are more interested in singularity theory, and do not wish to be immersed in Turaev's theory at the first reading, can take the topological invariant $\mathcal{T}_{M,\sigma_{\mathrm{can}}}$ as the invariant which is "defined" by those combinatorial relations 3.7. (In fact, those formulas resonate very much—although they are more complex—with A'Campo type formulas for the zeta function of the monodromy actions, well known in singularity theory.)

3. Topological preliminaries

3.1. Oriented knots in rational homology spheres. Let M be an oriented 3-manifold which is a *rational homology sphere*. Fix an oriented knot $K \subset M$, denote by $T(K)$ a small tubular neighborhood of K in M, and let $\partial T(K)$ be its oriented boundary with its natural orientation. The natural *oriented meridian* of K, situated in $\partial T(K)$, is denoted by m. We fix an oriented *parallel* ℓ in $\partial T(K)$ (i.e. $\ell \sim K$ in $H_1(T(K))$). If $\langle\, , \rangle$ denotes the intersection form in $H_1(\partial T(K))$, then $\langle m, \ell\rangle = 1$ (cf. e.g. Lescop's book [22, p. 104]; we will use the same notations m and ℓ for some geometric realizations of the meridian and parallel as primitive simple curves, respectively for their homology classes in $H_1(\partial T(K))$).

Obviously, the choice of ℓ is not unique. In all our applications, ℓ will be characterized by some precise additional geometric construction.

Assume that the order of the homology class of K in $H_1(M)$ is $o > 0$. Consider an oriented surface F_{oK} with boundary oK, and take the intersection $\lambda := F_{oK} \cap \partial T(K)$. Then λ is called the *longitude* of K. The homology class of λ in $H_1(\partial T(K))$ can be represented as $\lambda = o\ell + km$ for some integer k. Set $\gcd(o, |k|) = \delta > 0$. Then λ can be represented in $\partial T(K)$ as δ primitive torus curves of type $(o/\delta, k/\delta)$ with respect to ℓ and m.

3.2. Dehn fillings. Let $T(K)^\circ$ be the interior of $T(K)$. For any homology class $a \in H_1(\partial T(K))$ which can be represented by a primitive simple closed curve in $\partial T(K)$, one defines the *Dehn filling* of $M \setminus T(K)^\circ$ along a by

$$(M \setminus T(K)^\circ)(a) = M \setminus T(K)^\circ \coprod_f S^1 \times D^2,$$

where $f : \partial(S^1 \times D^2) \to \partial T(K)$ is a diffeomorphism which sends $\{*\} \times \partial D^2$ to a curve representing a.

3.3. Linking numbers. Consider two disjoint oriented knots $K, L \subset M$. Fix a Seifert surface F_{oK} of oK (cf. 3.1) and define the linking number $\mathrm{Lk}_M(K, L) \in \mathbb{Q}$ by the "rational" intersection $(F_{oK} \cdot L)/o$. In fact, $\mathrm{Lk}_M(K, \cdot) : H_1(M \setminus K, \mathbb{Q}) \to \mathbb{Q}$ is a well defined homeomorphism and $\mathrm{Lk}_M(K, L) = \mathrm{Lk}_M(L, K)$. For any oriented knot L on $\partial T(K)$ one has (see e.g. [22, 6.2.B])

$$\mathrm{Lk}_M(L, K) = \langle L, \lambda\rangle/o. \tag{1}$$

For any oriented knot $K \subset M$ one has the obvious exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\alpha} H_1(M \setminus T(K)) \xrightarrow{j} H_1(M) \to 0, \tag{2}$$

where $\alpha(1_{\mathbb{Z}}) = m$. If $K \subset M$ is homologically trivial then this sequence splits. Indeed, let ϕ be the restriction of $\mathrm{Lk}_M(K, \cdot)$ to $H_1(M \setminus K) = H_1(M \setminus T(K))$. Then ϕ has integer values and $\phi \circ \alpha = 1_{\mathbb{Z}}$. This provides automatically a morphism $s : H_1(M) \to H_1(M \setminus T(K))$ such that $j \circ s = 1$ and $\alpha \circ \phi + s \circ j = 1$; in particular with $\phi \circ s = 0$ too. In fact, $s(H_1(M)) = \mathrm{Tors}\, H_1(M \setminus T(K))$. Moreover, under the same assumption $o = 1$, one has the *isomorphisms*

$$H_2(M, K) \xrightarrow{\partial} H_1(K) = \mathbb{Z} \quad \text{and} \quad H_1(M) \to H_1(M, K). \tag{3}$$

Sometimes, in order to simplify the notations, we write H for the group $H_1(M)$.

The finite group H carries a natural symmetric bilinear form $b_M : H^{\otimes 2} \to \mathbb{Q}/\mathbb{Z}$ defined by $b_M([K],[L]) = \mathrm{Lk}_M(K,L) \pmod{\mathbb{Z}}$, where L and K are two representatives with $K \cap L = \emptyset$. If $\hat{H}$ denotes the Pontryagin dual $\mathrm{Hom}(H,S^1)$ of H, and $\exp(r) := e^{2\pi i r}$ for any rational number r, then $[K] \mapsto \exp(b_M([K],\cdot))$ is an isomorphism $H \to \hat{H}$.

3.4. (M,K) **represented by plumbing.** The main application of the present article involves algebraic links (M,K) which can be represented by plumbing. We recall the notations briefly (for more details, see e.g. [41] or [36]).

We will denote by $\Gamma(M,K)$ the plumbing graph of a link $K \subset M$. The vertices $v \in \mathcal{V}$ are decorated by the Euler numbers e_v (of the S^1-bundles over $E_v \approx S^2$ used in the plumbing construction). The components of the link K are represented by arrows in $\Gamma(M,K)$: if an arrow is attached to the vertex v then the corresponding component of K is a fixed fiber of the S^1-bundle over E_v. (We think about an arrow as an arrowhead connected to v by an edge.) If we delete the arrows then we obtain a plumbing graph $\Gamma(M)$ of M. Let δ_v (resp. $\bar{\delta}_v$) be the degree (i.e. the number of incident edges) of the vertex v in $\Gamma(M)$ (resp. in $\Gamma(M,K)$). Evidently $\bar{\delta}_v - \delta_v$ is exactly the number of arrows supported by the vertex v.

Let $\{I_{uv}\}_{u,v\in\mathcal{V}}$ be the intersection matrix associated with Γ; i.e. $I_{uu} = e_v$, and for $u \neq v$ the entry $I_{uv} = 1$ or 0 according as u and v are connected or not in Γ. Since M is a rational homology sphere, I is non-degenerate. In fact

$$|\det(I)| = |H|. \tag{4}$$

The main property of algebraic links (M,K) is that they can be represented by a plumbing by a *connected* plumbing graph Γ whose intersection matrix I is *negative definite* [20]. In fact, if M is a rational homology sphere, then Γ is a tree.

The generic oriented fiber of the S^1-bundle over E_v is denoted by g_v, and we use the same notation for its homology class in $H_1(M)$. By the above discussion, if $u \neq v$ then $\mathrm{Lk}_M(g_u, g_v)$ is well defined. If $u = v$ then we write $\mathrm{Lk}_M(g_u, g_u)$ for $\mathrm{Lk}_M(g_u, g_u')$ where g_u and g_u' are two different fibers of the S^1-bundle over E_v.

For any fixed vertex $u \in \mathcal{V}$, we denote by $\mathbf{b}_u$ the column (base) vector with entry 1 in place u and zero otherwise. We define the column vector $\mathbf{w}(u)$ (associated with the knot $g_u \subset M$ and its order $o(u)$) as the solution of the (non-degenerate) linear system $I \cdot \mathbf{w}(u) = -o(u)\mathbf{b}_u$. The entries $\{w_v(u)\}_{v\in\mathcal{V}}$ of $\mathbf{w}(u)$ are called the *weights* associated with g_u.

The following fact is well known, but for the convenience of the reader we provide all the details. Below we denote the vectors (cycles) by $\mathbf{x} = \sum_u x_u \mathbf{b}_u$, we write $\mathbf{x} \geq 0$ if $x_u \geq 0$ for any u, and we denote the support $\{u : x_u \neq 0\}$ of $\mathbf{x}$ by $|\mathbf{x}|$.

3.4.1. Lemma. *Assume that Γ is connected and I negative definite. Then:*

(a) *If $\mathbf{x}_1, \mathbf{x}_2 \geq 0$ and $|\mathbf{x}_1| \cap |\mathbf{x}_2| = \emptyset$, then $\mathbf{x}_1^t I \mathbf{x}_2 \geq 0$.*
(b) *If $I\mathbf{x} \geq 0$, then $\mathbf{x} \leq 0$. Additionally, if $\mathbf{x} \neq 0$, then $x_u < 0$ for any u.*
(c) *$I^{-1}_{uv} < 0$ for any u, v.*

Proof. (a) follows from the fact that any entry of I off the diagonal is non-negative. For (b), assume that $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1 > 0$, $\mathbf{x}_2 \geq 0$ and their supports are disjoint. Then $0 \leq \mathbf{x}_1^t I\mathbf{x} = \mathbf{x}_1^t I\mathbf{x}_1 - \mathbf{x}_1^t I\mathbf{x}_2 \leq \mathbf{x}_1^t I\mathbf{x}_1$ by (a), a fact which contradicts the negative definiteness of I. Assume now that $\mathbf{x} \neq 0$ but its support is not maximal. Then, since Γ is connected, there is a vertex $w \notin |\mathbf{x}|$ such that w is connected by an edge to a vertex $v \in |\mathbf{x}|$. Then, from the choice of w, and (a), $\mathbf{b}_w^t I\mathbf{x} < 0$. On the other hand, since $I\mathbf{x} \geq 0$, we get $\mathbf{b}_w^t I\mathbf{x} \geq 0$, a contradiction. For (c) apply (b) for $\mathbf{x} = I^{-1}\mathbf{b}_u$. □

3.4.2. Corollary. *Assume that Γ is connected and I negative definite. Then the inverse matrix I^{-1} of I, the set of weights $\{w_v(u)\}_{v\in\mathcal{V}}$, and the linking pairing* Lk_M *satisfy:*

$$-I_{uv}^{-1} = \frac{w_v(u)}{o(u)} = \mathrm{Lk}_M(g_u, g_v) \quad \text{for any } u, v \in \mathcal{V}. \tag{5}$$

In particular, $w_v(u)$ is a positive integer for any u and v.

Proof. The first identity follows from the definition of $\mathbf{w}(u)$; we hint at possible proof for the second one. Fix u, and take some positive integer t such that $tw_v(u)$ is an integer for any v. Then the "multilink" $to(u)g_u \subset M$ is analytic (see [14], or [11, 24.1]) with a possible embedded resolution graph $\Gamma(M, to(u)g_u)$. This means that there exist a space germ $(X, 0)$ and a map germ $f : (X, 0) \to (\mathbb{C}, 0)$ such that the link of the pair $(X, f^{-1}(0))$—where $f^{-1}(0)$ is counted with its multiplicity—is $(M, to(u)g_u)$. By Milnor's fibration theorem $\Gamma(M, to(u)g_u)$ is fibrable with a fiber F which satisfies $\partial F = to(u)g_u$. Let $\tilde{X}$ be a good resolution of $(X, 0)$ as above (with $\partial\tilde{X} = M$). Then, in fact, the homology class of F in $H_2(\tilde{X}, \partial\tilde{X})$ is $to(u)D_u + \sum_v \bar{w}_v E_v$ for some integers $\{\bar{w}_v\}_v$, where D_u is a transversal disc to E_u with $\partial D_u = g_u$. On the other hand, since F is given by an equation, it defines a principal divisor, hence $F \cdot E_w = 0$ for any w. This reads $to(u)\delta_{uw} + \sum_v \bar{w}_v I_{vw} = 0$ for any w. In vector notation, $I\bar{\mathbf{w}} + to(u)\mathbf{b}_u = 0$. Since I is non-degenerate, this shows that $\bar{\mathbf{w}} = t\mathbf{w}(u)$. Hence $F \cdot g_v$ (in M) $= F \cdot D_v$ (in $\tilde{X}$) $= \bar{w}_v = tw_v(u)$, which proves (5). (In fact, a posteriori, we see that $t = 1$ also works.) Compare also with [11, 11.1] and [9, A31]. □

3.5. (M, K) **represented by splice diagram.** If M is an *integral homology sphere*, and (M, K) has a plumbing representation, then there is an equivalent graph-codification of (M, K) in terms of the *splice* (or *Eisenbud–Neumann*) *diagram* (for details see [11]).

The splice diagram preserves the "shape" of the plumbing graph (e.g. there is a one-to-one correspondence between those vertices v with $\delta_v \neq 2$ of the splice, respectively of the plumbing graphs), but in the splice diagram one collapses into an edge each string of the plumbing graph. Moreover, the decorations are also different. In the splice diagram, each vertex has a sign $\epsilon = \pm 1$, which in all our cases will be $\epsilon = +1$, hence we omit them. Moreover, if an end of an edge is attached to a vertex v with $\delta_v \geq 3$, then it has a positive integer as its decoration. The arrows have the same significance.

One of the big advantages of the splice diagram is that it codifies in an ideal way the splicing decomposition of M into Seifert pieces. In fact, the numerical decorations are exactly the Seifert invariants of the corresponding Seifert splice-components.

Therefore, in some cases it is much easier and more suggestive to use them. (Nevertheless, we will use them only in those cases when we really want to emphasize this principle, e.g. in the proof of 5.10, or when it is incomparably easier to describe a construction with them, e.g. in 5.12.) The reader is invited to consult the book of Eisenbud and Neumann [11] for the needed properties: the criterion which guarantees that (M, K) is algebraic is given there in 9.4; the equivalence between the splice and plumbing graphs is described in Sections 20–22; the splicing construction appears in Section 8.

3.6. The Alexander polynomial. Assume that K *is a homologically trivial oriented knot* in M. Let V be an Alexander matrix of $K \subset M$, and V^* its transposed (cf. [22, p. 26]). The size of V is even, say $2r$.

In the literature one can find different *normalizations* of the Alexander "polynomial". The most convenient for us, which makes our formulae the simplest possible, is

$$\Delta^{\natural}_M(K)(t) := \det(t^{1/2}\, V - t^{-1/2}\, V^*).$$

In the surgery formula 4.4 we will need *Lescop's normalization* [22], in the present article denoted by $\Delta^L_M(K)(t)$. They are related by the identity (cf. [22, 2.3.13])

$$\Delta^{\natural}_M(K)(t) = \Delta^L_M(K)(t)/|H|.$$

Then (see e.g. [22, 2.3.1]) one has

$$\Delta^{\natural}_M(K)(1) = \Delta^L_M(K)(1)/|H| = 1. \tag{6}$$

We also prefer to think about the Alexander polynomial as a characteristic polynomial. For this, notice that V is invertible over $\mathbb{Q}$, hence one can define the "monodromy operator" $\mathcal{M} := V^{-1}V^*$. Then set

$$\Delta_M(K)(t) := \det(I - t\mathcal{M}) = \det(V^{-1}) \cdot t^r \cdot \Delta^{\natural}_M(K)(t). \tag{7}$$

If (M, K) can be represented by a connected negative definite plumbing graph, then by a theorem of Grauert [14], (M, K) is algebraic, hence by Milnor's fibration theorem, it is fibrable. In this case, $\mathcal{M}$ is exactly the monodromy operator acting on the first homology of the (Milnor) fiber. Moreover, $\Delta_M(K)(t)$ can be computed from the plumbing graph by A'Campo's theorem [1] as follows (see also [11] for a topological argument). Assume that $K = g_u$ for some $u \in \mathcal{V}$. Then

$$\frac{\Delta_M(g_u)(t)}{t-1} = \prod_{v\in\mathcal{V}} (t^{w_v(u)} - 1)^{\bar{\delta}_v - 2}. \tag{8}$$

Notice that (6) guarantees that $\Delta_M(K)(1) = \det(V^{-1})$, and from (8) we get $\Delta_M(K)(1) > 0$. Moreover, from the Wang exact sequence of the fibration one has

$|\Delta_M(K)(1)| = |H|$. Indeed, this exact sequence (for a later purpose written in a slightly more general form) is

$$H_1(F_{o(u)K}) \xrightarrow{I-\mathcal{M}} H_1(F_{o(u)K}) \to H_1(M \setminus K) \xrightarrow{\partial} \mathbb{Z} \to 0.$$

In this situation $o(u) = 1$, and ∂ can be identified with ϕ introduced after 3.3(2), hence this sequence combined with 3.3(2) gives $\operatorname{coker}(I - \mathcal{M}) = H_1(M)$. Therefore

$$\Delta_M(K)(1) = \det(V)^{-1} = |H|. \tag{9}$$

More generally, if (M, K) has a negative definite plumbing representation, and $K = g_u$ for some u, then for any character $\chi \in \hat{H}$ we define $\Delta_{M,\chi}(g_u)(t)$ via the identity

$$\frac{\Delta_{M,\chi}(g_u)(t)}{t-1} := \prod_{v\in\mathcal{V}} (t^{w_v(u)}\chi(g_v) - 1)^{\bar{\delta}_v - 2}, \tag{10}$$

and we write

$$\Delta_M^H(g_u)(t) := \frac{1}{|H|} \sum_{\chi\in\hat{H}} \Delta_{M,\chi}(g_u)(t). \tag{11}$$

In Section 6 we will need the following analog of (9) in the case when $K = g_u \subset M$ is not homologically trivial (but (M, K) has a negative definite plumbing representation):

$$\lim_{t\to 1} (t-1) \prod_{v\in\mathcal{V}} (t^{w_v(u)} - 1)^{\bar{\delta}_v - 2} = |H|/o(u). \tag{12}$$

In this case the "multilink" $(M, o(u)K)$ is fibrable, and (12) can be deduced again from the above Wang exact sequence of the monodromy $\mathcal{M}$. Indeed, we combine again this sequence with 3.3(2). Then $\partial \circ \alpha : \mathbb{Z} \to \mathbb{Z}$ is the multiplication by $o(u)$. Hence $\operatorname{coker}(I - \mathcal{M}) = \ker(\partial)$ can be inserted in an exact sequence $0 \to \ker(\partial) \to H_1(M) \to \mathbb{Z}_{o(u)} \to 0$. (For a different argument, see [36, A10(b)]. In fact, $|H|/o(u)$ has the geometric meaning of $|\operatorname{Tors} H_1(M \setminus K)|$.)

3.7. The Reidemeister–Turaev torsion. Assume that M is a rational homology 3-sphere and $\sigma \in \operatorname{Spin}^c(M)$. Below we will present a formula for the sign-refined Reidemeister–Turaev torsion (function) $\mathcal{T}_{M,\sigma}$ in terms of Fourier transform. Recall that a function $f : H \to \mathbb{C}$ and its Fourier transform $\hat{f} : \hat{H} \to \mathbb{C}$ satisfy

$$\hat{f}(\chi) = \sum_{h\in H} f(h)\bar{\chi}(h), \qquad f(h) = \frac{1}{|H|} \sum_{\chi\in\hat{H}} \hat{f}(\chi)\chi(h). \tag{13}$$

Here $\hat{H}$ denotes the Pontryagin dual of H as above. Notice that $\hat{f}(1) = \mathfrak{aug}(f)$, in particular (since $\mathfrak{aug}(\mathcal{T}_{M,\sigma}) = 0$, cf. 2.4.2) $\hat{\mathcal{T}}_{M,\sigma}(1) = 0$. Therefore,

$$\mathcal{T}_{M,\sigma}(1) = \frac{1}{|H|} \sum_{\chi\in\hat{H}\setminus\{1\}} \hat{\mathcal{T}}_{M,\sigma}(\chi). \tag{14}$$

Now, assume that M is represented by a *negative definite connected plumbing graph.* Fix a non-trivial character $\chi \in \hat{H} \setminus \{1\}$ and an arbitrary vertex $u \in \mathcal{V}$ with $\chi(g_u) \neq 1$. Set

$$\hat{P}_{M,\chi,u}(t) := \prod_{v\in\mathcal{V}} (t^{w_v(u)}\chi(g_v) - 1)^{\delta_v - 2}, \tag{15}$$

where $t \in \mathbb{C}$ is a free variable. Then, by one of the main results of [36], Theorem (5.8), the Fourier transform $\hat{\mathfrak{T}}_{M,\sigma_{\mathrm{can}}}$ of $\mathfrak{T}_{M,\sigma_{\mathrm{can}}}$ is given by

$$\hat{\mathfrak{T}}_{M,\sigma_{\mathrm{can}}}(\bar{\chi}) = \lim_{t\to 1} \hat{P}_{M,\chi,u}(t). \tag{16}$$

This limit is independent of the choice of u, as long as $\chi(g_u) \neq 1$. In fact, even if $\chi(g_u) = 1$, but u is adjacent to some vertex v with $\chi(g_v) \neq 1$, then u does the same job.

4. Some general splicing formulae

4.1. The splicing data. We will consider the following geometric situation. We start with two oriented 3-manifolds M_1 and M_2, both *rational homology spheres.* For $i = 1, 2$, we fix an oriented knot K_i in M_i, and we use the notations of 3.1 with the corresponding indices $i = 1, 2$. In this article we will consider a particular splicing, which is motivated by the geometry of the suspension singularities.

On the pair (M_2, K_2) we impose no additional restrictions. But, for $i = 1$, we will consider the following *working assumption*:

WA1: *Assume that $o_1 = 1$, i.e. K_1 is homologically trivial in M_1. Moreover, we fix the parallel ℓ_1 to be exactly the longitude λ_1. Evidently, $k_1 = 0$.*

Finally, by splicing, we define a 3-manifold M (for details, see e.g. [13]):

$$M = M_1 \setminus T(K_1)^\circ \coprod_A M_2 \setminus T(K_2)^\circ,$$

where A is an identification of $\partial T(K_2)$ with $-\partial T(K_1)$ determined by

$$A(m_2) = \lambda_1 \quad \text{and} \quad A(\ell_2) = m_1. \tag{1}$$

4.2. The closures $\overline{M}_i$. Once the splicing data is fixed, one can consider the *closures* $\overline{M}_i$ of $M_i \setminus T(K_i)^\circ$ $(i = 1, 2)$ with respect to A (cf. [5] or [13]) by the following Dehn fillings:

$$\overline{M}_2 = (M_2 \setminus T(K_2)^\circ)(A^{-1}(y_1)), \quad \overline{M}_1 = (M_1 \setminus T(K_1)^\circ)(A(y_2)),$$

where $\delta_i y_i := \lambda_i$ $(i = 1, 2)$. Using (1) one has $A^{-1}(y_1) = m_2$, hence

$$\overline{M}_2 = M_2.$$

Moreover, $A(y_2) = A((o_2\ell_2 + k_2 m_2)/\delta_2) = (o_2 m_1 + k_2\lambda_1)/\delta_2$, hence

$$\overline{M}_1 = (M_1 \setminus T(K_1)^\circ)(\mu), \quad \text{where} \quad \mu := (o_2 m_1 + k_2\lambda_1)/\delta_2.$$

In fact, $\overline{M}_1$ can be represented as a (p, q)-surgery of M_1 along K_1. The integers (p, q) can be determined as in [22, p. 8]: μ is homologous to qK_1 in $T(K_1)$, hence $q = k_2/\delta_2$. Moreover, $p = \mathrm{Lk}_{M_1}(\mu, K_1)$, which via 3.3(1) equals $\langle (o_2m_1 + k_2\lambda_1)/\delta_2, \lambda_1\rangle = o_2/\delta_2$. Therefore,

$$\overline{M}_1 = M_1(K_1, p/q) = M_1(K_1, o_2/k_2). \tag{2}$$

4.3. Fujita's splicing formula for the Casson–Walker invariant. If we use the above expressions for the closures, then formula (1.1) of [13], in the case of the above splicing (with $A = f^{-1}$), reads

$$\lambda_{\mathrm{W}}(M) = \lambda_{\mathrm{W}}(M_2) + \lambda_{\mathrm{W}}(M_1(K_1, o_2/k_2)) + \boldsymbol{s}(k_2, o_2). \tag{3}$$

Here, $\boldsymbol{s}(\cdot,\cdot)$ denotes Dedekind sums, defined by the same convention as in [45, 13, 22, 36]:

$$\boldsymbol{s}(q,p) = \sum_{l=0}^{p-1}\left(\left(\frac{l}{p}\right)\right)\left(\left(\frac{ql}{p}\right)\right), \quad \text{where} \quad ((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R}\setminus\mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Additionally, if we assume that K_2 is homologically trivial in M_2 (i.e. $o_2 = 1$), and we fix ℓ_2 as λ_2 (i.e. $k_2 = 0$), then (3) transforms into

$$\lambda_{\mathrm{W}}(M) = \lambda_{\mathrm{W}}(M_1) + \lambda_{\mathrm{W}}(M_2). \tag{4}$$

4.4. Walker–Lescop surgery formula. Now, we will analyze the manifold $M_1(K_1, p/q)$ obtained by p/q-surgery, where $p = o_2/\delta_2 > 0$ and $q = k_2/\delta_2$ (not necessarily positive). First notice (cf. [22, 1.3.4]) that $|H_1(M_1(K_1, p/q))| = p\cdot|H_1(M_1)|$. Using this, the surgery formula (T2) from [22, p. 13], and the identification $\lambda_{\mathrm{W}}(\cdot) = 2\lambda(\cdot)/|H_1(\cdot,\mathbb{Z})|$, one gets

$$\lambda_{\mathrm{W}}(M_1(K_1, p/q)) = \lambda_{\mathrm{W}}(M_1) + \mathrm{Cor}, \tag{5}$$

where the correction term Cor is

$$\mathrm{Cor} := \frac{q}{p}\cdot\frac{\Delta^L_{M_1}(K_1)''(1)}{|H_1(M_1)|} - \frac{p^2+1+q^2}{12pq} + \mathrm{sign}(q)\left(\frac{1}{4} + \boldsymbol{s}(p,q)\right).$$

Using (3), (5) and the reciprocity law for Dedekind sums (for $p > 0$) [45]:

$$\boldsymbol{s}(q,p) + \mathrm{sign}(q)\boldsymbol{s}(p,q) = -\frac{\mathrm{sign}(q)}{4} + \frac{p^2+1+q^2}{12pq},$$

one gets the following formula:

4.5. Theorem (Splicing formula for the Casson–Walker invariant). *Consider a splicing manifold M characterized by the data described in* 4.1. *Then*

$$\lambda_{\mathrm{W}}(M) = \lambda_{\mathrm{W}}(M_1) + \lambda_{\mathrm{W}}(M_2) + \frac{k_2}{o_2}\cdot\Delta^{\natural}_{M_1}(K_1)''(1).$$

4.6. The splicing property of the group $H_1(M,\mathbb{Z})$. In the next paragraphs we analyze the behavior of $H_1(\cdot,\mathbb{Z})$ under the splicing construction 4.1.

First notice that by excision, for any q, one has

$$H_q(M, M_2\setminus T(K_2)^\circ) = H_q(M_1\setminus T(K_1)^\circ, \partial T(K_1)) = H_q(M_1, K_1).$$

Therefore, the long exact sequence of the pair $(M, M_2 \setminus T(K_2)^\circ)$ reads

$$0 \to H_2(M_1, K_1) \xrightarrow{\partial_1} H_1(M_2 \setminus T(K_2)^\circ) \to H_1(M) \to H_1(M_1, K_1) \to 0.$$

Using the isomorphisms 3.3(3), ∂_1 can be identified with $\partial_1(1_{\mathbb{Z}}) = m_2$, hence $\operatorname{coker}(\partial_1) = H_1(M_2)$ (cf. 3.3(2)). Therefore, 3.3(3) gives the exact sequence

$$0 \to H_1(M_2) \xrightarrow{i} H_1(M) \xrightarrow{p} H_1(M_1) \to 0. \tag{6}$$

This exact sequence splits. Indeed, let $\bar{s}$ be the composition of $s_1 : H_1(M_1) \to H_1(M_1 \setminus T(K_1))$ (cf. 3.3) and $H_1(M_1 \setminus T(K_1)) \to H_1(M)$ (induced by the inclusion). Then $p \circ \bar{s} = 1$. In particular,

$$H_1(M) = \operatorname{Im}(i) \oplus \operatorname{Im}(\bar{s}) \approx H_1(M_2) \times H_1(M_1). \tag{7}$$

Notice that any $[K] \in H_1(M_1)$ can be represented (via s_1) by a representative K in $M_1 \setminus T(K_1)$ providing a class in $\operatorname{Tors} H_1(M_1 \setminus T(K_1))$. Write o for its order, and take a Seifert surface F, sitting in $M_1 \setminus T(K_1)$ with $\partial F = oK$. If $L \subset M_1 \setminus T(K_1)$ with $L \cap K = \emptyset$ then obviously $\mathrm{Lk}_M(K, L) = \mathrm{Lk}_{M_1}(K, L)$. Moreover, since F has no intersection points with any curve $L \in M_2 \setminus T(K_2)$, for such an L one gets

$$\mathrm{Lk}_M(K, L) = 0, \quad \text{hence} \quad b_M(\operatorname{Im}(\bar{s}), \operatorname{Im}(i)) = 0. \tag{8}$$

By a similar argument,

$$b_M(\bar{s}(x), y) = b_{M_1}(x, p(y)) \quad \text{for any } x \in H_1(M_1) \text{ and } y \in H_1(M). \tag{9}$$

In the next sections we need $\mathrm{Lk}_M(K, \cdot)$ for general $K \subset M_1 \setminus K_1$ which is not a torsion element in $H_1(M_1 \setminus K_1)$.

To compute this linking number consider another oriented knot $L \subset M_1 \setminus K_1$ with $K \cap L = \emptyset$. Our goal is to compare $\mathrm{Lk}_{M_1}(K, L)$ and $\mathrm{Lk}_M(K, L)$. Assume that the order of the class K in $H_1(M_1)$ is o. Let F_{oK} be a Seifert surface in M_1 with $\partial F_{oK} = oK$ which intersects L and K_1 transversally. It is clear that F_{oK} intersects L exactly in $o \cdot \mathrm{Lk}_{M_1}(K, L)$ points (counted with sign). On the other hand, it intersects K_1 in $o \cdot \mathrm{Lk}_{M_1}(K, K_1)$ points. For each intersection point with sign $\epsilon = \pm 1$, we cut out from F_{oK} the disc $F_{oK} \cap T(K_1)$, whose orientation depends on ϵ. Its boundary is ϵm_1 which, by the splicing identification, corresponds to $\epsilon \ell_2$ in $M_2 \setminus T(K_2)$. Using rational coefficients, $\ell_2 = (1/o_2)\lambda_2 - (k_2/o_2)m_2$. Some multiple of λ_2 can be extended to a surface in $M_2 \setminus K_2$ which clearly has no intersection with L. By splicing, m_2 is identified with λ_1 which has a Seifert surface in $M_1 \setminus K_1$ which intersects L in $\mathrm{Lk}_{M_1}(K_1, L)$ points. This shows that for $K, L \subset M_1 \setminus T(K_1)$,

$$\mathrm{Lk}_M(K, L) = \mathrm{Lk}_{M_1}(K, L) - \mathrm{Lk}_{M_1}(K, K_1) \cdot \mathrm{Lk}_{M_1}(L, K_1) \cdot k_2/o_2. \tag{10}$$

Assume now that $K \in M_2 \setminus T(K_2)$, and let o be the order of its homology class in $H_1(M_2)$. Assume that the Seifert surface F of oK intersects K_2 transversally in t_2 points. Then $F \cap \partial T(K_2) = t_2 m_2$, and after the splicing identification this becomes $t_2 \lambda_1$. Let F_1 be the Seifert surface of λ_1 in $M_1 \setminus T(K_1)^\circ$. Then (after some natural identifications) $F \setminus T(K_2)^\circ \amalg t_2 F_1$ is a Seifert surface of oK in M.

Therefore,

$$\mathrm{Lk}_M(K,L)=\mathrm{Lk}_{M_2}(K,K_2)\cdot \mathrm{Lk}_{M_1}(K_1,L)\qquad \text{for } K\subset M_2\setminus T(K_2) \text{ and } L\subset M_1\setminus T(K_1). \tag{11}$$

By a similar argument,

$$\mathrm{Lk}_M(K,L)=\mathrm{Lk}_{M_2}(K,L)\quad \text{for } K,L\subset M_2\setminus T(K_2). \tag{12}$$

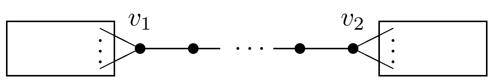
4.7. Splicing plumbing manifolds. Some of the results of this article about Reidemeister–Turaev torsion can be formulated and proved in the context of general (rational homology sphere) 3-manifolds, and arbitrary spin^c structures. Nevertheless, in this article we are mainly interested in algebraic links, therefore we restrict ourselves to plumbed manifolds.

This can be formulated in the second *working assumption*:

WA2: (M_i,K_i) $(i=1,2)$ *can be represented by a negative definite connected plumbing graph. Moreover, if M is the result of the splicing (satisfying* WA1*, cf.* 4.1*), then M can also be represented by a negative definite connected plumbing graph.*

Assume that the plumbing graphs $\Gamma(M_1,K_1)$ and $\Gamma(M_2,K_2)$ have the following schematic form (with $g_{v_1}=K_1$ and $g_{v_2}=K_2$):

Then it is not difficult to see that (a possible) plumbing graph $\Gamma(M)$ for M has the following form (where v_1 and v_2 are connected by a string):



If $\mathcal{V}_i$ $(i=1,2)$, respectively $\mathcal{V}$, represent the set of vertices of $\Gamma(M_i)$, resp. of $\Gamma(M)$, then $\mathcal{V}=\mathcal{V}_1\cup\mathcal{V}_2$ modulo some vertices with $\delta=2$. (In particular, in any formula like 3.7(15), $\mathcal{V}$ behaves like the union $\mathcal{V}_1\cup\mathcal{V}_2$.)

4.8. Remark. If one wants to compute $\mathcal{T}_{M,\sigma_{\mathrm{can}}}$ for such plumbed manifolds, then one can apply 3.7(15) and (16). For this, one has to analyze the supports of the characters, and the corresponding weights $w_v(u)$. These weights are closely related to the corresponding linking numbers $\mathrm{Lk}_M(g_u,g_v)$ (cf. 3.4(5)), hence the relations 4.6(10)-(11)-(12) are crucial. For characters of type $\chi\in\hat{p}(H_1(M_1)\hat{\ })$ (cf. 4.6(6) or the next proof) we have $u\in\mathcal{V}_1$, hence 4.6(10) should be applied for any $v\in\mathcal{V}_1$. But this is rather unpleasant due to the term $\mathrm{Lk}_{M_1}(g_u,K_1)\cdot\mathrm{Lk}_{M_1}(g_v,K_1)\cdot k_2/o_2$. The description is more transparent if either $H_1(M_1)=0$ or $k_2=0$.

Therefore, we will consider first these particular cases only. They, as guiding examples, already contain all the illuminating information and principles we need to proceed. For the link of $\{f(x,y)+z^n=0\}$, the splicing formula will be made very explicit in 7.13 (based on a detailed and complete classification of the characters and the regularization terms $\hat{P}$, which is rather involved).

4.9. Theorem (Splicing formulae for the Reidemeister–Turaev torsion). *Assume that M satisfy* WA1 (4.1) *and* WA2 (4.7)*. Then the following hold:*

(A) *Assume that $K_2 \subset M_2$ is also homologically trivial (i.e. $o_2 = 1$), and $\ell_2 = \lambda_2$ (i.e. $k_2 = 0$). Then*

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1) = \mathcal{T}_{M_1,\sigma_{\mathrm{can}}}(1) + \mathcal{T}_{M_2,\sigma_{\mathrm{can}}}(1).$$

(B) *Assume that M_1 is an integral homology sphere (i.e. $H_1(M_1) = 0$). Then*

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1) = \sum_{\chi_2 \in H_1(M_2)\hat{}\setminus\{1\}} \frac{\hat{\mathcal{T}}_{M_2,\sigma_{\mathrm{can}}}(\bar{\chi}_2)}{|H_1(M_2)|} \cdot \Delta_{M_1}(K_1)(\chi_2(K_2)).$$

In particular, if $K_2 \subset M_2$ is homologically trivial, then $\chi_2(K_2) = 1$ for any χ_2, hence by 3.6(9) *one gets*

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1) = \mathcal{T}_{M_2,\sigma_{\mathrm{can}}}(1) \quad (\textit{and evidently } \ \mathcal{T}_{M_1,\sigma_{\mathrm{can}}}(1) = 0).$$

This is true for any choice of ℓ_2, i.e. even if k_2 is non-zero.

Proof. The theorem is a consequence of 3.7 and 4.6. For this, we have to analyze the characters χ of $H_1(M)$. The dual of the exact sequence 4.6(6) is

$$0 \to H_1(M_1)\hat{} \xrightarrow{\hat{p}} H_1(M)\hat{} \xrightarrow{\hat{i}} H_1(M_2)\hat{} \to 0.$$

First, consider a character χ of $H_1(M)$ of the form $\chi = \hat{p}(\chi_1)$ for some $\chi_1 \in H_1(M_1)\hat{}$. Since any $\chi_1 \in H_1(M_1)\hat{}$ can be represented as $\exp(b_{M_1}(x,\cdot))$ for some $x \in H_1(M_1)$ (cf. 3.3), and $\hat{p}(b_{M_1}(x,\cdot)) = b_M(\bar{s}(x),\cdot)$ (cf. 4.6(9)), property 4.6(8) guarantees that $\chi(g_v) = 1$ for any $v \in \mathcal{V}_2$. In particular, for $\chi = \hat{p}(\chi_1)$ with $\chi_1 \in H_1(M_1)\hat{}\setminus\{1\}$, and for some $u \in \mathcal{V}_1$ with $\chi_1(g_u) \neq 1$ (which works for M as well), one gets

$$\hat{P}_{M,\chi,u}(t) = \prod_{v\in\mathcal{V}_1} (t^{w_v(u)}\chi_1(g_v) - 1)^{\bar{\delta}_v - 2} \cdot \prod_{v\in\mathcal{V}_2} (t^{w_v(u)} - 1)^{\bar{\delta}_v - 2}.$$

Here, for $v \in \mathcal{V}_i$, $\bar{\delta}_v$ means the number of adjacent edges of v in $\Gamma(M_i, K_i)$ $(i = 1, 2)$ (cf. 3.4).

By 4.6(11), for any $v \in \mathcal{V}_2$ one has

$$w_v(u) = o(u)\,\mathrm{Lk}_M(g_u, g_v) = o(u)\,\mathrm{Lk}_{M_1}(g_u, g_{v_1}) \cdot \mathrm{Lk}_{M_2}(g_{v_2}, g_v),$$

hence by 3.6(8),

$$\hat{P}_{M,\chi,u}(t) = \hat{P}_{M_1,\chi_1,u}(t) \cdot \Delta_{M_2}(g_{v_2})(t^{o(u)\,\mathrm{Lk}_{M_1}(g_u,g_{v_1})}).$$

Taking the limit $t \to 1$, and using 3.6(9), one gets

$$\hat{\mathcal{T}}_{M,\sigma_{\mathrm{can}}}(\bar{\chi}) = \hat{\mathcal{T}}_{M_1,\sigma_{\mathrm{can}}}(\bar{\chi}_1) \cdot |H_1(M_2)|. \tag{13}$$

Now, we prove (A). In this case M_1 and M_2 are symmetric, hence there is a term similar to (13) for characters $\chi_2 \in H_1(M_2)\hat{}$.

On the other hand, if $\chi = \chi_1\chi_2$ for two non-trivial characters $\chi_i \in H_1(M_i)\hat{}$ $(i = 1, 2)$, then one can show that $\hat{P}_{M,\chi,u}(t)$ has a root (of multiplicity at least two) at $t=1$, hence $\hat{\mathcal{T}}_{M,\sigma_{\mathrm{can}}}(\bar{\chi})=0$. Now, use 3.7(14) and $|H_1(M)|=|H_1(M_1)|\cdot|H_1(M_2)|$ (cf. 4.6(7)).

For (B), fix a non-trivial character $\chi_2\in H_1(M_2)\hat{}$. The relations in 4.6 guarantee that if we take $u\in\mathcal{V}_2$ with $\chi_2(g_u)\neq 1$ (considered as a property of M_2) then $\chi(g_u)\neq 1$ as well for $\chi=\hat{i}^{-1}(\chi_2)$. Moreover, for any $v\in\mathcal{V}_1$ one has $\chi(g_v)=\chi_2(K_2)^{\mathrm{Lk}_{M_1}(K_1,g_v)}$ (cf. 4.6(11)). Therefore

$$\begin{aligned}\hat{P}_{M,\chi,u}(t)&=\prod_{v\in\mathcal{V}_1}(t^{w_v(u)}\chi(g_v)-1)^{\bar{\delta}_v-2}\cdot\prod_{v\in\mathcal{V}_2}(t^{w_v(u)}\chi(g_v)-1)^{\bar{\delta}_v-2}\\&=\hat{P}_{M_2,\chi_2,u}(t)\cdot\Delta_{M_1}(g_{v_1})(t^{o(u)\,\mathrm{Lk}_{M_2}(g_u,g_{v_2})}\chi_2(K_2)).\end{aligned}$$ $\square$

4.10. Remarks. (1) A similar proof provides the following formula as well (which will be not used later). Assume that M satisfies WA1 and WA2, and $k_2=0$. Then

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)=\mathcal{T}_{M_1,\sigma_{\mathrm{can}}}(1)+\sum_{\chi_2\in H_1(M_2)\hat{}\setminus\{1\}}\frac{\hat{\mathcal{T}}_{M_2,\sigma_{\mathrm{can}}}(\bar{\chi}_2)}{|H_1(M_2)|}\cdot\Delta^H_{M_1}(K_1)(\chi_2(K_2)).$$

(2) The obstruction term (see also 5.5(1)) which measures the non-additivity of the Casson–Walker invariant (under the splicing assumption WA1) is given by $(k_2/o_2)\cdot\Delta^{\natural}_{M_1}(K_1)''(1)$ (cf. 4.5). On the other hand, if $H_1(M_1)=0$, then the obstruction term for the non-additivity of the Reidemeister–Turaev torsion (associated with σ_{can}) is

$$\sum_{\chi_2\in H_1(M_2)\hat{}\setminus\{1\}}\frac{\hat{\mathcal{T}}_{M_2,\sigma_{\mathrm{can}}}(\bar{\chi}_2)}{|H_1(M_2)|}\cdot(\Delta_{M_1}(K_1)(\chi_2(K_2))-1).$$

Notice that they "look rather different" (even under the extra assumption $H_1(M_1)=0$). In fact, even their nature is different: the first depends essentially on the choice of the parallel ℓ_2 (see the coefficient k_2 in its expression), while the second does not. In particular, one cannot really hope (in general) for the additivity of $\mathbf{sw}^{\mathrm{TCW}}_M(\sigma_{\mathrm{can}})$.

Therefore, it is really remarkable and surprising that in some of the geometric situations discussed in the next sections, this Seiberg–Witten invariant is additive (though the invariants λ_{W} and $\mathcal{T}_{\sigma_{\mathrm{can}}}(1)$ are not additive, their obstruction terms cancel each other).

5. The basic topological example

5.1. Recall that in our main applications (for algebraic singularities) the 3-manifolds involved are plumbed manifolds. In particular, they can be constructed inductively from Seifert manifolds by splicing (cf. also 3.5). The present section has a double role. First, we work out explicitly the splicing results obtained in the previous section for the case when M_2 is a Seifert manifold (with special Seifert invariants). On the other hand, the detailed study of this splicing formula provides us a better understanding of the subtlety of the behavior of the (modified)

Seiberg–Witten invariant with respect to splicing and cyclic covers. They will be formulated in some "almost-additivity" properties, where the non-additivity will be characterized by a new invariant $\mathcal{D}$ constructed from the Alexander polynomial of (M_1, K_1).

5.2. A short digression on Seifert invariants ([16, 39, 41])**.** Any Seifert fibration $\pi : M \to S^2$ is characterized by the set of (unnormalized) Seifert invariants $\{(\alpha_i, \beta_i)\}_{i=1}^{\nu}$ (with $\alpha_i > 0$ and $\gcd(\alpha_i, \beta_i) = 1$), and the *orbifold Euler number*

$$e := -\sum \beta_i/\alpha_i.$$

Above the collection of β_i's is not canonical, one can change each β_i within its residue class modulo α_i in such a way that the sum $e = -\sum_i(\beta_i/\alpha_i)$ is constant.

M is a link of singularity if and only if $e < 0$. A (possible) plumbing graph of M is a star-shaped graph with ν arms, corresponding to the number of Seifert invariants. The (absolute value of the) determinant of each leg is the corresponding α_i.

We will distinguish those vertices v of the graph which have $\delta_v \neq 2$. We will denote by $\bar{v}_0$ the central vertex, and by $\bar{v}_i$ the end-vertex of the i^{th} arm for all $1 \leq i \leq \nu$. Then $g_{\bar{v}_0}$ is exactly the class of the generic fiber of the Seifert fibration. Let $\alpha := \operatorname{lcm}(\alpha_1, \ldots, \alpha_\nu)$. The order of the group $H = H_1(M)$ and the order of the subgroup $\langle g_{\bar{v}_0} \rangle$ can be determined by (cf. [39])

$$|H| = \alpha_1 \cdots \alpha_\nu |e|, \qquad |\langle g_{\bar{v}_0} \rangle| = \alpha |e|.$$

In fact, if $|\langle g_{\bar{v}_0} \rangle| = 1$ (the situation we need), then the abelian group H (written additively) has the following presentation (see e.g. [39]):

$$H = \Big\langle g_{\bar{v}_1}, \ldots g_{\bar{v}_\nu} \;\Big|\; \sum_{i=1}^{\nu} \omega_i g_{\bar{v}_i} = 0,\ \alpha_i g_{\bar{v}_i} = 0 \text{ for all } i \Big\rangle.$$

5.3. The splicing component M_2 and the link K_2. Assume that M_2 is the link

$$\Sigma = \Sigma(p, a, n) := \{(x, y, z) \in \mathbb{C}^3 : x^p + y^a + z^n = 0,\ |x|^2 + |y|^2 + |z|^2 = 1\}$$

of the Brieskorn hypersurface singularity $X_2 := \{(x, y, z) \in \mathbb{C}^3 : x^p + y^a + z^n = 0\}$, where $\gcd(n, a) = 1$ and $\gcd(p, a) = 1$. Set $d := \gcd(n, p)$. Then M_2 is a rational homology sphere Seifert 3-manifold. The natural action of $S^1 = \{w \in \mathbb{C} : |w| = 1\}$ on Σ is given by $w * (x, y, z) = (xw^{na/d}, yw^{np/d}, zw^{pa/d})$. Clearly, the special orbits are given by the vanishing of the coordinates x, y, z. The properties (e.g. Seifert invariants) of Σ are classical well known facts (see e.g. [16, 39, 41]).

In this section we will slightly modify the construction of 4.1: we will fix a link with d connected components (instead of a knot), and perform splicing along each connected component.

This link $K_2 \subset M_2$ is given by the d special Seifert orbits given by the equation $\{y = 0\}$ in M_2. It is known that their Seifert invariant is $\alpha = a$. The components of K_2 will be denoted by $K_2^{(i)}$ $(i = 1, \ldots, d)$, and their tubular neighborhoods by $T(K_2^{(i)})$ as in 3.1.

Apart fram K_2, there are two more special orbits in $M_2 = \Sigma$, namely $Z := \{z = 0\}$ (with Seifert invariant $\alpha = n/d$) and $X := \{x = 0\}$ (with Seifert invariant $\alpha = p/d$). Moreover, let O be the generic fiber of the Seifert fibration of Σ (i.e. $O = g_{\bar{v}_0}$, and $X, Z, K_2^{(i)}$ are the elements $g_{\bar{v}_i}$, $i > 0$).

Here we collect some of the properties needed in the following: for (a)–(c) see [16, 39, 41] and the above discussion, (d) was determined in [36], but for the convenience of the reader the argument is essentially reproduced here in the proof of 5.7 (for more details, and for a more complete list of the relevant invariants, see e.g. [36, Section 6]).

(a) The Seifert invariants of Σ are: n/d, p/d, a, $a, \dots, a$ (a appearing d times, hence altogether there are $\nu = d + 2$ special fibers); these numbers also give (up to a sign) the determinants of the corresponding arms of the plumbing graph of Σ.
(b) The orbifold Euler characteristic is $e = -d^2/(npa)$.
(c) $|H| = a^{d-1}$ and $|\langle O\rangle| = 1$ (here use 5.2 and (b); or, in reverse order, first one can compute $|H|$ from an Alexander polynomial, then e from 5.2 and (a).)

In fact, using 5.2 and by an elementary arithmetical computation, one also sees that the link components $\{K_2^{(i)}\}_i$ generate $H_1(M_2)$; in fact $H_1(M_2)$ has the following presentation (written additively):

$$H_1(M_2) = \langle [K_2^{(1)}], \dots, [K_2^{(d)}] \mid a[K_2^{(i)}] = 0 \text{ for each } i, \\ \text{and } [K_2^{(1)}] + \cdots + [K_2^{(d)}] = 0\rangle. \tag{1}$$

In particular, $H_1(M_2) = \mathbb{Z}_a^{d-1}$.

(d) $\mathcal{T}_{M_2,\sigma_{\mathrm{can}}}(1) = \frac{np}{24da}(d-1)(a^2-1)$ (a fact which also follows from $(*)$ in the proof of 5.7).

5.3.1. Lemma. *One has the following linking numbers:*

(a) $\mathrm{Lk}_{M_2}(K_2^{(i)}, K_2^{(j)}) = np/(d^2 a)$ *for any* $i \neq j$*;*
(b) $\mathrm{Lk}_{M_2}(K_2^{(i)}, O) = np/d^2$*;* $\mathrm{Lk}_{M_2}(K_2^{(i)}, Z) = p/d$ *for any* i*;*
(c) $\mathrm{Lk}_{M_2}(O, O) = npa/d^2$ *and* $\mathrm{Lk}_{M_2}(O, Z) = pa/d$*;*
(d) $\mathrm{Lk}_{M_2}(X, Z) = a$*.*

Proof. This follows from 3.4.2(5) and [36, 5.5(1)]; but for the convenience of the reader we provide an argument.

We apply 3.4.2(5) for two different $g_{\bar{v}_i}$ $(0 \le i \le \nu)$; cf. 5.2 for the notation. Namely, $\mathrm{Lk}_\Sigma(g_{\bar{v}_i}, g_{\bar{v}_i}) = -I^{-1}_{g_{\bar{v}_i} g_{\bar{v}_i}}$. On the other hand, since I is the intersection matrix associated with a *tree*, I^{-1}_{vw} can be computed as follows (a fact which can be verified easily by the reader): Delete all the vertices situated on the unique geodesic path connecting u and v, including u and v, and all the edges adjacent to these vertices. Let this graph be Γ_{uv}. Then $-I^{-1}_{uv} = |\det(\Gamma_{uv})|/|H|$. For example, in (a), Γ_{uv} contains $\nu = d$ legs with corresponding Seifert invariants n/d, p/d, and the others with a. Hence, $|\det(\Gamma_{uv})| = a^{d-2}np/d^2$. This divided by $|H| = a^{d-1}$ provides (a). $\square$

Notice that $K_2 \subset M_2$ is fibrable. Indeed, $K_2 = \{y = 0\}$ is the link associated with the algebraic germ $y : (X_2, 0) \to (\mathbb{C}, 0)$, hence one can take its Milnor fibration. Let F be the fiber with $\partial F = K_2$ (equivalently, take any minimal Seifert surface F with $\partial F = K_2$). Then for each $i = 1, \dots, d$, we define the *parallel* $\ell_2^{(i)}$ in $\partial T(K_2^{(i)})$ by $F \cap \partial T(K_2^{(i)})$.

Let $\lambda_2^{(i)}$ be the *longitude* of $K_2^{(i)} \subset M_2$, and consider the invariants $o_2^{(i)}$, $k_2^{(i)}$, etc. as in 3.1 with the corresponding sub- and superscripts added.

5.3.2. Lemma. *For each $i = 1, \dots, d$ one has*

$$o_2^{(i)} = a \quad \textit{and} \quad k_2^{(i)} = \frac{np(d-1)}{d^2}.$$

Proof. The first identity is clear (cf. 5.3(1)). For the second, notice that (cf. 3.3(1))

$$-k_2^{(i)}/o_2^{(i)} = \langle \ell_2^{(i)}, \lambda_2^{(i)} \rangle / o_2^{(i)} = \mathrm{Lk}_{M_2}(\ell_2^{(i)}, K_2^{(i)}).$$

Moreover,

$$\mathrm{Lk}_{M_2}\Big(\sum_j \ell_2^{(j)}, K_2^{(i)}\Big) = 0,$$

hence

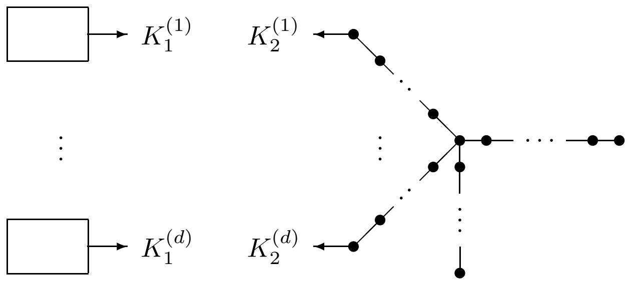
$$k_2^{(i)}/a = \sum_{j \neq i} \mathrm{Lk}_{M_2}(K_2^{(j)}, K_2^{(i)}).$$

Then use 5.3.1(a). $\square$

5.4. The manifold M. Next, we consider d manifolds $M_1^{(i)}$ with knots $K_1^{(i)} \subset M_1^{(i)}$ $(i = 1, \dots, d)$, each satisfying the assumption WA1 (i.e. $o_1^{(i)} = 1$, $\ell_1^{(i)} = \lambda_1^{(i)}$, and $k_1^{(i)} = 0$, cf. 4.1). Then, for each $i = 1, \dots, d$, we consider the splicing identification of $\partial T(K_2^{(i)})$ with $-\partial T(K_1^{(i)})$ (similarly to 4.1(1)):

$$A^{(i)}(m_2^{(i)}) = \lambda_1^{(i)} \quad \text{and} \quad A^{(i)}(\ell_2^{(i)}) = m_1^{(i)}.$$

Schematically:



In what follows we denote by WA1$'$ *the assumption which guarantees that the manifold M is constructed by this splicing procedure. Moreover,* WA2$'$ *guarantees that all the* 3*-manifolds involved have plumbing representations by negative definite connected plumbing graphs.*

Here some comments are in order.

(1) Assume that for some i, $M_1^{(i)} = S^3$, and $K_1^{(i)}$ is the unknot S^1 in S^3. Then performing splicing along $K_2^{(i)}$ with (S^3, S^1) is equivalent to putting back $T(K_2^{(i)})$ unmodified, hence it has no effect. In this case, one also has $\Delta_{S^3}(S^1)(t) \equiv \Delta^{\natural}_{S^3}(S^1)(t) \equiv 1$.

(2) Assume that we have already performed the splicing along the link-components $K_2^{(i)}$ for $i \leq k-1$, but not along the other ones. Denote the result of this partial modification by $M^{(k-1)}$. Consider $K_2^{(k)}$ in $M^{(k-1)}$ (in a natural way). Then all the invariants (e.g. $o_2^{(k)}$, $\lambda_2^{(k)}$, $k_2^{(k)}$, etc.) associated with $K_2^{(k)}$ in M_2 or in $M^{(k-1)}$ are the same. (This follows from the discussion in 4.6, and basically, it is a consequence of WA1$'$.)

In particular, performing splicing at place i does not affect the splicing data of place j ($j \neq i$). Therefore, using induction, the computation of the invariants can be easily reduced to the formulae established in the previous section.

5.5. Definitions/Notations. (1) In order to simplify the exposition, for any 3-manifold invariant $\mathcal{I}$, we write

$$\mathcal{O}(\mathcal{I}) := \mathcal{I}(M) - \mathcal{I}(M_2) - \sum_{i=1}^{d} \mathcal{I}(M_1^{(i)})$$

for the "additivity obstruction" of $\mathcal{I}$ (with respect to the splicing construction WA1$'$). For example, using 4.6 and 5.4(2) one has $\mathcal{O}(\log|H_1(\cdot)|) = 0$. Moreover, in all our Alexander invariant notations (e.g. in $\Delta_{M_1^{(i)}}(K_1^{(i)})(t)$), we omit the link $K_1^{(i)}$ (e.g. we simply write $\Delta_{M_1^{(i)}}(t)$).

When comparing $\mathcal{O}(\mathcal{T}_{\cdot,\sigma_{\mathrm{can}}})$ with $\mathcal{O}(\lambda_{\mathrm{W}}(\cdot))$, the following terminology will be helpful.

(2) For any set of integers $c_1, \ldots, c_r$, define

$$\mathcal{D}(c_1, \ldots, c_r) := \sum_{i,j=1}^{r} c_i c_j \min(i,j) - \sum_{i=1}^{r} i c_i.$$

(3) Define $\mathcal{D}(\Delta^{\natural}(t))$ to be $\mathcal{D}(c_1, \ldots, c_r)$ for any symmetric polynomial

$$\Delta^{\natural}(t) = 1 + \sum_{i=1}^{r} c_i (t^i + t^{-i} - 2) \quad \text{(for some } c_i \in \mathbb{Z}).$$

(4) A set $\{c_i\}_{i \in I}$ ($I \subset \mathbb{N}$) is called *alternating* if $c_i \in \{-1, 0, +1\}$ for any $i \in I$; and if $c_i \neq 0$ then $c_i = (-1)^{n_i}$, where $n_i = \#\{j : j > i \text{ and } c_j \neq 0\}$.

5.6. Corollary. *Assume that M satisfies* WA1$'$. *Then*

$$\mathcal{O}(\lambda_{\mathrm{W}})=\frac{np(d-1)}{ad^2}\sum_{i=1}^{d}(\Delta^{\natural}_{M_1^{(i)}})''(1).$$

Proof. Use 4.5, 5.3.2 and 5.4(2). □

5.7. Corollary. *Assume that M satisfies* WA1$'$ *and* WA2$'$, *and $M_1^{(i)}$ is an integral homology sphere for any i. Identify $\mathbb{Z}_a:=\{\xi\in\mathbb{C}:\xi^a=1\}$ and write $\mathbb{Z}_a^*:=\mathbb{Z}_a\setminus\{1\}$. Then*

$$\mathcal{O}(\mathcal{T}_{\cdot,\sigma_{\mathrm{can}}}(1))=\frac{np}{ad^2}\sum_{i\neq j}\sum_{\xi\in\mathbb{Z}_a^*}\frac{\Delta_{M_1^{(i)}}(\xi)\cdot\Delta_{M_1^{(j)}}(\bar\xi)-1}{(\xi-1)(\bar\xi-1)}.$$

Proof. We recall first how one computes the torsion for the manifold M_2 (cf. 3.7, for the original version see [36]). The point is (see the last sentence of 3.7) that for any character $\chi\in H_1(M_2)\hat{}\setminus\{1\}$, one can choose the central vertex of the star-shaped graph for the vertex u in order to generate the weights in $\hat P$. Then by 5.3.1 (see also [36]) one gets

$$\hat P_{M_2,\chi,u}(t)=\frac{(t^{\alpha}-1)^d}{(t^{d\alpha/n}-1)(t^{d\alpha/p}-1)\prod_i(t^{\alpha/a}\chi(K_2^{(i)})-1)},$$

where $\alpha:=npa/d^2$. The limit of this expression as $t\to 1$ always exists. In particular $\#\{i:\chi(K_2^{(i)})\neq 1\}\geq 2$ (cf. also 5.3(1)). If this number is strictly greater than 2, then the limit is zero. If $\chi(K_2^{(i)})\neq 1$ for exactly two indices i and j, then by 5.3(1) clearly $\chi(K_2^{(i)})\chi(K_2^{(j)})=1$. Since there are exactly $d(d-1)/2$ such pairs, one gets

$$\begin{aligned}\mathcal{T}_{M_2,\sigma_{\mathrm{can}}}(1)&=\frac{1}{|H_1(M_2)|}\lim_{t\to1}\frac{(t^{\alpha}-1)^d}{(t^{d\alpha/n}-1)(t^{d\alpha/p}-1)(t^{\alpha/a}-1)^{d-2}}\\&\quad\cdot\sum_{i\neq j}\sum_{\xi\in\mathbb{Z}_a^*}\frac{1}{(\xi-1)(\bar\xi-1)}\\&=\frac{np}{ad^2}\sum_{i\neq j}\sum_{\xi\in\mathbb{Z}_a^*}\frac{1}{(\xi-1)(\bar\xi-1)}\\&=\frac{np}{ad^2}\frac{d(d-1)}{2}\sum_{\xi\in\mathbb{Z}_a^*}\frac{1}{(\xi-1)(\bar\xi-1)}=\frac{np(d-1)(a^2-1)}{24ad},\end{aligned}\qquad(*)$$

since

$$\sum_{\xi\in\mathbb{Z}_a^*}\frac{1}{(\xi-1)(\bar\xi-1)}=\frac{a^2-1}{12}.\qquad(**)$$

Consider now the manifold M. Then using 4.9(B) (and/or its proof), by the same argument as above, one obtains

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)=\frac{np}{ad^2}\sum_{i\neq j}\sum_{\xi\in\mathbb{Z}_a^*}\frac{\Delta_{M_1^{(i)}}(\xi)\cdot\Delta_{M_1^{(j)}}(\bar\xi)}{(\xi-1)(\bar\xi-1)}.\qquad(*{*}*)$$

Finally, taking the difference between $(*{*}*)$ and $(*)$ one gets the result. □

5.8. Example/Discussion. Assume that M satisfies WA1$'$ and WA2$'$, and additionally $(M_1^{(i)}, K_1^{(i)}) = (M_1, K_1)$ for some *integral homology sphere* M_1. Then

$$\mathcal{O}(\lambda_{\mathrm{W}})/2 = \frac{np(d-1)}{2ad}(\Delta^{\natural}_{M_1})''(1),$$
$$\mathcal{O}(\mathcal{T}_{\cdot,\sigma_{\mathrm{can}}}(1)) = \frac{np(d-1)}{2ad} \cdot \sum_{\xi \in \mathbb{Z}_a^*} \frac{\Delta_{M_1}(\xi)\cdot \Delta_{M_1}(\bar{\xi}) - 1}{(\xi - 1)(\bar{\xi} - 1)}.$$

Recall that the modified Seiberg–Witten invariant $\mathbf{sw}^{\mathrm{TCW}}_M(\sigma_{\mathrm{can}})$ is defined by the difference $\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1) - \lambda_{\mathrm{W}}(M)/2$ (cf. 2.4.2(4)). Notice the remarkable fact that in the above expressions the coefficients before the Alexander invariants have become the same. Hence

$$\mathcal{O}(\mathbf{sw}^{\mathrm{TCW}}_{\cdot}(\sigma_{\mathrm{can}})) = \frac{np(d-1)}{2ad}\mathcal{D}_a, \tag{$\mathcal{D}$}$$

where

$$\mathcal{D}_a := \sum_{\xi \in \mathbb{Z}_a^*} \frac{\Delta_{M_1}(\xi)\cdot \Delta_{M_1}(\bar{\xi}) - 1}{(\xi - 1)(\bar{\xi} - 1)} - (\Delta^{\natural}_{M_1})''(1).$$

Recall that $\Delta^{\natural}_{M_1}(t)$ is a symmetric polynomial (cf. [22, 2.3.1]) with $\Delta^{\natural}_{M_1}(1) = 1$ (cf. 3.6(6)). In what follows we will compute $\mathcal{D}_a$ explicitly, provided that a is sufficiently large, in terms of the coefficients $\{c_i\}_{i=1}^r$ of $\Delta^{\natural}_{M_1}(t)$ (cf. 5.5(3)).

The contribution $(\Delta^{\natural}_{M_1})''(1)$ is easy: it is $\sum_{i=1}^r 2i^2 c_i$. By 3.6(9), $\det(V) = 1$, hence by 3.6(7), $\Delta_{M_1}(t) = t^r \Delta^{\natural}_{M_1}(t)$. In particular, $\Delta_{M_1}(\xi)\cdot\Delta_{M_1}(\bar{\xi}) = \Delta^{\natural}_{M_1}(\xi)\cdot \Delta^{\natural}_{M_1}(\bar{\xi})$. Then write

$$\frac{\Delta^{\natural}_{M_1}(t)}{1-t} = \frac{1}{1-t} - \sum_{i=1}^r c_i(1 + t + \cdots + t^{i-1}) + \sum_{i=1}^r c_i(t^{-1} + \cdots + t^{-i}).$$

An elementary computation using the identity $\sum_{\xi\in\mathbb{Z}_a^*} 1/(1-\xi) = (a-1)/2$ gives

$$\sum_{\xi \in \mathbb{Z}_a^*} \frac{\Delta^{\natural}_{M_1}(\xi)\cdot \Delta^{\natural}_{M_1}(\bar{\xi}) - 1}{(1-\xi)(1-\bar{\xi})} = \sum_{i=1}^r 2i^2 c_i + 2a\cdot \mathcal{D}(\Delta^{\natural}_{M_1}(t)), \quad \text{provided that } a \geq 2r.$$

In particular, if $a \geq 2r$, then $\mathcal{D}_a = 2a\cdot\mathcal{D}(\Delta^{\natural}_{M_1}(t))$. Hence

$$\mathcal{O}(\mathbf{sw}^{\mathrm{TCW}}_{\cdot}(\sigma_{\mathrm{can}})) = \frac{np(d-1)}{d}\mathcal{D}(\Delta^{\natural}_{M_1}(t)).$$

This raises the following natural question: for what Alexander polynomials is the expression $\mathcal{D}(\Delta^{\natural}_{M_1}(t))$ zero? The next lemma provides such an example (the proof is elementary and it is left to the reader).

5.9. Lemma. *If $\{c_i\}_{i=1}^r$ is an alternating set then $\mathcal{D}(c_1, \ldots, c_r) = 0$.*

The above discussions have the following topological consequence:

Consider a knot $L_1 \subset K_1$, and fix two relatively prime positive integers p and a. Let K be the primitive simple curve in $\partial T(L_1)$ with homology class $am_1 + p\lambda_1$. Let M denote the n-cyclic cover of N_1 branched along K. Set $d := \gcd(n, p)$, and let M_1 be the (n/d)-cyclic cover of N_1 branched along L_1. Denote by K_1 the preimage of L_1 via this cover. Finally, let $\Delta^\natural_{M_1}(t)$ be the normalized Alexander polynomial of (M_1, K_1).

5.10. Corollary. *Consider the above data. Additionally, assume that* WA2′ *is satisfied and M is a rational homology sphere. Then:*

(A) $(d-1)\cdot(\gcd(n,a)-1) = 0$.

(B) *If $d = 1$, then*

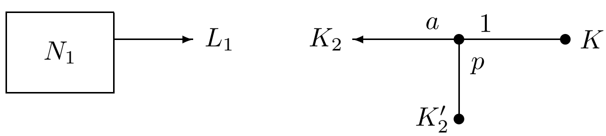
$$\mathbf{sw}^{\mathrm{TCW}}_M(\sigma_{\mathrm{can}}) = \mathbf{sw}^{\mathrm{TCW}}_{M_1}(\sigma_{\mathrm{can}}) + \mathbf{sw}^{\mathrm{TCW}}_{\Sigma(p,a,n)}(\sigma_{\mathrm{can}}).$$

(C) *If $\gcd(n,a) = 1$, $a \geq \deg \Delta_{M_1}(t)$, and M_1 is an integral homology sphere, then*

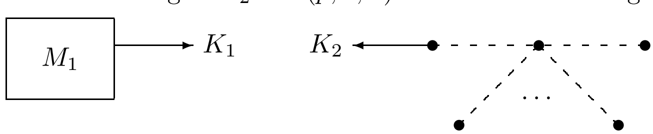
$$\mathbf{sw}^{\mathrm{TCW}}_M(\sigma_{\mathrm{can}}) = d\cdot \mathbf{sw}^{\mathrm{TCW}}_{M_1}(\sigma_{\mathrm{can}}) + \mathbf{sw}^{\mathrm{TCW}}_{\Sigma(p,a,n)}(\sigma_{\mathrm{can}}) + \frac{np(d-1)}{d}\mathcal{D}(\Delta^\natural_{M_1}(t)).$$

If the coefficients of $\Delta^\natural_{M_1}(t)$ form an alternating set, then $\mathcal{D}(\Delta^\natural_{M_1}(t)) = 0$.

Proof. Consider the following schematic splicing of splice diagrams (cf. 3.5):



The result of the splicing can be identified with N_1 (and under this identification L_1 is identified with K_2'). The advantage of this splicing representation is that it emphasizes the position of the knot K in the Seifert component $\Sigma(p, a, 1)$. If M is a rational homology sphere then the n-cyclic cover of $\Sigma(p, a, 1)$ branched along K (which is $\Sigma(p, a, n)$) should be a rational homology sphere, hence (A) follows. If $d = 1$ then M has a splice decomposition of the following schematic plumbing diagrams (where on the right $M_2 = \Sigma(p, a, n)$ and the dots mean $\gcd(n, a)$ arms):



Here $o_1 = o_2 = 1$ and $k_1 = k_2 = 0$, and A is the identification $\lambda_2 = m_1$, $m_2 = \lambda_1$. Therefore, part (B) follows from 4.3(4) and 4.9(A). The last case corresponds exactly to the situation treated in 5.8. □

5.11. Remarks. (1) Our final goal (see the following sections) is to prove the additivity result $\mathcal{O}(\mathbf{sw}^{\mathrm{TCW}}(\sigma_{\mathrm{can}})) = 0$ for any (M_1, K_1), which can be represented as a cyclic cover of S^3 branched along the link $K_f \subset S^3$ of an arbitrary irreducible (complex) plane curve singularity (even if M_1 is not an integral homology sphere), provided that a is sufficiently large. This means that from the above Corollary, part (C), we will need to eliminate the assumption about the vanishing of $H_1(M_1)$. The assumption about a will follow from the special property 6.1(6)) of irreducible plane curve singularities (cf. also 6.4, especially (7)).

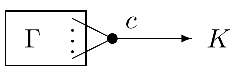
The proof of the vanishing of the $\mathcal{D}$-correction will take up most of the last section of the paper. It relies in a crucial manner on the alternating nature of the Alexander polynomial $\Delta^{\natural}_{S^3}(K_f)(t)$ of any irreducible plane curve singularity f, a fact which will be established in Proposition 6.2.

(2) It is really interesting and remarkable that the behavior of the modified Seiberg–Witten invariant with respect to (some) splicing and cyclic covers (constructions which are basically topological in nature) definitely gives preference to the Alexander polynomials of some algebraic links. The authors hope that a better understanding of this phenomenon would lead to some deep properties of the Seiberg–Witten invariant.

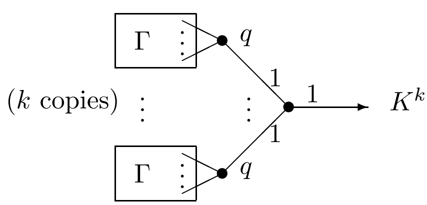
5.12. Example. In general, in 5.10, the invariant $\mathcal{D}(\Delta^{\natural}_{M_1}(K_1)(t))$ does not vanish. In order to see this, start for example with a pair (N_1, L_1) with non-zero $\mathcal{D}(\Delta^{\natural}_{N_1}(L_1)(t))$, and consider the case when $d\,|\,n$. (If $d \neq 1$, then the coefficient of $\mathcal{D}(\Delta^{\natural}_{N_1}(L_1)(t))$ in 5.10(C) will be non-zero as well.)

Next, we show how one can construct a pair (N, L) which satisfies WA2, $H_1(N) = 0$, but $\mathcal{D}(\Delta^{\natural}_N(L)(t)) \neq 0$. First, we notice the following fact.

If the Alexander polynomial $\Delta^{\natural}_M(K)(t)$ is realizable for some pair (M, K) (satisfying WA2 and $H_1(M) = 0$), then the k-power of this polynomial is also realizable for some pair (M^k, K^k) (satisfying WA2 and $H_1(M^k) = 0$). Indeed, assume that (M, K) has a schematic splice diagram of the following form:



Then let (M^k, K^k) be given by the following schematic splice diagram:

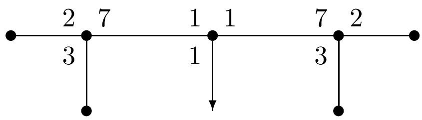


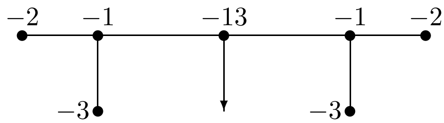
Here, we take q sufficiently large (in order to ensure that the new edges will also satisfy the algebraicity condition [11, 9.4]), and also q should be relatively prime to some integers which appear as decorations of Γ (see [loc. cit.]). Obviously, by construction, M^k is an integral homology sphere. Then, by [loc. cit., 12.1], one can easily verify that

$$\Delta^{\natural}_{M^k}(K^k)(t) = \Delta^{\natural}_M(K)(t)^k.$$

For example, if $(M, K) = (S^3, K_f)$, where K_f is the (2,3)-torus knot (or, equivalently, the knot of the plane curve singularity $f = x^2 + y^3$, cf. 6.1), then $\Delta^{\natural}_{S^3}(K_f)(t)$

$= t-1+1/t$ (see 6.1(5)). Now, if we take $k=2$ and $q=7$ then (M^2, K^2) has the following splice, respectively plumbing graph:





Then (M^2, K^2) is algebraic, $H_1(M^2) = 0$. But $\Delta^{\natural}_{M^2}(K^2)(t) = (t-1+1/t)^2$, whose coefficients are not alternating. In fact, $r=2$, $c_1=-2$ and $c_2=1$; in particular $\mathcal{D}(\Delta^{\natural}_{M^2}(K^2)(t)) = 2$.

We end this section with the following property which is needed in the last section.

5.13. Lemma. *Assume that M satisfies* WA1$'$ *and* WA2$'$ *with $(M_1^{(i)}, K_1^{(i)}) = (M_1, K_1)$. Let Γ denote the plumbing graph of M. Let v be the central vertex of M_2 considered in M, and let Γ_- be the connected component of $\Gamma\setminus\{v\}$ which contains the vertices of $M_1^{(1)}$. Then $|\det(\Gamma_-)| = a\cdot|H_1(M_1)|$.*

Proof. If I denotes the intersection matrix of M, then

$$-I^{-1}_{vv} \overset{3.4(5)}{=} \mathrm{Lk}_M(g_v, g_v) \overset{4.6(12)}{=} \mathrm{Lk}_{\Sigma(p,a,n)}(O,O) \overset{5.3.1(c)}{=} npa/d^2.$$

On the other hand, I^{-1}_{vv} can be computed from the determinants of the components of $\Gamma\setminus\{v\}$, hence (cf. also 5.3)

$$-|H_1(M)|\cdot I^{-1}_{vv} = |\det(\Gamma_-)|^d \cdot pn/d^2.$$

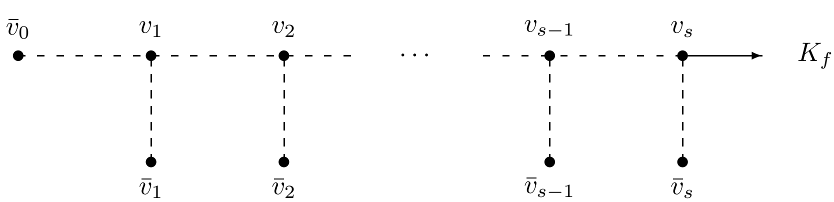
By 4.6(7) and 5.3, $|H_1(M)| = |H_1(M_1)|^d\cdot a^{d-1}$, hence the result follows. $\square$

6. Properties of irreducible plane curve singularities

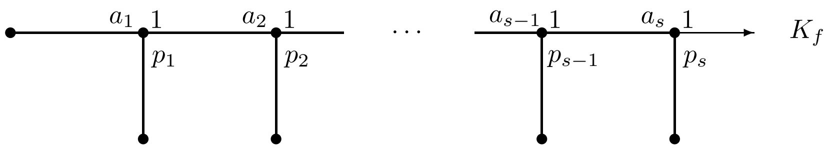
6.1. The topology of an irreducible plane curve singularity. Consider an irreducible plane curve singularity $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ with Newton pairs $\{(p_k,q_k)\}_{k=1}^s$ (cf. [11, p. 49]). Clearly $\gcd(p_k,q_k)=1$ and $p_k,q_k\geq 2$. Define the integers $\{a_k\}_{k=1}^s$ by

$$a_1 = q_1 \quad \text{and} \quad a_{k+1} = q_{k+1} + p_{k+1}p_k a_k \quad \text{if } k\geq 1. \tag{1}$$

Then again, $\gcd(p_k,a_k)=1$ for any k. The minimal (good) embedded resolution graph of the pair $(\mathbb{C}^2,\{f=0\})$ has the following schematic form:



This can be identified with the plumbing graph $\Gamma(S^3, K_f)$, where K_f is the link of f (with only one component) in the Milnor sphere S^3. In the above diagram we emphasized only those vertices $\{\bar v_k\}_{k=0}^s$ and $\{v_k\}_{k=1}^s$ which have $\bar\delta \neq 2$. We denote the set of those vertices by $\mathcal{V}^*$. The dashed line between two such vertices replaces a string $-\!\bullet\!-\!\!-\!\bullet\!- \cdots -\!\bullet\!-$. In our discussion the corresponding Euler numbers will not be important (the interested reader can find the complete description of the graph in [11, Section 22], or in [32]). The above numerical data $\{(p_k, a_k)\}_k$ and the set $\mathcal{V}^*$ of vertices are codified in the splice diagram (cf. [11]):



The knot $K_f \subset S^3$ defines a set $\{w_v(u)\}_{v\in\mathcal{V}^*}$ of weights as in 3.4 (where $K_f = g_u$ and $o(u) = 1$). In terms of the resolution, $w_v(u)$ (or simply w_v) is exactly the vanishing order (multiplicity) of $f \circ \pi$ along the exceptional divisor codified by v, where π denotes the resolution map. Then (cf. [11, Section 10])

$$\begin{aligned} w_{v_k} &= a_k p_k p_{k+1} \cdots p_s \qquad \text{for } 1 \le k \le s; \\ w_{\bar v_0} &= p_1 p_2 \cdots p_s; \\ w_{\bar v_k} &= a_k p_{k+1} \cdots p_s \qquad \text{for } 1 \le k \le s. \end{aligned} \tag{2}$$

Recall that the characteristic polynomial $\Delta(f)(t) := \Delta_{S^3}(K_f)(t)$ of the monodromy acting on the first homology of the Milnor fiber of f is given by A'Campo's formula 3.6(8):

$$\frac{\Delta_{S^3}(K_f)(t)}{t-1} = \prod_{v\in\mathcal{V}^*} (t^{w_v} - 1)^{\bar\delta_v - 2}. \tag{3}$$

In inductive proofs and constructions (over the number of Newton pairs of f), it is convenient to use the notation $f_{(l)}$ for an irreducible plane curve singularity with Newton pairs $\{(p_k, q_k)\}_{k=1}^l$, where $1 \le l \le s$. Evidently $f_{(s)} = f$, and $f_{(1)}$ can be taken as the Brieskorn singularity $x^{p_1} + y^{a_1}$. We write $\Delta(f_{(l)})$ for the characteristic polynomial associated with $f_{(l)}$. Then from (2) and (3) one gets

$$\Delta(f_{(l)})(t) = \Delta(x^{p_l} + y^{a_l})(t) \cdot \Delta(f_{(l-1)})(t^{p_l}) \qquad \text{for } l \ge 2, \tag{4}$$

where

$$\Delta(x^p + y^a)(t) = \frac{(t^{pa} - 1)(t-1)}{(t^p - 1)(t^a - 1)}. \tag{5}$$

By induction, using the identities (1), one can prove (see e.g. [31, 5.2])

$$a_l > p_l \cdot \deg \Delta(f_{(l-1)}) \qquad \text{for any } l \ge 2. \tag{6}$$

6.2. Proposition. (a) *$\Delta(f)(0) = \Delta(f)(1) = 1$, and the degree of $\Delta(f)(t)$ is even (say $2r$).*

(b) *If $\Delta(f)(t) = \sum_{i=0}^{2r} b_i t^i$, then the set $\{b_i\}_{i=0}^{2r}$ is alternating (cf.* 5.5(4)*).*

(c) *The coefficients $\{c_i\}_{i=1}^r$ of $\Delta^\natural(f)(t) := t^{-r}\Delta(f)(t)$ (cf.* 5.5(3)*) are alternating as well.*

Proof. (a) is clear from (4) and (5), and (c) follows easily from (b). We will prove (b) by induction over s. For each $1 \leq l \leq s$ we verify that there exist

(i) a_l-residue classes $\{r_1, \ldots, r_t\} \subset \{1, 2, \ldots, a_l - 1\}$ (where t may depend on l);
(ii) integers $n_1, \ldots, n_t \in \mathbb{N}$ such that

$$\Delta(f_{(l)})(t) = 1 + \sum_{i=1}^{t} \sum_{j=0}^{n_i} t^{r_i + ja_l}(t-1).$$

It is clear that the coefficients of a polynomial of this form are alternating.

Let us start with the case $l = 1$. Write $(p_1, a_1) = (p, a)$. Then (cf. (5))

$$\Delta(x^p + y^a)(t) = (t^{p(a-1)} + \cdots + t^p + 1)/Q(t), \qquad Q(t) := t^{a-1} + \cdots + t + 1.$$

For each $i = 0, 1, \ldots, a-1$ write pi in the form $x_i a + r_i$ for some $r_i \in \{0, \ldots, a-1\}$. Since $\gcd(p, a) = 1$, $\{r_i\}_i = \{0, \ldots, a-1\}$, and $p \,|\, r_i$ if and only if $x_i = 0$. Therefore,

$$\sum_{i=0}^{a-1} t^{pi} = Q(t) + \sum_{i:\, p \nmid r_i} t^{r_i}(t^{x_i a} - 1) = Q(t) \cdot \left[1 + \sum_{i:\, p \nmid r_i} \sum_{j=0}^{x_i - 1} t^{r_i + ja}(t-1) \right].$$

Now we prove that $\Delta(f_{(l)}(t)$ has a similar form. By the inductive step, assume that

$$\Delta(f_{(l-1)})(t) = 1 + \sum_{i=1}^{t} \sum_{j=0}^{n_i} t^{r_i + ja_{l-1}}(t-1).$$

Then, using 6.1(4) and (5), for $\Delta(f_{(l)})(t)$ one gets

$$\frac{(t^{p_l a_l} - 1)(t-1)}{(t^{p_l} - 1)(t^{a_l} - 1)} + \sum_{i=1}^{t} \sum_{j=0}^{n_i} t^{(r_i + ja_{l-1})p_l} \frac{(t^{p_l a_l} - 1)(t-1)}{t^{a_l} - 1}.$$

Let $\{s_j\}_{j=0}^{a_l - 1}$ be the set of a_l-residue classes. Then (using the result of case $l = 1$) the above expression reads

$$1 + \sum_{j:\, p_l \nmid s_j} \sum_{k=0}^{x_j - 1} t^{s_j + ka_l}(t-1) + \sum_{i=1}^{t} \sum_{j=0}^{n_i} \sum_{k=0}^{p_l - 1} t^{(r_i + ja_{l-1})p_l + ka_l}(t-1).$$

Notice that 6.1(6) guarantees that for each i and j one has $(r_i + ja_{l-1})p_l < a_l$, hence these numbers can be considered as (non-zero) a_l-residue classes. Moreover, they are all different from the residue classes $\{s_j : p_l \nmid s_j\}$ since they are all divisible by p_l. $\square$

The "alternating property" of the coefficients of the Alexander polynomial of any irreducible plane curve singularity will be crucial in the computation of the Reidemeister–Turaev torsion of $\{f + z^n = 0\}$. The key algebraic fact is summarized in the next property:

6.3. Algebraic Lemma. *In the expressions below, t is a free variable and a is a positive integer. $\mathbb{Z}_a$ is identified with the a-roots of unity. Assume that the coefficients of a polynomial $\Delta(t) \in \mathbb{Z}[t]$ form an alternating set, $\Delta(1) = 1$, and $a \geq \deg \Delta$. Then:*

(a) *For an arbitrary complex number A one has*

$$\frac{1}{a} \sum_{\xi \in \mathbb{Z}_a} \frac{\Delta(\xi t)}{1 - \xi t} \cdot \frac{\Delta(\bar{\xi} A t)}{1 - \bar{\xi} A t} = \frac{(1 - A^a t^{2a}) \cdot \Delta(A t^2)}{(1 - t^a)(1 - A^a t^a)(1 - A t^2)}.$$

(b) *For arbitrary integers $d \geq 2$ and $k \geq 1$ one has*

$$\frac{1}{a^{d-1}} \sum_{\substack{\xi_1, \dots, \xi_d \in \mathbb{Z}_a \\ \xi_1 \cdots \xi_d = 1}} \frac{\Delta(\xi_1 t)}{1 - \xi_1 t} \cdots \frac{\Delta(\xi_{d-1} t)}{1 - \xi_{d-1} t} \cdot \frac{\Delta(\xi_d t^k)}{1 - \xi_d t^k}$$
$$= \frac{(1 - t^{a(d+k-1)}) \cdot \Delta(t^{d+k-1})}{(1 - t^a)^{d-1}(1 - t^{ak})(1 - t^{d+k-1})}.$$

Proof. The assumption about $\Delta(t)$ guarantees that one can write $\Delta(t) = 1 - R(t)(1-t)$ for some $R(t) = \sum_{j \geq 1} \tilde{b}_j t^j$ with $\tilde{b}_j \in \{0, 1\}$ for all j. Then the left hand side of (a) is

$$\frac{1}{a} \sum_{\xi \in \mathbb{Z}_a} \frac{1}{(1 - \xi t)(1 - \bar{\xi} A t)} - \frac{1}{a} \sum_{\xi \in \mathbb{Z}_a} \frac{R(\bar{\xi} A t)}{1 - \xi t} - \frac{1}{a} \sum_{\xi \in \mathbb{Z}_a} \frac{R(\xi t)}{1 - \bar{\xi} A t} + \frac{1}{a} \sum_{\xi \in \mathbb{Z}_a} R(\xi t) \cdot R(\bar{\xi} A t).$$

The first sum (with the coefficient $1/a$) can be written in the form

$$\frac{1}{a}(1 + \xi t + \xi^2 t^2 + \cdots)(1 + \bar{\xi} A t + \bar{\xi}^2 A^2 t^2 + \cdots) = \frac{1}{a} \sum_{n \geq 0} \sum_{j=0}^{n} \xi^{n-2j} A^j t^n.$$

By an elementary computation this is

$$\frac{1 - A^a t^{2a}}{(1 - t^a)(1 - A^a t^a)(1 - A t^2)}.$$

The second term gives $R(At^2)/(1 - t^a)$. In order to prove this, first notice that the formula is additive in the polynomial R, hence it is enough to verify it for $R(t) = t^k$ for all $0 \leq k < a$. The case $k = 0$ is easy, it is equivalent to the identity

$$\frac{1}{a} \sum_{\xi} \frac{1}{1 - \xi t} = \frac{1}{1 - t^a}. \qquad (*)$$

If $1 \leq k < a$, then write

$$\frac{1}{a} \sum_{\xi} \frac{\bar{\xi}^k A^k t^k}{1 - \xi t} = \frac{A^k t^{2k}}{a} \left[\sum_{\xi} \frac{1}{1 - \xi t} + \sum_{\xi} (\xi t)^{-1} + \cdots + (\xi t)^{-k} \right].$$

Since $k < a$ the last sum is zero (here $k < a$ is crucial!), hence $(*)$ gives the claimed identity.

By a similar method, the third term is $R(At^2)/(1-A^a t^a)$. Finally, the fourth one is $R(At^2)$ (here one needs to apply the alternating property, namely that $\tilde{b}_j^2 = \tilde{b}_j$ for any j).

For part (b), use (a) and induction over d. For this, write ξ_d as $\bar{\xi}_1 \cdots \bar{\xi}_{d-1}$ and use (a) for $\xi = \xi_{d-1}$ and $A = \bar{\xi}_1 \cdots \bar{\xi}_{d-2} t^{k-1}$. Then apply the inductive step. □

6.4. Remarks. (1) Let $\Delta(t)$ and a be as in 6.3. The expression in 6.3(a) with $A = 1$ has a pole of order 2 at $t = 1$. This comes from the pole of the summand given by $\xi = 1$. Therefore

$$\begin{aligned}\frac{1}{a}\sum_{\xi\in\mathbb{Z}_a^*}\frac{\Delta(\xi)}{1-\xi}\cdot\frac{\Delta(\bar{\xi})}{1-\bar{\xi}} &= \lim_{t\to 1}\left[\frac{(1-t^{2a})\cdot\Delta(t^2)}{(1-t^a)^2(1-t^2)} - \frac{\Delta(t)^2}{a(1-t)^2}\right]\\ &= \frac{a^2-1}{12a} + \frac{1}{a}[\Delta'(1) - \Delta'(1)^2 + \Delta''(1)],\end{aligned}$$

where the first equality follows from 6.3, and the second by a computation.

(2) Assume that $\Delta(t)$ is an arbitrary symmetric polynomial of degree $2r$, and write $\Delta^\natural(t) = t^{-r}\Delta(t)$. Then it is easy to show that

$$\Delta'(1) = r\Delta(1), \qquad (\Delta^\natural)'(1) = 0, \qquad (\Delta^\natural)''(1) = (r - r^2) + \Delta''(1)/\Delta(1).$$

(3) If one combines (1) and (2), then for a symmetric polynomial $\Delta(t)$ with alternating coefficients and with $\Delta(1) = 1$ one gets

$$\sum_{\xi\in\mathbb{Z}_a^*}\frac{\Delta(\xi)}{1-\xi}\cdot\frac{\Delta(\bar{\xi})}{1-\bar{\xi}} = \frac{a^2-1}{12} + (\Delta^\natural)''(1) \qquad (\text{for } a \geq \deg\Delta).$$

This reproves the vanishing of $\mathcal{D}_a$ in 5.8($\mathcal{D}$) for such polynomials (cf. also 5.7(∗∗)).

(4) Although the Alexander polynomial $\Delta(f)$ of the algebraic knot (S^3, K_f) (f an irreducible plane curve singularity) is known since 1932 [7, 50], and it was studied intensively (see e.g. [21, 1, 11]), the property 6.2 remained hidden (to the best of the authors' knowledge).

On the other hand, similar properties were intensively studied in number theory: namely, in the 40's, 50's and 60's a considerable number of articles were published about the coefficients of cyclotomic polynomials. Here we mention only a few results. If ϕ_n denotes the n^{th} cyclotomic polynomial, then it was proved that the coefficients of ϕ_n have values in $\{-1, 0, +1\}$ for $n = 2^\alpha p^\beta q^\gamma$ (p and q distinct odd primes) (a result which goes back to the work of I. Schur); if n is a product of three distinct primes pqr ($p < q < r$ and $p + q > r$) then the coefficient of t^r in ϕ_n is -2 (a result of V. Ivanov); later Erdős proved interesting estimates for the growth of the coefficients; and G. S. Kazandzidis provided an exact formula for them. The interested reader can consult [23, pp. 404–411] for a large list of articles about this subject. (Reading these reviews shows that apparently the alternating property was not perceived in this area either.)

Clearly, the above facts are not independent of our problem: by 6.1(3) the Alexander polynomial $\Delta(f)$ is a *product* of cyclotomic polynomials.

(5) In fact, there is a recent result [15] in the theory of singularities which implies the alternating property 6.2(b). For any irreducible curve singularity, using its normalization, one can define a semigroup $S \subset \mathbb{N}$ with $0 \in S$ and $\mathbb{N} \setminus S$ finite. Then, in [15], based on some results of Zariski, for an irreducible plane curve singularities it is proved that $\Delta(f)(t)/(1-t) = \sum_{i\in S} t^i$. This clearly implies the alternating property.

(6) Are the irreducible plane curve singularities unique with the alternating property? The answer is negative. In order to see this, consider the Seifert integral homology sphere $\Sigma = \Sigma(a_1, a_2, \dots, a_{k+1})$ (where $\{a_i\}_i$ are pairwise coprime integers). Let K be the special orbit associated with the last arm (with Seifert invariant a_{k+1}). Then the Alexander polynomial $\Delta_\Sigma(K)$ has alternating coefficients. Indeed, write $a := a_1 \cdots a_k$, $a'_i := a/a_i$ for any $1 \le i \le k$, and let $S \subset \mathbb{N}$ be the semigroup (with $0 \in S$) generated by $a'_1, \dots, a'_k$. Then

$$\Delta_\Sigma(K)(t) = \frac{(1-t^a)^{k-1}(1-t)}{\prod_{i=1}^k (1-t^{a'_i})} = (1-t)\sum_{i\in S} t^i.$$

The first equality follows e.g. by [11, Section 11]; the second by an induction over k. This implies the alternating property as above.

(7) We can ask the following natural question: which property distinguishes (M, K_f) (where f is an irreducible plane curve singularity), or (Σ, K) given in (6), from the example described in 5.12? Why is the $\mathcal{D}$-invariant zero in the first case, but not in the second? Can this be connected with some property of the semigroups S associated with the curve whose link is K? (We believe that the validity of an Abhyankar–Azevedo type theorem for this curve plays an important role in this phenomenon.)

(8) Examples show that the assumptions of 6.3 are really essential (cf. also 7.8(2)).

7. The link of $\{f(x, y) + z^n = 0\}$

7.1. Preliminaries. The present section is more technical than the previous ones, and some of the details are left to the reader, which might cost the reader some work.

Fix an irreducible plane curve singularity $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and let $K_f \subset S^3$ be its link as in the previous section. Fix an integer $n \ge 1$, and consider the "suspension" germ $g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by $g(x, y, z) = f(x, y) + z^n$. Its link (i.e. $\{g = 0\} \cap S^5_\epsilon$ for $\epsilon \ll 1$) will be denoted by M. *We will assume that M is a rational homology sphere* (cf. 7.2(c)).

First, we recall/fix some numerical notations. We set:

- the Newton pairs $\{(p_k, q_k)\}_{k=1}^s$ of f;
- the integers $\{a_k\}_{k=1}^s$ defined as in 6.1(1); recall that $\gcd(p_k, a_k) = 1$ for any k;
- $d_k := \gcd(n, p_{k+1}p_{k+2}\cdots p_s)$ for $0 \le k \le s-1$, and $d_s := 1$;

- $h_k := d_{k-1}/d_k = \gcd(p_k, n/d_k)$ for $1 \le k \le s$;
- $\tilde{h}_k := \gcd(a_k, n/d_k)$ for $1 \le k \le s$.

For any integer $1 \le l \le s$, let $M_{(l)}$ be the link of the suspension singularity $g_{(l)}(x,y,z) := f_{(l)}(x,y) + z^{n/d_l}$. Evidently, $M_{(s)} = M$, and $M_{(1)} = \Sigma(p_1, a_1, n/d_1)$.

7.2. Some properties of the 3-manifolds $\{M_{(l)}\}_l$

(a) [10, 17, 38, 46] For each $1 \le l \le s$, $M_{(l)}$ is the (n/d_l)-cyclic cover of S^3 branched along $\{f_{(l)} = 0\}$. Let $K_{(l)} \subset M_{(l)}$ be the preimage of $\{f_{(l)} = 0\}$ with respect to this cover.

(b) $(M_{(l)}, K_{(l)})$ can be represented by a ("canonical") plumbing graph (or resolution graph) which is compatible with the above cover. This is done explicitly in [30] (based on the idea of [20]); see also 7.3 here. Using this graph one obtains the following inductive picture.

For any $2 \le l \le s$, $M_{(l)}$ can be obtained by splicing, as described in Section 4, the 3-manifold $M_2 = \Sigma(p_l, a_l, n/d_l)$ along $K_2 = \{y = 0\}$ with h_l copies of $M_{(l-1)}$ along the link $K_{(l-1)}$ (with the same splicing data $\{A^{(i)}\}_i$ as in Section 4). (In order to prove this, one needs to determine the invariant M_w used in [30]; this is done in [31, proof of (3.2)].)

(c) Part (b) ensures that M is a rational homology sphere if and only if for each $1 \le l \le s$ the Seifert 3-manifold $\Sigma(p_l, a_l, n/d_l)$ is a rational homology sphere. Since $\gcd(p_l, a_l) = 1$, this happens if and only if

$$(h_l - 1)(\tilde{h}_l - 1) = 0 \quad \text{for any } l \text{ (cf. [6] or [30])}.$$

(d) Using (b) and 4.6, one has

$$|H_1(M_{(l)})| = |H_1(\Sigma(p_l, a_l, n/d_l))| + h_l \cdot |H_1(M_{(l-1)})| \quad \text{for any } 2 \le l \le s,$$

or

$$|H_1(M)| = \sum_{l=1}^{s} d_l \cdot |H_1(\Sigma(p_l, a_l, n/d_l))|.$$

In fact, one can give a complete description of the group $H_1(M)$ and the character group $H_1(M)\hat{}$ using 4.6 (we will come back to this in 7.3).

(e) As a parallelism, let us recall some similar formulae for other numerical invariants: let $\mu_{(l)}$, respectively $\sigma_{(l)}$, be the Milnor number, respectively the signature of the Milnor fiber of $g_{(l)}$. Similarly, let $\mu(p_l, a_l, n/d_l)$ and $\sigma(p_l, a_l, n/d_l)$ be the Milnor number and the signature of the Brieskorn singularity $x^{p_l} + y^{a_l} + z^{n/d_l}$. Then, by [31], for any $2 \le l \le s$,

$$\sigma_{(l)} = \sigma(p_l, a_l, n/d_l) + h_l \cdot \sigma_{(l-1)} \quad \text{or, equivalently,} \quad \sigma_{(s)} = \sum_{l=1}^{s} d_l \cdot \sigma(p_l, a_l, n/d_l).$$

By contrast, for the Milnor numbers one has $\mu_{(l)} = \mu(p_l, a_l, n/d_l) + p_l \cdot \mu_{(l-1)}$ (involving p_l versus h_l, which follows e.g. from 6.1(4)).

Our goal is to establish an inductive formula for $\mathbf{sw}^{\rm TCW}_M(\sigma_{\rm can})$, similar to $|H_1|$ or to the signature $\sigma_{(l)}$. (For $\lambda_{\rm W}$ or $\mathcal{T}_{M,\sigma_{\rm can}}(1)$ such a formula does not hold, see below.)

(f) [10, 17, 38, 46] $(S^3, K_{f_{(l)}})$ is fibrable. Let $\mathcal{M}_{{\rm geom},(l)}: F_{(l)} \to F_{(l)}$ (respectively $\mathcal{M}_{(l)}$) be a geometric (respectively the algebraic) monodromy acting on the Milnor fiber $F_{(l)}$ (respectively on $H_1(F_{(l)})$). Then $(M_{(l)}, K_{(l)})$ is also fibrable, and its open book decomposition has the same fiber $F_{(l)}$ and geometric monodromy $\mathcal{M}^{n/d_l}_{{\rm geom},(l)}$. In particular, the (normalized) Alexander polynomial of $(M_{(l)}, K_{(l)})$ is (the normalization of)

$$\Delta_{M_{(l)}}(t) = \Delta_{M_{(l)}}(K_{(l)})(t) = \det(1 - t\mathcal{M}^{n/d_l}_{(l)}).$$

Therefore, using 3.6(7) and $2r_l := \operatorname{rank} H_1(F_{(l)})$ we obtain

$$\Delta^\natural_{M_{(l)}}(t) = \frac{1}{|H_1(M_{(l)})|}\cdot t^{-r_l}\cdot \det(1-t\mathcal{M}^{n/d_l}_{(l)}) \quad \text{with} \quad |H_1(M_{(l)})| = |\det(1-\mathcal{M}^{n/d_l}_{(l)})|.$$

Notice that $\Delta^\natural_{M_{(l)}}(t)$ can be deduced from the Alexander polynomial

$$\Delta(f_{(l)})(t) = \det(1 - t\mathcal{M}_{(l)})$$

of $f_{(l)}$ (cf. Section 5). Indeed, for any polynomial $\Delta(t)$ of degree $2r$ and of the form $\Delta(t) = \prod_v (1 - t^{m_v})^{n_v}$, and for any positive integer k, define

$$\Delta^{c(k)}(t) := \prod_v (1 - t^{m_v/\gcd(m_v,k)})^{n_v\cdot \gcd(m_v,k)}.$$

Let $\Delta^{c(k),\natural}(t) = t^{-r}\Delta^{c(k)}(t)/\Delta^{c(k)}(1)$ denote the normalization of $\Delta^{c(k)}(t)$. An eigenvalue argument then proves

$$\Delta_{M_{(l)}}(t) = \Delta(f_{(l)})^{c(n/d_l)}(t) \quad \text{and} \quad \Delta^\natural_{M_{(l)}}(t) = \Delta(f_{(l)})^{c(n/d_l),\natural}(t).$$

(g) The inductive formula 6.1(4) reduces the computation of the Alexander invariants to the Seifert case. Clearly (from 6.1(5), 5.3 and (f) above),

$$\Delta(x^{p_l} + y^{a_l})^{c(n/d_l),\natural}(t) = \frac{1}{a_l^{h_l-1}p_l^{\tilde h_l-1}}\cdot t^{-(a_l-1)(p_l-1)/2}\cdot \frac{(t^{p_la_l/(h_l\tilde h_l)}-1)^{h_l\tilde h_l}(t-1)}{(t^{p_l/h_l}-1)^{h_l}(t^{a_l/\tilde h_l}-1)^{\tilde h_l}}.$$

(Recall that $(h_l-1)(\tilde h_l-1) = 0$ for any l.) Then, by a computation, one can show that

$$(\Delta(x^{p_l} + y^{a_l})^{c(n/d_l),\natural})''(1) = \frac{1}{12}\left(\frac{a_l^2}{\tilde h_l} - 1\right)\left(\frac{p_l^2}{h_l} - 1\right).$$

(h) Using (f) and 6.1(4) one gets

$$\Delta^\natural_{M_{(l)}}(t) = \Delta(x^{p_l} + y^{a_l})^{c(n/d_l),\natural}(t)\cdot [\Delta^\natural_{M_{(l-1)}}(t^{p_l/h_l})]^{h_l}.$$

Then, using $(\Delta^\natural)'(1) = 0$ (cf. 6.4(2)) and the result from (g), one obtains

$$(\Delta^\natural_{M_{(l)}})''(1) = \frac{1}{12}\left(\frac{a_l^2}{\tilde h_l} - 1\right)\left(\frac{p_l^2}{h_l} - 1\right) + \frac{p_l^2}{h_l}\cdot (\Delta^\natural_{M_{(l-1)}})''(1).$$

Therefore

$$(\Delta^\natural_{M_{(l)}})''(1)=\sum_{k=1}^{l}\frac{1}{12}\Big(\frac{a_k^2}{\tilde h_k}-1\Big)\Big(\frac{p_k^2}{h_k}-1\Big)\cdot\frac{(p_{k+1}\cdots p_l)^2}{h_{k+1}\cdots h_l}.$$

(i) For any $2\leq l\leq s$, one has

$$\begin{aligned}\lambda_{\mathrm{W}}(M_{(l)})&=\lambda_{\mathrm{W}}(\Sigma(p_l,a_l,n/d_l))+h_l\cdot\lambda_{\mathrm{W}}(M_{(l-1)})\\&\quad+\frac{np_l(h_l-1)}{d_la_lh_l}\sum_{k=1}^{l-1}\frac{1}{12}\Big(\frac{a_k^2}{\tilde h_k}-1\Big)\Big(\frac{p_k^2}{h_k}-1\Big)\frac{(p_{k+1}\cdots p_{l-1})^2}{h_{k+1}\cdots h_{l-1}}.\end{aligned}$$

Indeed, if $h_l=1$ then we have additivity as in 4.3(4) (cf. also the proof of 5.10, part B). If $\tilde h_l=1$, then apply 5.6 (see also 5.8) and (h) above.

For the value of the Casson–Walker invariant $\lambda_{\mathrm{W}}(\Sigma(p,a,n))$ of a Seifert manifold, see [22, 6.1.1] or [36, 5.4].

(j) Below in (k), we will compute $(\Delta^\natural_{M_{(l)}})''(1)$ in terms of $\{(\Delta(f_{(k)})^\natural)''(1)\}_{k\leq l}$. Clearly, one can obtain an inductive formula for $(\Delta(f_{(k)})^\natural)''(1)$ similar to the one in (h) by taking $n=1$. More precisely, for any $2\leq l\leq s$,

$$(\Delta(f_{(l)})^\natural)''(1)=(a_l^2-1)(p_l^2-1)/12+p_l^2\cdot(\Delta(f_{(l-1)})^\natural)''(1).$$

(k) The next (rather complicated) identity looks very artificial, but it is one of the most important formulae in this list. Basically, its validity is equivalent to the fact that the two correction terms $\mathcal{O}(\mathcal{T})$ and $\mathcal{O}(\lambda_{\mathrm{W}}/2)$ are the same (cf. 7.13 and 7.14). For any $2\leq l\leq s$ one has

$$(\Delta^\natural_{M_{(l)}})''(1)=(\Delta(f_{(l)})^\natural)''(1)-\sum_{k=1}^{l}\frac{a_k^2p_k^2}{\tilde h_k^2h_k^2}\cdot\frac{p_{k+1}^2\cdots p_l^2}{h_{k+1}\cdots h_l}\cdot A_k,$$

where

$$A_k:=\frac{h_k(h_k-1)}{a_k^2}\cdot\Big[\frac{a_k^2-1}{12}+(\Delta(f_{(k-1)})^\natural)''(1)\Big]+\frac{\tilde h_k(\tilde h_k-1)}{p_k^2}\cdot\frac{p_k^2-1}{12}.$$

For the proof proceed as follows. Let $E_{(l)}$ be the difference between the left and the right hand side of the identity. Then, using the inductive formulae (h) and (j) and the property $(h_k-1)(\tilde h_k-1)=0$, by an elementary computation one can verify that $E_{(l)}-(p_l^2/h_l)\cdot E_{(l-1)}=0$, and $E_{(1)}=0$. Then $E_{(l)}=0$ by induction.

(l) The last invariant we wish to determine is $\mathcal{T}_{M_{(l)},\sigma_{\mathrm{can}}}(1)$. The computation is more involved and it is given in the next subsections. The inductive formula for $\mathcal{T}_{M_{(l)},\sigma_{\mathrm{can}}}(1)$ is given in 7.13.

7.3. Characters of $H_1(M)$. In the computation of $\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)$ we plan to use 3.7(16). For this, we need to describe the characters $\chi\in\hat H=H_1(M)\hat{}$.

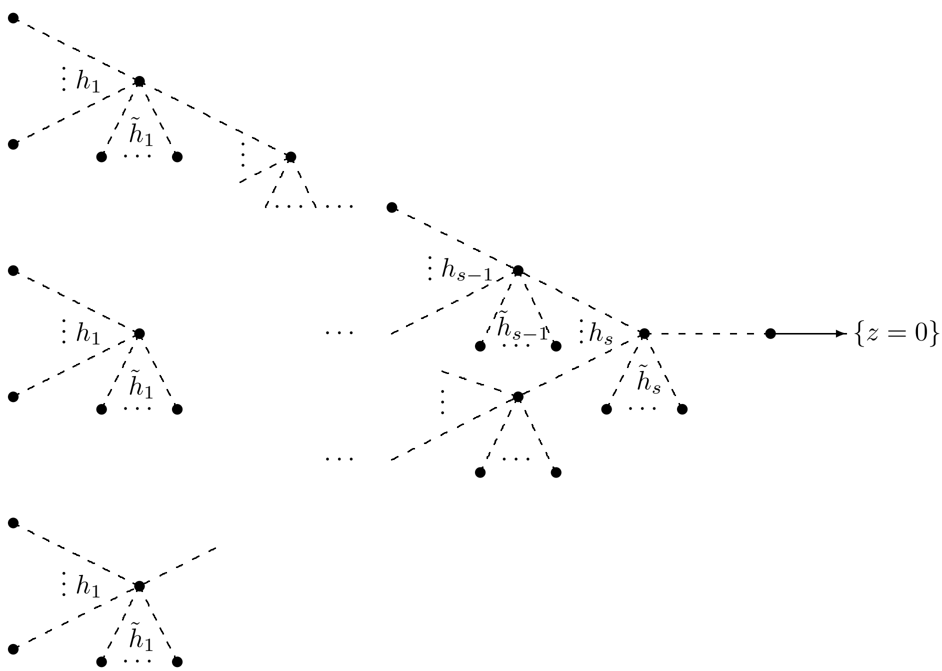
The group H can be determined in many different ways. For example, using the monodromy operator $\mathcal{M}=\mathcal{M}_{(s)}$ of f, H can be identified, as an abstract group, with $\operatorname{coker}(1-\mathcal{M}^n)$. The homology of the Milnor fiber of f and $\mathcal{M}$ have a direct sum decomposition with respect to the splicing (see e.g. [11] or [31]). This can be used to provide an inductive description of H.

Nevertheless, we prefer to use 4.6. The main reason is that, in fact, we have to understand $\hat{H}$ (rather than H) together with the description of the supports $\{v:\ \chi(g_v)\neq 1\}$ for each character $\chi\in\hat{H}$. For this, the discussion from Section 3 is more suitable.

We consider again the "canonical" plumbing graph $\Gamma(M)$ of M provided by the algorithm of [30] (published also in [33]; cf. also 7.2(b) here). In fact, that algorithm provides $\Gamma := \Gamma(M, K_z)$, the plumbing graph of the 3-dimensional link $M=\{f(x,y)+z^n=0\}$ with the knot $K_z:=\{z=0\}$ in it. Again, if we replace the strings by dashed lines, then one can represent Γ as a covering graph of $\Gamma(S^3,K_f)$; for details, see [loc. cit.], cf. also 7.2(b). If we denote this graph-projection by π, then

$$\begin{aligned}
&\#\pi^{-1}(v_k) = h_{k+1}\cdots h_s, && 1\le k\le s,\\
&\#\pi^{-1}(\bar{v}_k) = \tilde{h}_k h_{k+1}\cdots h_s, && 1\le k\le s,\\
&\#\pi^{-1}(\bar{v}_0) = h_1\cdots h_s
\end{aligned}$$

(see 6.1 for notations concerning $\Gamma(S^3,K_f)$). In fact, there is a $\mathbb{Z}_n$-action on Γ which acts transitively on each fiber of π, hence all the vertices above a given vertex $v\in\mathcal{V}^*$ of $\Gamma(S^3,K_f)$ are symmetric in Γ. In particular, their decorations and their numerical invariants (computed from the graph Γ) are the same. Therefore, Γ has the following schematic form:



Now, we consider the Seifert manifold $\Sigma := \Sigma(p,a,n)$ with $\gcd(p,a)=1$.

If $\gcd(a,n)=1$ and $\gcd(p,n)=d$, then by 5.3, $H_1(\Sigma)$ is generated by the homology classes of $\{K_2^{(i)}\}_{i=1}^d$. For an arbitrary character χ, we write $\chi(K_2^{(i)})=\xi_i$. Here ξ_i is an a-root of unity in $\mathbb{C}$ (briefly $\xi_i\in\mathbb{Z}_a$). Then χ is completely characterized by the collection $\{\xi_i\}_{i=1}^d$ which satisfies $\xi_i\in\mathbb{Z}_a$ for any i and $\xi_1\cdots\xi_d=1$ (cf. 5.3(1)).

Notice that $\chi(O)=1$, and χ is supported by those d "arms" of the star-shaped graph which have Seifert invariant a (i.e. $\chi(g_v)=1$ for the vertices v situated on the other arms). In [36] it is proved that for each non-trivial character χ, in $\hat{P}_{\Sigma,\chi,u}(t)$ one can take for u the central vertex O. Moreover, $\lim_{t\to 1}\hat{P}_{\Sigma,\chi,u}(t)$ can be non-zero only if χ is supported exactly on two arms, i.e. $\xi_i\neq 1$ exactly for two values of i, say i_1 and i_2 (hence $\xi_{i_1}=\bar{\xi}_{i_2}$).

If $\gcd(n,p)=1$, but $\gcd(n,a)\neq 1$, then clearly we have a symmetric situation; in this case we use the notation η instead of ξ.

In both situations, for any character χ, $\chi(g_v)=1$ for any v situated on the arm with Seifert invariant $n/\gcd(n,ap)$.

These properties proved for the building block Σ will generate all the properties of $H=H_1(M)$ via the splicing properties 4.6 and linking relations 5.3.1. For example, one can prove by induction, that for any character χ of H, $\chi(g_v)=1$ for any vertex v situated on the string which supports the arrow of K_z.

Consider the splicing decomposition

$$M=M_{(s)}=h_sM_{(s-1)}\amalg\Sigma(p_s,a_s,n).$$

As in the previous inductive arguments, assume that we understand the characters χ of $H_1(M_{(s-1)})$. By 4.6, they can be considered in a natural way as characters of $H_1(M_{(s)})$ satisfying additionally $\chi(g_v)=1$ for any vertex v of $\Sigma(p_s,a_s,n)$ (cf. also the first paragraph of the proof of 4.9). We say that these characters *do not propagate* from $M_{(s-1)}$ into $\Sigma(p_s,a_s,n)$.

In the "easy" case when $h_s=1$ (even if $\tilde{h}_s\neq 1$), the splicing invariants are $o_1=o_2=1$ and $k_1=k_2=0$, hence H (together with its linking form) is a direct sum in a natural way, hence the characters of $\Sigma(p_s,a_s,n)$ (described above) will not propagate into $M_{(s-1)}$ either.

On the other hand, if $h_s>1$, then the *non-trivial characters* of $\Sigma(p_s,a_s,n)$ do propagate into the h_s copies of $M_{(s-1)}$. If χ is such a character, with $\chi(K_2^{(i)})=\xi_i$, then analyzing the properties of the linking numbers in 4.6 we deduce that χ does propagate into M in such a way that $\chi(g_{v^{(i)}})=\xi_i^l$ for any vertex $v^{(i)}$ of $M_{(s-1)}^{(i)}$, where $l=\mathrm{Lk}_{M_{(s-1)}^{(i)}}(g_{v^{(i)}},K_1^{(i)})$. This can be proved as follows. Write $\exp(r)$ for $e^{2\pi ir}$, and assume that $\chi=\exp(\mathrm{Lk}_\Sigma(L,\cdot))$ for some fixed $L\subset\Sigma$. Then $\xi_i=\chi(K_2^{(i)})=\exp(\mathrm{Lk}_\Sigma(L,K_2^{(i)}))$. Therefore, χ extended into M as $\exp(\mathrm{Lk}_M(L,\cdot))$ satisfies (cf. 4.6(11))

$$\chi(g_{v^{(i)}})=\exp(\mathrm{Lk}_M(L,g_{v^{(i)}}))=\exp(\mathrm{Lk}_\Sigma(L,K_2^{(i)})\cdot\mathrm{Lk}_{M_{s-1}^{(i)}}(K_1^{(i)},g_{v^{(i)}}))=\xi_i^l.$$

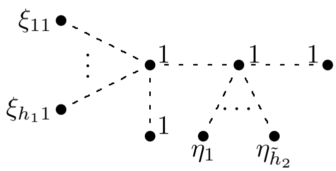
Clearly, all these linking numbers $\mathrm{Lk}_{M^{(i)}_{(s-1)}}(g_{v^{(i)}}, K_1^{(i)})$ can be determined inductively by the formulae provided in 4.6 and 5.3.1.

Moreover, if we multiply such an extended character χ with a character χ' supported by $M_{(s-1)}$, then $\chi\chi'(g_v) = \chi(g_v)$ for any vertex v situated above the vertices v_s or v_{s-1} or above any vertex on the edge connecting v_{s-1} with v_s (since the support of χ' does not contain these vertices). We will say that such a character $\chi\chi'$ is *born at level* s (provided that the original χ of Σ was non-trivial). (The interested reader can reformulate the above discussion in the language of an exact sequence of dual groups, similar to the one used in the first paragraph of the proof of 4.9.)

Below we provide some examples. In these diagrams, for any fixed character χ, we put the complex number $\chi(g_v)$ at the vertex v.

7.4. Example. Assume that $s = 2$. Then basically one has two different cases: $\tilde{h}_1 = h_2 = 1$ or $\tilde{h}_1 = \tilde{h}_2 = 1$ (since the first pairs (p_1, a_1) can be permuted).

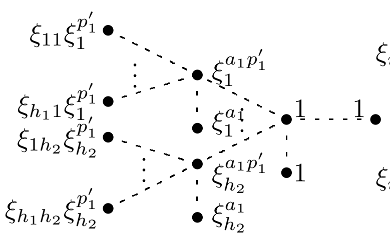
In the first ("easy") case, the schematic diagram of the characters is:



$\eta_{i_2} \in \mathbb{Z}_{p_2}$ for $1 \le i_2 \le \tilde{h}_2$, and $\prod_{1\le i_2\le \tilde{h}_2} \eta_{i_2} = 1$;

$\xi_{i_1 1} \in \mathbb{Z}_{a_1}$ for $1 \le i_1 \le h_1$, and $\prod_{1\le i_1\le h_1} \xi_{i_1 1} = 1$.

In the second case $\tilde{h}_1 = \tilde{h}_2 = 1$, with the notation $p_1' := p_1/h_1$, the characters $\hat{H}$ are:



$\xi_{i_2} \in \mathbb{Z}_{a_2}$ for $1 \le i_2 \le h_2$, and $\prod_{1\le i_2\le h_2} \xi_{i_2} = 1$;

and for any fixed i_2:

$\xi_{i_1 i_2} \in \mathbb{Z}_{a_1}$ for $1 \le i_1 \le h_1$, and $\prod_{1\le i_1\le h_1} \xi_{i_1 i_2} = 1$.

7.5. Using the discussion 7.3, the above example for $s = 2$ can be generalized inductively to arbitrary s. For the convenience of the reader, we make this explicit. In order to have a uniform notation, we consider the case $\tilde{h}_k = 1$ for any $1 \le k \le s$. The interested reader is invited to write down a similar description of the characters in those cases when $\tilde{h}_k \ne 1$ for some k, using the present model and 7.4.

More precisely, for any character χ, we will indicate $\chi(g_{v'})$ for any vertex $v' \in \pi^{-1}(\mathcal{V}^*)$.

It is convenient to introduce the index set $(i_1, \ldots, i_s)$, where $1 \le i_l \le h_l$ for any $1 \le l \le s$. As already mentioned, this set can be considered as the index set of $\pi^{-1}(\bar{v}_0)$. Moreover, for any $1 \le k \le s-1$, $(i_{k+1}, \ldots, i_s)$ (where $1 \le i_l \le h_l$ for any $k+1 \le l \le s$) is the index set of $\pi^{-1}(v_k)$ (and of $\pi^{-1}(\bar{v}_k)$ since $\tilde{h}_k = 1$). Moreover, for any l we write $p_l' := p_l/h_l$.

Next, we consider a system of roots of unity as follows:

- $\xi_{i_s} \in \mathbb{Z}_{a_s}$ with $\prod_{1\leq i_s\leq h_s} \xi_{i_s} = 1$;
- for any fixed i_s a collection $\xi_{i_{s-1}i_s} \in \mathbb{Z}_{a_{s-1}}$ with $\prod_{1\leq i_{s-1}\leq h_{s-1}} \xi_{i_{s-1}i_s} = 1$; and, more generally, if $1 \leq k \leq s-1$:
- for any fixed $(i_{k+1},\dots,i_s)$ a collection $\xi_{i_k\cdots i_s} \in \mathbb{Z}_{a_k}$ with $\prod_{1\leq i_k\leq h_k} \xi_{i_k\cdots i_s} = 1$.

Then, any character χ can be characterized by the following properties:

- $\pi^{-1}(v_s)$ contains exactly one vertex, say v'. Then $\chi(g_{v'}) = 1$. The same is valid for $\pi^{-1}(\bar{v}_s)$ and for the vertex in Γ which supports the arrow $\{z = 0\}$.
- For any $1 \leq k \leq s-1$, if $v'(i_{k+1},\dots,i_s)$ is the vertex in $\pi^{-1}(v_k)$ corresponding to $(i_{k+1},\dots,i_s)$, then
$$\chi(g_{v'(i_{k+1},\dots,i_s)}) = \xi_{i_{k+1}\cdots i_s}^{a_k p'_k} \cdot \xi_{i_{k+2}\cdots i_s}^{a_k p'_k p'_{k+1}} \cdots \xi_{i_s}^{a_k p'_k\cdots p'_{s-1}}.$$
- Similarly, if $\bar{v}'(i_{k+1},\dots,i_s)$ is the vertex in $\pi^{-1}(\bar{v}_k)$ corresponding to $(i_{k+1},\dots,i_s)$, then
$$\chi(g_{\bar{v}'(i_{k+1},\dots,i_s)}) = \xi_{i_{k+1}\cdots i_s}^{a_k} \cdot \xi_{i_{k+2}\cdots i_s}^{a_k p'_{k+1}} \cdots \xi_{i_s}^{a_k p'_{k+1}\cdots p'_{s-1}}.$$
- Finally, if $\bar{v}'(i_1,\dots,i_s)$ is the vertex in $\pi^{-1}(\bar{v}_0)$ corresponding to the index $(i_1,\dots,i_s)$, then
$$\chi(g_{\bar{v}'(i_1,\dots,i_s)}) = \xi_{i_1\cdots i_s} \cdot \xi_{i_2\cdots i_s}^{p'_1} \cdots \xi_{i_s}^{p'_1\cdots p'_{s-1}}.$$

If $\xi_{i_s} \neq 1$ for some i_s then χ is born at level s. If $\xi_{i_s} = 1$ for all i_s, but $\chi_{i_{s-1}i_s} \neq 1$ for some (i_{s-1}, i_s), then χ is born at level $s-1$, etc. In general, a character χ is born at level k ($1 \leq k \leq s$) if for any $l \geq k$ and $v' \in \pi^{-1}(v_l)$, one has $\chi(g_{v'}) = 1$, but there exists at least one vertex $v' \in \pi^{-1}(v_k)$ which is adjacent (in the graph Γ) to the support of χ.

The next result will be crucial when we apply the Fourier inversion formula 3.7(14). It is a really remarkable property of the links associated with irreducible plane curve singularities. It is the most important qualitative ingredient in our torsion computation (see also 7.8 for another powerful application).

7.6. Proposition. *Consider (S^3, K_f) associated with an irreducible plane curve singularity f. Let $\Delta_{S^3}(K_f)(t)$ be its Alexander polynomial. For any integer $n \geq 1$, consider (M, K_z), i.e. the link M of $\{f(x,y)+z^n = 0\}$ and the knot $K_z := \{z = 0\}$ in it. Let $\Delta_M^H(K_z)(t)$ be the Alexander invariant defined in* 3.6(10–11) *with $H = H_1(M)$. Then*
$$\Delta_M^H(K_z)(t) = \Delta_{S^3}(K_f)(t).$$

Proof. First we notice that in 3.6(10), $g_u = K_z$ and $o(u) = 1$. Then, by 3.4(5), $m_{v'}(u) = \mathrm{Lk}_M(g_{v'}, K_z)$ for any vertex v' of Γ. By the algorithm of [30], for any vertex $v' \in \pi^{-1}(v)$, where $v \in \mathcal{V}^*(\Gamma(S^3, K_f))$, the linking number $\mathrm{Lk}_M(g_{v'}, K_z)$ is given by $w_{v'} = w_v/\gcd(w_v, n)$. Recall that for any $v \in \mathcal{V}^*(\Gamma(S^3, K_f))$, the corresponding weights w_v are given in 6.1(2). In particular, this discussion provides all the weights $w_{v'}(u)$ needed in the definition 3.6(10) of $\Delta_{M,\chi}(K_z)(t)$. On the other hand, for characters $\chi \in \hat{H}$ we can use the above description. Then the proposition follows (after some computation) inductively using the Algebraic Lemma 6.3(b), for the Alexander polynomials $\Delta(f_{(l)}(0))$ $(0 \le l \le s-1$, where $\Delta(f_{(0)})(t) \equiv 1)$. Notice that this lemma can be applied thanks to Proposition 6.2 (which ensures that the coefficients of $\Delta(f_{(l)})(t)$ are alternating), and to the inequality 6.1(6) (which ensures that a is "sufficiently large"). (For an expression of $\Delta(f_{(l)})(t)$, see 6.1.)

In the next example we make this argument explicit for the case $s = 2$ and $\tilde{h}_1 = \tilde{h}_2 = 1$. Using this model, the reader can complete the general case easily. $\square$

7.7. Example. Assume that $s = 2$ and $\tilde{h}_1 = \tilde{h}_2 = 1$. Then, with the notation of 7.4, $\Delta_{M,\chi}(K_z)(t)$ equals

$$\prod_{i_1,i_2} \frac{1}{1 - t^{p_1' \, p_2'} \xi_{i_1 i_2} \, \xi_{i_2}^{p_1'}} \cdot \prod_{i_2} \frac{(1 - t^{a_1 \, p_1' \, p_2'} \xi_{i_2}^{a_1 \, p_1'})^{h_1}}{1 - t^{a_1 p_2'} \xi_{i_2}^{a_1}} \cdot \frac{(1 - t^{a_2 \, p_2'})^{h_2} (1-t)}{1 - t^{a_2}}.$$

First, for each fixed index i_2, we make a sum over $\xi_{i_1 i_2} \in \mathbb{Z}_{a_1}$. Using 6.3(b) for $\Delta \equiv 1$, $t = t^{p_1' \, p_2'} \xi_{i_2}^{p_1'}$ and $a = a_1$ and $d = h_1$, the above expression transforms (after some simplifications) into

$$\prod_{i_2} \frac{1 - t^{a_1 \, p_1 \, p_2'} \xi_{i_2}^{a_1 \, p_1}}{(1 - t^{p_1 \, p_2'} \xi_{i_2}^{p_1}) \cdot (1 - t^{a_1 p_2'} \xi_{i_2}^{a_1})} \cdot \frac{(1 - t^{a_2 \, p_2'})^{h_2} \cdot (1-t)}{1 - t^{a_2}}.$$

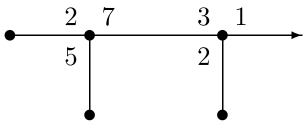
The expression in the product is exactly $\Delta(f_{(1)})(t^{p_2'})/(1 - t^{p_2'})$. Therefore, 6.3(b) can be applied again, now for $\Delta = \Delta(f_{(1)})$, $t = t^{p_2'}$, $a = a_2$ and $d = h_2$. Then the expression transforms into $\Delta(f_{(2)})$.

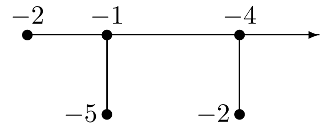
7.8. Remarks. (1) If the link M of $\{f(x,y) + z^n = 0\}$ is a rational homology sphere, then in [28] we prove the following facts. Using the combinatorics of the plumbing graph of M, one can recover the knot K_z in it. Then, by the above proposition, from the pair (M, K_z) one can recover the Alexander polynomial $\Delta_{S^3}(K_f)$ of f. It is well known that this is equivalent to the equisingular type of the plane singularity f. Moreover, analyzing again the graph of (M, K_z), one can recover the integer n as well. In particular, from M, we can recover not only the geometric genus of $\{f + z^n = 0\}$ (which is proved in this article), but also its *multiplicity*, and in fact, any numerical invariant which can be computed from the Newton (or Puiseux) pairs of f and from the integer n (e.g. even all the equivariant Hodge numbers associated with the vanishing cohomology of the hypersurface singularity $g = f + z^n$, or even the *embedded* topological type (S^5, M) of g with its integral Seifert matrix). (In fact, in [28], one obtains the Newton pairs by a more direct argument.)

(2) 7.6 suggests the following question. Let N be an integral homology sphere, and $L \subset N$ a knot in it such that (N, L) can be represented by a (negative definite) plumbing. Let (M, K) be the n-cyclic cover of (N, L) (branched along L) such

that M is a rational homology sphere with $H_1(M) = H$. Is it then true that $\Delta_M^H(K)(t) = \Delta_N(L)(t)$?

The answer is negative: one can easily construct examples (satisfying even the algebraicity condition) when the identity 7.6 fails. For example, consider (N, L) given by the following splice, respectively plumbing diagram:





Then one can show that e.g. for $n = 2$ the identity $\Delta_M^H(K)(t) = \Delta_N(L)(t)$ fails.

This example also shows that in the Algebraic Lemma 6.3 the assumption $a \geq \deg \Delta$ is crucial. Indeed, in this example $H = \mathbb{Z}_3$; and in order to determine $\Delta_M^H(K)(t)$, one needs to compute a sum like the one in 6.3(a) with $a = 3$, $A = 1$ and $\Delta = t^4 - t^3 + t^2 - t + 1$ (i.e. with $a < \deg \Delta$). But for these data, the identity in 6.3(a) fails.

7.9. The Reidemeister–Turaev sign-refined torsion. Now we will start to compute $\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)$ associated with $M = M_{(s)}$ and the canonical spin^c structure σ_{can} of M. As above, we write $H = H_1(M)$. Using 3.7(14), $\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)$ can be determined by the Fourier inversion formula from $\{\hat{\mathcal{T}}_{M,\sigma_{\mathrm{can}}}(\chi) : \chi \in \hat{H} \setminus \{1\}\}$. On the other hand, each $\hat{\mathcal{T}}_{M,\sigma_{\mathrm{can}}}(\bar{\chi})$ is given by the limit $\lim_{t\to 1} \hat{P}_{M,\chi,u}(t)$ for some convenient u (cf. 3.7(16)).

In the discussion below, the following terminology is helpful. Fix an integer $1 \leq k \leq s$ and a vertex $v'(I) := v'(i_{k+1}, \ldots, i_s) \in \pi^{-1}(v_k)$. Consider the graph $\Gamma \setminus \{v'(I)\}$. If $\tilde{h}_k = 1$ then it has $h_k + 2$ connected components: h_k (isomorphic) subgraphs $\Gamma_-^{i_k}(v'(I))$ $(1 \leq i_k \leq h_k)$ which contain vertices at level $k-1$, a string $\Gamma_{\mathrm{st}}(v'(I))$ containing a vertex above $\bar{v}_k$, and the component $\Gamma_+(v'(I))$ which supports the arrow $\{z = 0\}$. Similarly, if $h_k = 1$, then $\Gamma \setminus \{v'(I)\}$ has $\tilde{h}_k + 2$ connected components: $\Gamma_-(v'(I))$ contains vertices at level $k-1$, $\Gamma_+(v'(I))$ supports the arrow $\{z = 0\}$, and $\tilde{h}_k$ other (isomorphic) components $\Gamma_{\mathrm{st}}^{j_k}(v'(I))$ $(1 \leq j_k \leq \tilde{h}_k)$, which are strings, and each of them contains exactly one vertex staying above $\bar{v}_k$.

Just as for the Seifert manifold $\Sigma(p, n, n)$ (see the discussion in 7.3 after the diagram), for a large number of characters χ, the limit $\lim_{t\to 1} \hat{P}_{M,\chi,u}(t)$ is zero. Analyzing the structure of the graph of M and the supports of the characters, one can deduce that a *non-trivial* character χ, with the above limit non-zero, should satisfy one of the following structure properties.

E(asy) case: The character χ is born at level k (for some $1 \leq k \leq s$) with $\tilde{h}_k > 1$. For any vertex $v' \in \pi^{-1}(v_k)$ one has $\chi(g_{v'}) = 1$, but there is exactly one vertex $v'(I) := v'(i_{k+1}, \ldots, i_s) \in \pi^{-1}(v_k)$ which is adjacent to the support of χ. Moreover, χ is supported by exactly two components of type $\Gamma_{\mathrm{st}}^{j_k}(v'(I))$, say for indices j'_k and j''_k. Let $v'(j'_k)$ be the unique vertex in $\Gamma_{\mathrm{st}}^{j_k}(v'(I)) \cap \pi^{-1}(\bar{v}_k)$ (similarly for j''_k). Then $v'(j'_k)$ and $v'(j''_k)$ are the only vertices v' of the graph of M with $\delta_{v'} \neq 2$

and $\chi(g_{v'}) \neq 1$. Moreover, $\chi(g_{v'(j'_k)}) = \bar{\chi}(g_{v'(j''_k)}) = \eta \in \mathbb{Z}^*_{p_k}$. Therefore, with fixed $(i_{k+1}, \ldots, i_s)$ and (j'_k, j''_k), there are exactly $p_k - 1$ such characters.

D(ifficult) case: The character χ is born at level k (for some $1 \leq k \leq s$) with $h_k > 1$. For any vertex $v' \in \pi^{-1}(v_k)$, $\chi(g_{v'}) = 1$, but there is exactly one vertex $v'(I) := v'(i_{k+1}, \ldots, i_s) \in \pi^{-1}(v_k)$ which is adjacent to the support of χ. The character χ is supported by exactly two components of type $\Gamma_-^{i_k}(v'(I))$, say for indices i'_k and i''_k. Using the previous notations, this means that $\xi_{i_k i_{k+1} \cdots i_s} = 1$ except for $i_k = i'_k$ or $i_k = i''_k$. (Evidently, $\xi_{i_t \cdots i_s} = 1$ for any $t > k$.) For $t < k$, the values $\xi_{i_t \cdots i_{k+1} \cdots i_s}$ are arbitrary. In particular, with indices $(i_{k+1}, \ldots, i_s)$ and (i'_k, i''_k) fixed, there are exactly $(a_k - 1) \cdot |H_1(M_{(k-1)})|^2$ such characters. Here, $a_k - 1$ stands for $\xi_{i'_k\, i_{k+1} \cdots i_s} = \bar{\xi}_{i''_k\, i_{k+1} \cdots i_s} \in \mathbb{Z}^*_{a_k}$, and $|H_1(M_{(k-1)})|^2$ for the arbitrary characters born at level $< k$ on the two branches corresponding to $(i'_k, i_{k+1}, \ldots, i_s)$ and $(i''_k, i_{k+1}, \ldots, i_s)$.

In both cases (E) or (D), if such a character χ is born at level k (i.e. if it satisfies the above characterization for k), then we write $\chi \in B_k$.

Now, we fix a non-trivial character χ. Let $S(\chi)$ be the support of χ and $\bar{S}(\chi)$ its complement. Then

$$\frac{1}{|H|} \cdot \hat{\mathcal{T}}_{M,\sigma_{\mathrm{can}}}(\bar{\chi}) = \mathrm{Loc}(\bar{\chi}) \cdot \mathrm{Reg}(\bar{\chi}),$$

where

$$\mathrm{Loc}(\bar{\chi}) := \prod_{v' \in S(\chi)} (\chi(g_{v'}) - 1)^{\delta_{v'} - 2}, \qquad \mathrm{Reg}(\bar{\chi}) := \frac{1}{|H|} \cdot \lim_{t \to 1} \prod_{v' \in \bar{S}(\chi)} (t^{w_{v'}(u)} - 1)^{\delta_{v'} - 2}.$$

We will call $\mathrm{Loc}(\bar{\chi})$ the *local contribution*, while $\mathrm{Reg}(\bar{\chi})$ the *regularization contribution*.

By the above discussion, $\mathrm{Reg}(\bar{\chi}) = 0$ unless χ is not of the type (E) or (D) described above. If χ is of type (E) or (D), then in $\hat{P}_{M,\chi,u}(t)$ (cf. 3.7(15)) one can take $v'(I)$. Moreover, if $\chi \in B_k$, then by the symmetry of the plumbing graph of M, $\mathrm{Reg}(\bar{\chi})$ does not depend on the particular choice of χ, but only on the integer k. We write $\mathrm{Reg}(k)$ for $\mathrm{Reg}(\chi)$ for some (any) $\chi \in B_k$.

In particular,

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1) = \sum_{k=1}^{s} \mathrm{Reg}(k) \cdot \sum_{\chi \in B_k} \mathrm{Loc}(\bar{\chi}). \tag{$\mathcal{T}$}$$

7.10. Proposition. *For any fixed $1 \leq k \leq s$ one has:*

(E) *If $h_k = 1$ then*

$$\sum_{\chi \in B_k} \mathrm{Loc}(\bar{\chi}) = d_k \cdot \frac{\tilde{h}_k(\tilde{h}_k - 1)}{2} \cdot \frac{p_k^2 - 1}{12}.$$

(D) *If $\tilde{h}_k = 1$ then*

$$\sum_{\chi \in B_k} \mathrm{Loc}(\bar{\chi}) = d_k \cdot \frac{h_k(h_k - 1)}{2} \cdot |H_1(M_{(k-1)})|^2 \cdot \left[\frac{a_k^2 - 1}{12} + (\Delta(f_{(k-1)})^\natural)''(1) \right].$$

Proof. In case (E), $d_k = h_{k+1}\cdots h_s$ is the cardinality of the index set $(i_{k+1},\dots,i_s)$, and $\tilde h_k(\tilde h_k-1)/2$ is the number of possibilities to choose the indices (j'_k, j''_k). The last term comes from a formula of type 5.7(∗∗), where the sum is over $\eta\in\mathbb{Z}^*_{p_k}$.

In case (D), $d_k h_k(h_k-1)/2$ has the same interpretation. Fix the branch $(i'_k, i_{k+1},\dots,i_s)$ and consider the sum over all the characters born at level $< k$. Then 7.6, applied for $(M_{(k-1)}, K_{(k-1)})$ as a covering of $(S^3, f_{(k-1)}=0)$, provides $|H_1(M_{(k-1)})|^2\cdot\Delta(f_{(k-1)})(t)/(t-1)$ evaluated at $t=\xi_{i'_k\, i_{k+1}\cdots i_s}$. The same is true for the other index i''_k. Then apply 6.4(3) for $\Delta=\Delta(f_{(k-1)})$ and $a=a_k$. This can be done because of 6.2 and 6.1(6). □

7.11. The regularization contribution $\mathrm{Reg}(k)$**.** Fix a character $\chi\in B_k$ of type (E) or (D) as in 7.9. Recall that one can take $u=v'(I)$. Consider the connected components of $\Gamma\setminus\{v'(I)\}$ (as in 7.9), where we add to each component an arrow corresponding to the edge which connects the component to $v'(I)$. For these graphs, if one applies 3.6(12), one finds that $\mathrm{Reg}(k)$ is

$$\mathrm{Reg}(k)=\begin{cases} -\det(\Gamma_-)\cdot\det(\Gamma_{\mathrm{st}})^{\tilde h_k-2}\cdot\det(\Gamma_+)/\det(\Gamma) & \text{in case (E)},\\ -\det(\Gamma_-)^{h_k-2}\cdot\det(\Gamma_{\mathrm{st}})\cdot\det(\Gamma_+)/\det(\Gamma) & \text{in case (D)}.\end{cases}$$

Let I be the intersection matrix of M, and $I_k^{-1}:=I^{-1}_{v'v'}$ for any $v'\in\pi^{-1}(v_k)$. Then, by the formula which provides the entries of an inverse matrix, one gets

$$\det\Gamma\cdot I^{-1}_{v'v'}=\begin{cases}\det(\Gamma_-)\det(\Gamma_{\mathrm{st}})^{\tilde h_k}\det(\Gamma_+) & \text{in case (E)},\\ \det(\Gamma_-)^{h_k}\det(\Gamma_{\mathrm{st}})\det(\Gamma_+) & \text{in case (D)}.\end{cases}$$

These two facts combined show that

$$\mathrm{Reg}(k)=\begin{cases} -I_k^{-1}/\det(\Gamma_{\mathrm{st}})^2=-I_k^{-1}/p_k^2 & \text{in case (E)},\\ -I_k^{-1}/\det(\Gamma_-)^2=-I_k^{-1}/(a_k\cdot|H_1(M_{(k-1)})|)^2 & \text{in case (D)}\end{cases}$$

since in case (E), $|\det(\Gamma_{\mathrm{st}})|=p_k$ e.g. from 5.3, and in case (D), $|\det(\Gamma_-)|=a_k\cdot|H_1(M_{(k-1)})|$ by 5.13.

This together with 7.9($\mathfrak{T}$) and 7.10 yields

$$\mathfrak{T}_{M,\sigma_{\mathrm{can}}}(1)=-\sum_{k=1}^{s} I_k^{-1}\cdot d_k\cdot A_k/2, \tag{$*$}$$

where A_k is defined in 7.2(k) in terms of the numerical invariants of $f_{(k)}$.

7.12. The computation of I_k^{-1}**.** For any $l\geq k$, let $I_k^{-1}(M_{(l)})$ be the (v',v')-entry of the inverse of the intersection form $I(M_{(l)})$ associated with $M_{(l)}$, where v' is any vertex above v_k. For example $I_k^{-1}(M_{(s)})$ is I_k^{-1} used above.

By 3.4(5), $-I_k^{-1}(M_{(l)})=\mathrm{Lk}_{M_{(l)}}(g_{v'},g_{v'})$. If $l=k$, by 4.6(12) this is $\mathrm{Lk}_{\Sigma(p_k,a_k,n/d_k)}(O,O)$, hence by 5.3.1 it is $np_ka_k/(d_kh_k^2\tilde h_k^2)$.

Next, assume that $l>k$. If $h_l=1$, then the splicing

$$M_{(l)}=h_lM_{(l-1)}\amalg\Sigma(p_l,a_l,n/d_l)$$

is trivial (with $o_1 = o_2 = 1$ and $k_1 = k_2 = 0$), hence by 4.6(10) one gets

$$-I_k^{-1}(M_{(l)}) = -I_k^{-1}(M_{(l-1)}).$$

If $h_l > 1$, then 5.3.2 and an iterated application of 4.6(10) and 5.3.1 give

$$-I_k^{-1}(M_{(l)}) = -I_k^{-1}(M_{(l-1)}) - \left(\frac{a_k p_k}{\tilde{h}_k h_k} \cdot \frac{p_{k+1}}{h_{k+1}} \cdots \frac{p_{l-1}}{h_{l-1}}\right)^2 \cdot \frac{np_l(h_l-1)}{d_l a_l h_l^2}.$$

Indeed, by 3.4(5) and 4.6,

$$-I_k^{-1}(M_{(l)}) = -I_k^{-1}(M_{(l-1)}) - (\mathrm{Lk}_{M_{(l-1)}}(g_{v'}, K_{(l-1)}))^2 \cdot \frac{np_l(h_l-1)}{d_l a_l h_l^2},$$

and, again by 4.6, $\mathrm{Lk}_{M_{(l-1)}}(g_{v'}, K_{(l-1)})$ equals

$$\mathrm{Lk}_{\Sigma(p_k, a_k n/d_k)}(O, Z) \cdot \mathrm{Lk}_{\Sigma(p_{k+1}, a_{k+1} n/d_{k+1})}(K_2^{(i)}, Z) \cdots \mathrm{Lk}_{\Sigma(p_{l-1}, a_{l-1} n/d_{l-1})}(K_2^{(i)}, Z).$$

7.13. The splicing formula for $\mathcal{T}_{M\sigma_{\mathrm{can}}}(1)$. Using 7.11(∗) and 7.12 (and $d_s = 1$), we can write

$$\begin{aligned}
&\mathcal{T}_{M_{(s)},\sigma_{\mathrm{can}}}(1) - h_s \cdot \mathcal{T}_{M_{(s-1)},\sigma_{\mathrm{can}}}(1) \\
&\qquad = \frac{na_s p_s}{2\tilde{h}_s^2 h_s^2} \cdot A_s + \sum_{k=1}^{s-1} (-I_k^{-1}(M_{(s)})) \cdot h_{k+1} \cdots h_s \cdot A_k/2 \\
&\qquad\quad - h_s \sum_{k=1}^{s-1} (-I_k^{-1}(M_{(s-1)})) \cdot h_{k+1} \cdots h_{s-1} \cdot A_k/2 \\
&\qquad = \frac{na_s p_s}{2\tilde{h}_s^2 h_s^2} \cdot A_s - \sum_{k=1}^{s-1} h_{k+1} \cdots h_s \cdot \frac{a_k^2 p_k^2 \cdots p_{s-1}^2}{\tilde{h}_k^2 h_k^2 \cdots h_{s-1}^2} \cdot \frac{np_s(h_s-1)}{a_s h_s^2} \cdot A_k/2.
\end{aligned}$$

But by 5.3(d) one has

$$\frac{na_s p_s}{2\tilde{h}_s^2 h_s^2} \cdot A_s = \mathcal{T}_{\Sigma(p_s, a_s, n),\sigma_{\mathrm{can}}}(1) + \frac{na_s p_s}{2\tilde{h}_s^2 h_s^2} \cdot \frac{h_s(h_s-1)}{a_s^2} \cdot (\Delta(f_{(s-1)})^\natural)''(1).$$

Therefore, (using also $(h_s - 1)/\tilde{h}^2 = h_s - 1$) one gets

$$\begin{aligned}
\mathcal{O}(\mathcal{T}_{\cdot,\sigma_{\mathrm{can}}}(1)) &= \mathcal{T}_{M_{(s)},\sigma_{\mathrm{can}}}(1) - h_s \cdot \mathcal{T}_{M_{(s-1)},\sigma_{\mathrm{can}}}(1) - \mathcal{T}_{\Sigma(p_s,a_s,n),\sigma_{\mathrm{can}}}(1) \\
&= \frac{np_s(h_s-1)}{2h_s a_s}\left[(\Delta(f_{(s-1)})^\natural)''(1) - \sum_{k=1}^{s-1} \frac{a_k^2 p_k^2 \cdots p_{s-1}^2}{\tilde{h}_k^2 h_k^2 h_{k+1} \cdots h_{s-1}} \cdot A_k\right].
\end{aligned}$$

Now, using 5.8 for $M = M_{(s)}$, and 7.2(k) for $l = s-1$, one has the following consequences:

7.14. Theorem. *The additivity obstruction* $\mathcal{O}(\mathbf{sw}^{\mathrm{TCW}}_{\cdot}(\sigma_{\mathrm{can}}))$ *is* 0*, in other words:*

$$\mathbf{sw}^{\mathrm{TCW}}_{M_{(s)}}(\sigma_{\mathrm{can}}) = h_s \cdot \mathbf{sw}^{\mathrm{TCW}}_{M_{(s-1)}}(\sigma_{\mathrm{can}}) + \mathbf{sw}^{\mathrm{TCW}}_{\Sigma(p_s,a_s,n)}(\sigma_{\mathrm{can}}).$$

In particular, by induction for $M = M_{(s)}$ *one gets*

$$\mathbf{sw}^{\mathrm{TCW}}_{M}(\sigma_{\mathrm{can}}) = \sum_{k=1}^{s} d_k \cdot \mathbf{sw}^{\mathrm{TCW}}_{\Sigma(p_k,a_k,n/d_k)}(\sigma_{\mathrm{can}}).$$

7.15. Corollary. *Consider the hypersurface singularity* $g(x,y,z) = f(x,y) + z^n$*, where* f *is an irreducible plane curve singularity. Assume that its link* M *is a rational homology sphere. If* $\sigma(g)$ *denotes the signature of the Milnor fiber of* g*, then*

$$-\mathbf{sw}^{\mathrm{TCW}}_{M}(\sigma_{\mathrm{can}}) = \sigma(g)/8.$$

In particular, the geometric genus of $\{g = 0\}$ *is topological and it is given by*

$$p_g = \mathbf{sw}^{\mathrm{TCW}}_{M}(\sigma_{\mathrm{can}}) - (K^2 + s)/8,$$

where the invariant $K^2 + s$ *(associated with any connected negative definite plumbing graph of* M*) is defined in the introduction.*

Proof. By 7.14 and [31, (3.2)] (cf. also 7.2(e)), we only have to show that

$$-\mathbf{sw}^{\mathrm{TCW}}_{\Sigma(p_k,a_k,n/d_k)}(\sigma_{\mathrm{can}}) = \sigma(p_k,a_k,n/d_k)/8$$

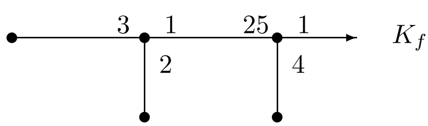
for any k. But this follows from [36, Section 6]. □

For an explicit formula for $\mathbf{sw}^{\mathrm{TCW}}_{\Sigma(p_k,a_k,n/d_k)}(\sigma_{\mathrm{can}})$, see [36] or [37].

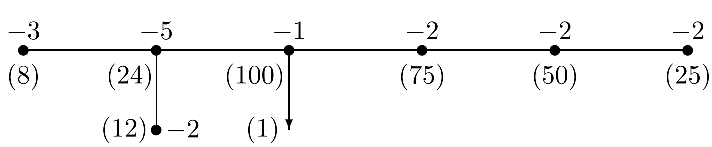
8. Appendix A

In this section we exemplify some of the theorems, formulas and invariant computations on a not very complicated, but still sufficiently representative example.

We start with a plane curve singularity f with two Newton pairs: $(p_1, q_1) = (2,3)$ and $(p_2, q_2) = (4,1)$. Then (cf. 6.1) $a_1 = 3$ and $a_2 = 25$, and the splice diagram of (S^3, K_f) is



The corresponding minimal good embedded resolution (or plumbing) graph—decorated with the self-intersections and the corresponding multiplicities—is

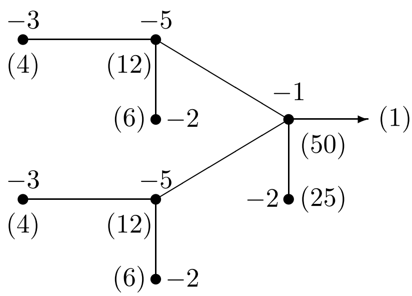


From this one can compute the Alexander polynomial $\Delta_{S^3}(K_f)(t)$ by 3.6(8), and one gets

$$\Delta_{S^3}(K_f)(t) = \frac{(t-1)(t^{24}-1)(t^{100}-1)}{(t^8-1)(t^{12}-1)(t^{25}-1)}.$$

Its degree $2r$ (or the Milnor number of f) is 80.

Now, we take $n = 2$. One gets the numerical invariants $d_0 = d_1 = 2$, $d_2 = 1$, $h_1 = 1$, $h_2 = 2$, and $\tilde{h}_1 = \tilde{h}_2 = 1$. Let M be the link of the hypersurface singularity $\{f(x,y) + z^2 = 0\}$ and K_z be the knot in M determined by $z = 0$. Then the graph of (M, K_z) is the following:

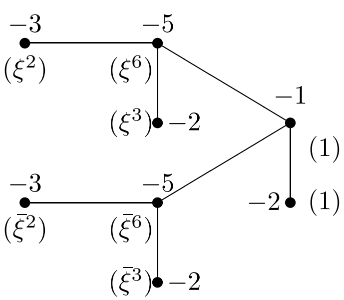


Then

$$\Delta_M(K_z)(t) = \frac{(t-1)(t^{12}-1)^2(t^{50}-1)^2}{(t^4-1)^2(t^6-1)^2(t^{25}-1)}.$$

In particular, $\Delta_M(K_z)(1) = 25$, which also equals the order $|H|$ of the group $H = H_1(M)$.

In fact, $H = \mathbb{Z}_{25}$, and the diagram of the characters is the following:



(where ξ is a 25-root of unity)

One can take for the vertex u (in 3.7) the -1-vertex. Therefore, $\hat{P}_{M,\xi,u}(t)$ equals

$$\frac{(t^{12}\xi^6-1)(t^{12}\bar{\xi}^6-1)(t^{50}-1)}{(t^4\xi^2-1)(t^4\bar{\xi}^2-1)(t^6\xi^3-1)(t^6\bar{\xi}^3-1)(t^{25}-1)}.$$

Then by 3.7(16) one has

$$\begin{aligned}\hat{\mathcal{T}}_{M,\sigma_{\mathrm{can}}}(\bar{\xi}) &= 2\cdot\frac{(\xi^6-1)(\bar{\xi}^6-1)}{(\xi^2-1)(\bar{\xi}^2-1)(\xi^3-1)(\bar{\xi}^3-1)}\\ &= 1+\frac{\xi}{\bar{\xi}-1}+\frac{\bar{\xi}}{\xi-1}+\frac{1}{(\xi-1)(\bar{\xi}-1)}.\end{aligned}$$

By a computation (via the identity $(**)$ used in the proof of 5.7), one gets

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)=\frac{1}{25}\sum_{\xi\neq 1=\xi^{25}}\hat{\mathcal{T}}_{M,\sigma_{\mathrm{can}}}(\xi)=\frac{108}{25}.$$

We continue with the additivity, respectively non-additivity formulas for the signature, torsion and Casson–Walker invariant.

For the signature, by 7.2(e), one has

$$\sigma(f+z^2)=2\cdot\sigma(2,3,1)+\sigma(4,25,2)=\sigma(4,25,2).$$

In fact, the value of $\sigma(a,b,c)$ was not needed in the body of the paper; for precise formulas see [36] or [37]. In our case, $\sigma(4,25,2)$ can be computed fast as follows. The Milnor number $\mu(4,25,2)$ of the Brieskorn singularity $(4,25,2)$ is $3\cdot 24\cdot 1=72$. Its geometric genus p_g is $\#\{i\geq 1\,:\,1/2+1/4+i/25<1\}$, which is 6. In particular, $\sigma(4,25,2)=4p_g-\mu=-48$. Hence $\sigma(f+z^2)=-48$ as well.

Since $\Sigma(2,3,1)=S^3$, its torsion is trivial. Nevertheless, it is not true (for the canonical spin^c structure) that $\mathcal{T}_M(1)=\mathcal{T}_{\Sigma(4,25,2)}(1)$. By 5.3(d) one has $\mathcal{T}_{\Sigma(4,25,2)}(1)=104/25$. Therefore,

$$\mathcal{T}_M(1)=\mathcal{T}_{\Sigma(4,25,2)}(1)+4/25.$$

For the Casson–Walker invariant, by 7.2(i), one has

$$-\lambda_{\mathrm{W}}(M)/2=-\lambda_{\mathrm{W}}(\Sigma(4,25,2))/2-4/25.$$

Therefore,

$$\mathbf{sw}^{\mathrm{TCW}}(M)=\mathbf{sw}^{\mathrm{TCW}}(\Sigma(4,25,2)).$$

In particular, the main identity for $f+z^2$ reduces to the corresponding identity valid for the Brieskorn singularity $(4,25,2)$ (which was verified in [36] and [37]).

In order to complete the discussion, we will compute the Casson–Walker invariant of M using the formula (5.3) of [36] (in terms of the plumbing graph of M):

$$-12\lambda_{\mathrm{W}}(M)=\sum_v e_v+3\cdot\#\mathcal{V}+\sum_v(2-\delta_v)I^{-1}_{vv}.$$

After computing the inverse matrix I^{-1}, one gets $-\lambda_{\mathrm{W}}(M)=42/25$. This shows that $\mathbf{sw}^{\mathrm{TCW}}(M)=108/25+42/25=6$, which is the same as $-\sigma(f+z^2)/8=48/8=6$, as the main result of the article predicts.

9. Appendix B: Index

A_k (7.2.k)
additivity obstruction $\mathcal{O}$ (5.5)
Alexander matrix V (3.6)
Alexander polynomial
  $\Delta^\natural_M(K)$, $\Delta^L_M(K)$, $\Delta_M(K)$, $\Delta_M(g_u)$ (3.6)
  twisted $\Delta_{M,\chi}(g_u)$ (3.6)
  $\Delta(f)$ (6.1)
  $\Delta^{c(k)}(t)$ (7.2.f)
alternating set (5.5)
canonical divisor K (2.1)
  $K^2 + s$ (2.1)
characters $\chi \in \hat{H}$ (3.6) (3.7) (7.3)
  born at level s (7.3)
  B_k (7.9)
  support $S(\chi)$, its complement $\bar{S}(\chi)$ (7.9)
closures $\bar{M}_i$ (4.2)
$\mathcal{D}(c_1, \dots, c_r)$, $\mathcal{D}(\Delta^\natural)$ (5.5)
$\mathcal{D}_a$ (5.8)
Dehn fillings (3.2)
invariants of oriented knots (3.1)
  longitude λ (3.1)
  oriented meridian m (3.1)
  tubular neighborhood $T(K)$ (3.1)
  parallel ℓ (3.1)
  o, δ (3.1)
invariants of singularities
  analytical (2.1)
  geometric genus (2.1)
  topological (2.1)
  smoothing (2.1)
invariants of 3-manifolds
  Casson–Walker invariant $\lambda(M)$, $\lambda_{\mathrm{W}}(M)$ (2.4)
    Walker–Lescop surgery formula (4.4)
    splicing formula (4.9)
  Seiberg–Witten invariant $\mathbf{sw}^*_M(\sigma)$ (2.4)
  Reidemeister–Turaev torsion $\mathcal{T}_{M,\sigma}$ (2.4)
    its Fourier transform $\hat{\mathcal{T}}_{M,\sigma}(\chi)$ (3.7)
    $\hat{P}_{M,\chi,u}(t)$ (3.7)
  $H = H_1(M, \mathbb{Z})$ (2.4)
link of a singularity (2.1)
linking form b_M (3.3)
linking numbers $\mathrm{Lk}_M(\cdot,\cdot)$ (3.3)
$\mathrm{Loc}(\chi)$ (7.9)
$(M_{(l)}, K_{(l)})$ (7.2)
Milnor number μ (2.1)
Milnor fiber F (2.1)
monodromy operator $\mathcal{M}$ (3.6)
  $\mathcal{M}_{(l)}$ (7.2)
singularities
  normal surface singularities (Sec. 2)
    elliptic, rational (2.3)
    hypersurface (Sec. 2)
    Gorenstein, $\mathbb{Q}$-Gorenstein (Sec. 2)
    suspension (1.2)
  plane curve singularities (6.1)
    Newton pairs $\{(p_i, q_i)\}_{i=1}^s$ (6.1)
    splice diagram decorations a_k (6.1)
    d_k, h_k, $\tilde{h}_k$ (7.1)
    A'Campo's formula (6.1)
plumbing graph Γ (2.1) (3.4)
  intersection matrix I (3.4)
  $\mathcal{V}$, e_v, δ_v, $\bar{\delta}_v$, g_v (3.4)
  weights $\{w_v(u)\}$ (3.4)
  Γ_+, Γ_-, Γ_{st} (7.9)
$\mathrm{Reg}(\chi)$, $\mathrm{Reg}(k)$ (7.9)
resolution $\tilde{X}$ (2.1)
resolution graph (2.1)
Seifert manifolds (5.2), $\Sigma(a,b,c)$ (Sec. 7)
  invariants α_i, β_i, α, e (5.2)
$\sigma_{(l)}$ (7.2)
signature $\sigma(F) = \mu_+ - \mu_-$ (2.1)
splice diagram (3.5)
splicing data (4.1)
splicing formulae
  Fujita's formula (4.3)
  other: (4.6) (4.7) (4.9)
Sylvester invariant (μ_0, μ_+, μ_-) (2.1)
working assumptions WA1 (4.1) and WA2 (4.7)

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