INTERSECTION THEORY

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ABSTRACT. I provide more details to the intersection theoretic results in [1].

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1. TRANSVERSALITY AND TUBULAR NEIGHBORHOODS

We need to introduce a bit of microlocal terminology. We begin by reviewing a few facts of linear algebra.

If V is a finite dimensional real vector space and U is a subspace of V, then its *annihilator* is the subspace

$$U^{\#} := \big\{ v^* \in V^*;\ \ \langle v^*, u\rangle = 0,\ \ \forall u \in U \,\big\}.$$

Note that

$$\dim U + \dim U^{\#} = \dim V = \dim V^*.$$

If we identify the bidual V^{**} with V in a canonical fashion, then we deduce

$$(U^{\#})^{\#} = U.$$

If U_0, U_1 are subspaces in V then

$$(U_0 + U_1)^{\#} = U_0^{\#} \cap U_1^{\#},\ \ (U_0 \cap U_1)^{\#} = U_0^{\#} + U_1^{\#}. \tag{1.1}$$

If $T : V_0 \to V_1$ is a linear map between finite dimensional vector spaces, then it has a dual

$$T^* : U_1^* \to U_0^*.$$

It satisfies the equalities

$$(\ker T)^{\#} = \mathbf{R}(T^*),\ \ \mathbf{R}(T)^{\#} = \ker(T^*) \tag{1.2}$$

where $\mathbf{R}$ denotes the range of a linear operator.

Two linear maps $T_i : U_0 \to V$, $i = 0, 1$ are said to be *transversal*, and we write this $T_0 \pitchfork T_1$, if

$$\mathbf{R}(T_0) + \mathbf{R}(T_1) = V.$$

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Using (1.1) and (1.2) we deduce that

$$T_0 \pitchfork T_1 \Longleftrightarrow \ker T_0^* \cap \ker T_1^* = 0, \tag{1.3}$$

i.e., $T_0 \pitchfork T_1$ if and only if the systems of linear equations $T_0^* v^* = 0$, $T_1^* v^* = 0$, $v^* \in V^*$, has only the trivial solution $v^* = 0$.

Suppose now that U_1 is a subspace in V and T_1 is the natural inclusion. In this case we say that T_0 *is transversal to* U_1 and we write this $T_0 \pitchfork U_1$. From (1.3) we deduce

$$T_0 \pitchfork U_1 \Longleftrightarrow \text{the restriction of } T_0^* \text{ to } U_1^\# \text{ is injective.} \tag{1.4}$$

Proposition 1.1. *Supposet that $T_0 \pitchfork U_1$. Then*

$$T_0^{-1}(U_1)^\# = T_0^*(U_1^\#).$$

In particular, the codimension of $T_0^{-1}(U_1)$ in U_0 is equal to the codimension of U_1 in V.

Proof. Fix surjection $T : V \to W$ such that $\ker T = U_1$. Then

$$T_0^{-1}(U_1) = \ker TT_0.$$

Using (1.2) we deduce

$$T_0^{-1}(U_1)^\# = \mathbf{R}(T_0^* T^*) = T_0^*(\mathbf{R}(T^*)).$$

On the other hand

$$\mathbf{R}(T^*) = (\ker T)^\# = U_1^\#.$$

□

Suppose now that M is a smooth manifold of dimension m and S is a closed submanifold of M. The tangent bundle of S is naturally a subbundle of the restriction to S of the tangent bundle of M,

$$TS \hookrightarrow (TM)|_S.$$

The quotient bundle $(TM)|_S/TS$ is called the *normal bundle of S in M* and it is denoted by $T_S M$. In particular, we have a short exact sequence of vector bundles over S

$$0 \to TS \to (TM)|_S \to T_S M \to 0. \tag{1.5}$$

Dualizing this sequence we obtain a short exact sequence

$$0 \to (T_S M)^* \to (T^*M)|_S \to T^*S \to 0.$$

The bundle $(T_S M)^*$ is called the *conormal bundle of S in M* and it is denoted by $T_S^* M$. It can be given the following intuitive description. Suppose that $(x^1, \dots, x^m)$ are local coordinates on an open set $\mathcal{O} \subset M$ such that in these coordinates the intersection $\mathcal{O} \cap S$ has the simple description

$$x^{s+1} = \cdots = x^m = 0,$$

i.e., in these coordinates $\mathcal{O} \cap S$ can be identified with the coordinate plane $(x^1, \dots, x^s, 0, \dots, 0)$. We can regard the differentials $dx^{s+1}, \dots, dx^m$ as defining a local trivialization of $T_S^* M$ on $\mathcal{O} \cap S$. Note that for any $x \in S$ we have

$$(T_S^*)_x = (T_x S)^\#. \tag{1.6}$$

Any Riemann metric on M defines an isomorphism between the normal and conormal bundles of S.

From the exact sequence (1.5) we deduce that if two of the bundles TS, $(TM)|_S$ $T_S M$ are orientable , then so is the third, and orientations on two induce orientations on the third via the *base-first convention*

$$\boldsymbol{or}(TM)|_S = \boldsymbol{or}\, TS \wedge \boldsymbol{or}\, T_S M. \tag{1.7}$$

A *co-orientation* on a closed submanifold $S \subset M$ is by definition an orientation on the normal bundle $T_S M$. Thus, an orientation on the ambient space M together with a co-orientation on S induce an orientation on S.

Definition 1.2. Suppose that S is a closed submanifold of the smooth manifold M. A *tubular neighborhood* of S in M is a pair $(\mathcal{N}, \Psi)$, where $\mathcal{N}$ is an open neighborhood of S in M, and $\Psi : T_S M \to \mathcal{N}$ is a diffeomorphism such that its restriction to $S \hookrightarrow T_S N$ coincides with the identity map $\mathbb{1}_S : S \to S$. □

We have the following fundamental result, [2, 3].

Theorem 1.3. *(a) Let S and M be as in the above definition. Then for any open neighborhood $\mathcal{U}$ of S in M there exists a tubular neighborhood $(\mathcal{N}, \Psi)$ such that $\mathcal{N} \subset \mathcal{U}$.*
(b) If $(\mathcal{N}_0, \Psi_0)$ and $(\mathcal{N}_1, \Psi_1)$ are two tubular neighborhoods of S, then there exists a smooth map

$$H : [0,1] \times M \to M, \;\; x \mapsto H_t(x),$$

with the following properties.[1]

(i) $H_0 = \mathbb{1}_M$.
(ii) H_t *is a diffeomorphism of* M.
(iii) $H_t(x) = x \; \forall x \in S$, $t \in [0,1]$.
(iv) $H_1(\mathcal{N}_0) = \mathcal{N}_1$.
(v) *The diagram below is commutative*

$$\begin{tikzcd} & T_S M \arrow[dl, "\Psi_0"'] \arrow[dr, "\Psi_1"] & \\ \mathcal{N}_0 \arrow[rr, "H_1"'] & & \mathcal{N}_1 \end{tikzcd}$$

□

If $(\mathcal{N}, \Psi)$ is a tubular neighborhood of S in M, then we have a natural projection

$$\pi_{\mathcal{N}} : \mathcal{N} \xrightarrow{\Psi} T_S M \xrightarrow{\pi} M.$$

We will denote by $\mathcal{N}_x$ the fiber of $\pi_{\mathcal{N}}$ over $x \in S$. Observe that $x \in \mathcal{N}_x$.

Remark 1.4. When S is compact, we can construct a tubular neighborhood of S as follows.

- Fix a Riemann metric g on M.
- Then for any $\varepsilon > 0$ sufficiently small, the set
$$\mathcal{N}_g^{\varepsilon} := \bigl\{ x \in M; \;\; \mathrm{dist}_g(x, S) < \varepsilon \bigr\}$$
determines a tubular neighborhood of S.
- The natural projection $\pi_{\mathcal{N}} : \mathcal{N}_g^{\varepsilon} \to S$ has the following description: for $x \in \mathcal{N}_g^{\varepsilon}$, the point $\pi_{\mathcal{N}}(x)$ is the point on S closest to x.

If $\mathcal{N}_g^{\varepsilon}$ is as above, then the diffeomorphism $\Psi : T_S M \to \mathcal{N}_g^{\varepsilon}$ can be constructed as follows. The normal bundle $T_S M$ can be identified with the orthogonal complement $(TS)^{\perp}$ of TS in $(TM)_S$ and if $x \in S$, $X \in (T_x S)^{\perp} \subset T_x M$, then

$$\Psi(X) = \exp_{g,x}\bigl(\beta(\, |X|_g \,) X \,\bigr),$$

[1] A map H with the properties (i), (ii), (iii) above is called an *isotopy* (or *diffeotopy*) of M rel S.

where $\exp_{g,x} : T_x M \to M$ is the exponential map defined by the metric g, $|X|_g$ is the g-length of X, and $\beta : [0, \infty) \to [0, \varepsilon)$ is a diffeomorphism such that $\beta(t) = t$ if $t < \frac{\varepsilon}{4}$. □

Suppose M (respectively R, S) is a smooth manifold of dimension m (respectively r, s) and $f : R \to M$, $g : S \to M$ are smooth maps. We say that f *is transversal to* g, and we write this $f \pitchfork g$, if for any $x \in R$ and $y \in S$ such that $f(x) = g(y) = z \in M$ the differentials $\dot{f}_x : T_x R \to T_z M$ and $\dot{g} : T_y S \to T_z M$ are transversal. When S is a closed submanifold of M and g is the natural inclusion, we say that f *is transversal to* S and we write this $f \pitchfork S$.

Proposition 1.5. *Suppose M is a manifold of dimension m, S is a closed submanifold of M of codimension c, R is a smooth manifold of dimension r and $f : R \to M$ is a smooth map transversal to S. Then $f^{-1}(S)$ is a closed submanifold of R of codimension c and there exists natural bundle isomorphism*

$$\dot{f}^* : f^* T_S^* M \to T^*_{f^{-1}(S)} R, \;\; \dot{f} : T_{f^{-1}(S)} \to f^* T_S M.$$

Proof. The first claim is local so it suffices to prove it in the special case when M coincides with a vector space of dimension V and S is a subspace U of V of codimension c. Consider a surjection $T : V \to \mathbb{R}^c$ such that $\ker T = U = S$.

Fix $x \in R$, set $y = f(x)$ and denote by $\dot{f}_x$ the differential of f at x, $\dot{f}_x : T_x R \to T_y M = V$. Observe that

$$f^{-1}(S) = \{z \in R; \;\; T(f(z)) = 0\}$$

The differential of $T \circ f$ at $x \in R$ is $T \circ \dot{f}_x$. Since

$$\dot{f}_x(T_x R) + T_y S = \dot{f}_x(T_x R) + U_0 = T_y M = V$$

and the linear map T is surjective with kernel U we deduce that $T \circ \dot{f}_x$ is surjective. We can now invoke the implict function theorem to conclude that that $f^{-1}(S)$ is a submanifold. Using Proposition 1.1 and (1.6) we deduce that the dual of $\dot{f}_x$ defines a linear isomorphism

$$\dot{f}_x^* : (T_S^* M)_{f(x)} \to (T^*_{S^{-1}(S)} R)_x, \;\; \forall x \in f^{-1}(S).$$

Observing that $(T_S^* M)_{f(x)}$ is the fiber at $x \in f^{-1}(S)$ of the pullback $f^*(T_S^* M)$ we deduce that these isomorphisms define a bundle isomorphism

$$\dot{f}^* : f^*(T_S^* M) \to T^*_{f^{-1}(S)} R.$$

The isomorphism $\dot{f} : T_{f^{-1}(S)} R \to f^* T_S M$ is the dual of the above isomorphism. □

2. The Poincaré dual of a submanifold

Let M be a finite type, oriented smooth manifold of dimension m. A (*Borel-Moore*) *cycle* of dimension s in M is defined to be a linear map map

$$C : H_c^s(M) \to \mathbb{R}, \;\; H_c^s(M) \ni \omega \mapsto \langle C, \omega \rangle.$$

The *Poincaré dual* of the cycle C is the cohomology class $\eta_C \in H^{m-s}(M)$ uniquely determined by the equality

$$\int_M \omega \wedge \eta_C = \langle C, \omega \rangle, \;\; \forall \omega \in H_c^s(M).$$

Suppose that S is a closed oriented submanifold of dimension s of the oriented manifold M of dimension m. It defines an s-dimensional cycle $[S]$ via the equality

$$\langle [S], \omega\rangle = \int_S \omega, \ \ \forall \omega \in H^s_c(M).$$

The *Poincaré dual of S in M* is the Poincaré dual of the cycle $[S]$, i.e., cohomology class $\eta_S = \eta_S^M \in H^{m-s}(M)$ uniquely determined by the equality

$$\int_M \omega \wedge \eta_S = \int_S \omega, \ \ \forall \omega \in H^s_c(M).$$

Theorem 2.1. *Suppose that $\pi : E \to S$ is an oriented real vector bundle of rank n over the oriented manifold S. Denote by Φ_E the Thom class of E and equip the total space of E using the base first convention. Denote by $i : S \hookrightarrow E$ the inclusion of S as zero section. Then*

$$\eta_S^E = \phi_E. \tag{2.1}$$

Moreover, if S is compact then

$$\int_E \omega \wedge \Phi_E = \int_S \omega, \ \ \forall \omega \in \Omega^s(E), \ \ d\omega = 0. \tag{2.2}$$

*We want to emphasize that in the above equality **we do not require that ω have compact support**.*

Proof. Define

$$H : E \times [0,1] \to E, \ \ E \times [0,1] \ni (z,t) \mapsto H_t(z) := (1-t) * z + t * i \circ \pi(z),$$

where $t*$ denotes the multiplication by t along the fibers of E. Observe that H is proper, $H_0 = \mathbb{1}_E$, $H_1 = i \circ \pi$. Thus for any closed, compactly supported $\omega \in \Omega^s_c(E)$ there exists a $\tau \in \Omega^{s-1}_c(E)$ such that

$$\omega - \pi^* i^* \omega = H_0^* \omega - H_1^* \omega = d\tau.$$

We have

$$\int_E \omega \wedge \Phi_E = \int_E \pi^*(i^*\omega) \wedge \Phi_E + \underbrace{\int_E d(\tau \wedge \Phi_E)}_{=0}$$

(use the projection formula)

$$= \int_S i^*\omega \wedge \pi_* \Phi_E = \int_S i^* \omega.$$

This proves (2.1). To prove (2.2) use the same argument and the fact that when S is compact, the Thom class has compact support. □

Corollary 2.2. *Suppose that S is a compact s-dimensional oriented submanifold of the smooth, finite type oriented manifold M of dimension m. Denote by $\Phi_{T_S N}$ the Thom class of the normal bundle with the orientation induced via the base-first convention (1.7). Fix a tubular neighborhood $(\mathcal{N}, \Psi)$ of S in M, denote by $\eta_S^{\mathcal{N}}$ the Poincaré dual of S in $\mathcal{N}$, and by j_* the extension by 0 morphsim $H^*_c(\mathcal{N}) \to H^*(M)$. Then*

$$\eta_S^N = (\Psi^{-1})^* \Phi_{T_S M}, \ \ \eta_S^M = j_* \eta_S^{\mathcal{N}}. \tag{2.3}$$

□

Remark 2.3. Let us point out a rather subtle phenomenon. Suppose that S is a compact s-dimensional oriented submanifold of the oriented manifold M of dimmension m. Suppose further that $\mathcal{N}$ is a tubular neighborhood of S in M Observe that S defines defines a linear map

$$H^s(M) \ni \omega \mapsto \int_S \omega \in \mathbb{R}.$$

From the Poincaré duality we deduce that there exists a *unique* compactly supported cohomology class $\eta_S^{\mathrm{comp}} \in H_c^{m-s}(M)$ such that

$$\int_M \omega \wedge \eta_S^{\mathrm{comp}} = \int_S \omega, \ \ \forall \omega \in H^s(M).$$

We have a natural morphism $j_\# : H_c^{m-s}(M) \to H^{m-s}(M)$ and the above corollary proves that $\eta_S = j_\#(\eta_S^{\mathrm{comp}})$. In particular, we deduce that two closed compactly supported forms η_0, η_1 representing η_S must be cohomologous *as compactly supported forms*, i.e., there exists a compactly supported form α such that $\eta_1 - \eta_0 = d\alpha$. □

Corollary 2.4. *Suppose that S is a compact s-dimensional oriented submanifold of the smooth finite type oriented manifold M of dimension m. A cohomology class $u \in H^{m-s}(M)$ is the Poincaré dual of S if and only if there exists a tubular neighborhood $(\mathcal{N}, \Psi)$ of S in M such that u is represented by a closed form η with compact support contained in $\mathcal{N}$ and such that*

$$\int_{\mathcal{N}_x} \eta = 1, \ \ \forall x \in S,$$

where $\mathcal{N}_x$ is oriented as the fiber of $(T_S M)_x$ via the diffeomorphism $\Psi : (T_S M)_x \to \mathcal{N}_x$. □

Definition 2.5. Let S, and L be closed, oriented submanifolds of the oriented manifold N such that $\dim S + \dim L = \dim N$

(a) If $\boldsymbol{p} \in L \cap S$ is a point where the two submanifolds intersect transversally, i.e., $T_{\boldsymbol{p}} L + T_{\boldsymbol{p}} S = T_{\boldsymbol{p}} N$, then we define $\epsilon(\boldsymbol{p}) = \epsilon(S, L, \boldsymbol{p}) \in \{\pm 1\}$ via the rule

$$\boldsymbol{or}(T_{\boldsymbol{p}} S) \wedge \boldsymbol{or}(T_{\boldsymbol{p}} L) = \epsilon(S, L, \boldsymbol{p})\, \boldsymbol{or}(T_{\boldsymbol{p}} N).$$

We will refer to $\boldsymbol{or}(L, S\boldsymbol{p})$ as the *local intersection number* of L and S at $\boldsymbol{p}$. Note that

$$\boldsymbol{or}(L, S, \boldsymbol{p}) = (-1)^{\dim L \cdot \dim S}\, \boldsymbol{or}(S, L, \boldsymbol{p}).$$

(b) If S is compact, and $L \pitchfork S$, then $L \cap S$ is finite, and we define the *intersection number* of the cycles L and S to be the integer

$$[S] \bullet [L] := \sum_{\boldsymbol{p} \in S \cap L} \epsilon(S, L, \boldsymbol{p}).$$ □

Theorem 2.6. *Suppose that S, L are closed oriented submanifolds of the smooth, finite type, oriented manifold N satisfying the following conditions.*

- *S is compact.*
- $\dim L + \dim S = \dim N$.
- $L \pitchfork S$.

If η_S is compactly supported closed form representing the Poincaré dual of S., and η_L is a closed form representing the Poincaré dual of $[L]$,Then

$$[S] \bullet [L] = \int_M \eta_L \wedge \eta_S = \int_L \eta_S. \tag{2.4}$$

Proof. Set $s := \dim S$, $\ell := \dim L$, $n = s + \ell = \dim N$. For simplicity, for any $\boldsymbol{p} \in S \cap L$ we will set $\epsilon(\boldsymbol{p}) = \epsilon(S, L, \boldsymbol{p})$.

From the transversality of the intersection we deduce that for any $\boldsymbol{p} \in L \cap S$ we can find local coordinates $(x^1, \dots, x^n)$ defined on an open neighborhood $U_{\boldsymbol{p}}$ of $\boldsymbol{p}$ such that

$$U_{\boldsymbol{p}} \cap S = \{ x^{s+1} = \cdots = x^{s+\ell} = 0 \}, \;\; U_{\boldsymbol{p}} \cap L = \{ x^1 = \cdots = x^s = 0 \},$$

the orientation of $S \cap U$ is described by the volume form $dx^1 \wedge \cdots \wedge dx^s$, the orientation of $L \cap U$ is given by $dx^{s+1} \wedge \cdots \wedge dx^{s+\ell}$. Under these assumptions we deduce that

$$\boldsymbol{or}(U_{\boldsymbol{p}}) = \epsilon(\boldsymbol{p}) dx^1 \wedge \cdots \wedge dx^n. \tag{2.5}$$

We can additionally assume that $U_{\boldsymbol{p}} \cap U_{\boldsymbol{q}} = \emptyset$ if $\boldsymbol{p} \neq \boldsymbol{q}$.

We can find neighborhoods $V_{\boldsymbol{p}}$ of $\boldsymbol{p}$ in $U_{\boldsymbol{p}}$ and a Riemann metric g on N such that along $V_{\boldsymbol{p}}$ the metric g has the Euclidean form $g = (dx^1)^2 + \cdots + (dx^n)^2$.

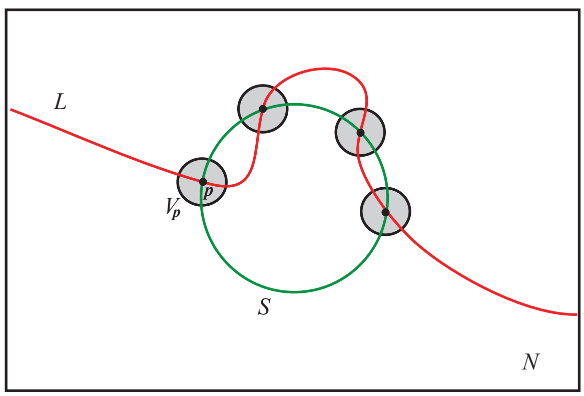

FIGURE 1. *Two manifolds of complementary dimensions intersecting transversally*

Construct a thin tubular neighborhood $(\mathcal{N}, \Psi)$ of S in N such that for every $\boldsymbol{p} \in L \cap S$ we have (see Figure 1)

$$\mathcal{N}_{\boldsymbol{p}} = (L \cap V_{\boldsymbol{p}}) \cap \mathcal{N} =: L_{\boldsymbol{p}}$$

where we recall that $\mathcal{N}_{\boldsymbol{p}}$ denotes the fiber of $\mathcal{N}$ over $\boldsymbol{p}$ with respect to the natural projection $\pi_{\mathcal{N}} : \mathcal{N} \to S$. We can choose $\mathcal{N}$ thin enough so that we have the aditional equality

$$L \cap \mathcal{N} = \bigcup_{\boldsymbol{p} \in S \cap L} \mathcal{N}_{\boldsymbol{p}}.$$

Consider a compactly supported closed form $\eta_S^{\mathcal{N}} \in \Omega_c^\ell(\mathcal{N})$ representing the Poincaré dual of S in $\mathcal{N}$.

The fiber $\mathcal{N}_{\boldsymbol{p}}$ is equipped with a natural orientation, given by the *co-orientation* of S in N. On the other hand $L_{\boldsymbol{p}}$ is equipped with an orientation induced by the orientation of L. The equality (2.5) implies

$$\boldsymbol{or}(L_{\boldsymbol{p}}) = \epsilon(\boldsymbol{p})\, \boldsymbol{or}(\mathcal{N}_{\boldsymbol{p}}).$$

Now observe that

$$\int_M \eta_S \wedge \eta_L = \int_L \eta_S = \sum_{\boldsymbol{p}\in S\cap L} \int_{L_{\boldsymbol{p}}} \eta_S = \sum_{\boldsymbol{p}\in S\cap L} \epsilon(\boldsymbol{p}) \int_{\mathcal{N}_{\boldsymbol{p}}} \eta_S = \sum_{\boldsymbol{p}\in S\cap L} \epsilon(\boldsymbol{p}).$$

□

3. SMOOTH CYCLES AND THEIR INTERSECTIONS

Suppose that M is a smooth, compact oriented manifold of dimension m, R is a smooth compact oriented manifold of dimension r and $f : R \to M$ is a smooth map. The pair (R, f) defines a linear map

$$H^r(M) \to \mathbb{R}, \quad H^r(M) \ni \omega \mapsto \int_R f^*\omega.$$

We will denote this element of $H^r(M)^*$ by $f_*[R]$ and we will refer to it as the *smooth cycle* of dimension r determined by the pair (R, f). From the Poincaré duality we deduce that there exists $\eta_{f_*[R]} \in H^{m-r}(M)$ such that

$$\int_R f^*\omega = \int_M \omega \wedge \eta_{f_*[R]}, \quad \forall \omega \in H^r(M).$$

We will refer to the cohomology class $\eta_{f_*[R]}$ as the *Poincaré dual* of the smooth cycle $f_*[R]$.

Suppose that R_0, R_1 are two compact oriented manifolds of the same dimension r and $f_i : R_i \to M$ are smooth maps, $i = 0, 1$. We say that the pairs (R_0, f), (R_1, f_1) are *cobordant*, and we write this $(R_0, f_0) \sim_c (R_1, f_1)$ if there exists a compact, oriented $(r+1)$-dimensional manifold with boundary $\widehat{R}$ and a smooth map $\widehat{f} : \widehat{R} \to M$ such that the following hold.

- $\partial\widehat{R} = R_0 \sqcup R_1$.
- The orientation induced by $\widehat{R}_1$ coincides with the orientation of R_1, while the orientation induced by $\widehat{R}$ on R_0 is the opposite orientation of R_0.
- $\widehat{f}|_{R_i} = f_i$, $i = 0, 1$.

The pair $(\widehat{R}, \widehat{f})$ is called an *oriented cobordism* between (R_0, f_0) and (R_1, f_1).

Proposition 3.1. *If* $(R_0, f_0) \sim (R_1, f_1)$ *then*

$$(f_0)_*[R_0] = (f_1)_*[R_1].$$

Proof. We have to show that for any $\omega \in H^r_c(M)$, $r = \dim R_0 = \dim R_1$ we have

$$\int_{R_0} f_0^*\omega = \int_{R_1} f_1^*\omega.$$

Let $(\widehat{R}, \widehat{f})$ be an oriented cobordism between (R_0, f_0) and (R_1, f_1) and $\omega \in H^r_c(M)$ then

$$\int_{R_1} f_1^*\omega - \int_{R_0} f_0^*\omega = \int_{\partial\widehat{R}} \widehat{F}^*\omega = \int_{\widehat{R}} d\widehat{f}^*\omega$$

$$= \int_{\widehat{R}} \widehat{f}^*(d\omega) = 0.$$

□

Suppose that S is a smooth, compact, oriented submanifold of M of codimension k. Its conormal bundle $T^*_S M$ is equipped with a natural orientation given by the rule

$$\boldsymbol{or}(TM)|_S = \boldsymbol{or}(T^*M)|_S = \boldsymbol{or}(S) \wedge \boldsymbol{or}(T^*_S M).$$

Denote by i the canonical inclusion $S \hookrightarrow M$. Observe that

$$\eta_S = \eta_{i_*[S]} \in H^k(M).$$

Two compact, oriented submanifolds $i_0 : S_0 \hookrightarrow M$, $(i_1 : S_1 \hookrightarrow M$ of the same dimension s are said to be *cobordant* if the pairs (S_0, i_0) and (S_1, i_1) are cobordant. We write this $S_0 \sim_c S_1$. Proposition 3.1 shows that

$$S_0 \sim_c S_1 \Rightarrow \eta_{S_0} = \eta_{S_1}.$$

Corollary 3.2. *Suppose that S_0, S_1 are two compact, oriented s-dimensional manifolds of the oriented, finite type smooth manifold M of dimension m, and L is a closed, oriented submanifold of M of dimension $m - s$ that intersects transversally both S_0 and S_1. If $S_0 \sim_c S_1$, then*

$$[S_0] \bullet [L] = [S_1 \bullet [L].$$

Proof. Since $S_0 \sim_c S_1$ we have $\eta_{S_0} = \eta_{S_1}$. The corollary now follows form Theorem 2.6. $\square$

Example 3.3. For $t \in [0, 1]$ define

$$F_t : \mathbb{CP}^1 \to \mathbb{CP}^2, \;\; \mathbb{CP}^1 \ni [z_0, z_1] \mapsto [tz_0, (1-t)z_1, z_1] \in \mathbb{CP}^2,$$

where $[z_0, z_1, \dots, z_n]$ denote the homogeneous coordinates in $\mathbb{CP}^n$. Observe that each F_t is an embedding. The image of F_0 is the projective line

$$H_0 = \big\{ [\zeta_0, \zeta_1, \zeta_2]; \;\; \zeta_0 = 0 \big\},$$

while the image of F_1 is the projective line

$$H_1 = \big\{ [\zeta_0, \zeta_1, \zeta_2] \in \mathbb{CP}^2; \;\; \zeta_1 = 0 \big\}.$$

We deduce that

$$\eta_{H_0} = \eta_{H_1} =: \eta_H.$$

Observe that $H_0 \pitchfork H_1$ and the unique intersection point is $\boldsymbol{p}_0[0, 0, 1]$. An lementary computation shows that $\epsilon(\boldsymbol{p}_0) = 1$. Hence

$$\int_{\mathbb{CP}^2} \eta_H \wedge \eta_H = [H_0] \bullet [H_1] = 1. \qquad \square$$

Suppose that R is a compact oriented manifold of dimension r, $S \subset M$ is a compact oriented submanifold of codimension k, and $f : R \to M$ is a smooth map transversal to S. Using Proposition 1.5 we deduce that the submanifold $f^{-1}(S)$ of R carries a co-orientation induced by the bundle isomorphism

$$\dot{f}^* : f^*(T^*_S M) \to T^*_{f^{-1}(S)} R.$$

We thus have an orientation on $f^{-1}(S)$. We obtain a smooth $(r - k)$-dimensional cycle

$$[S] \cap f_*[R] := f_*[f^{-1}(S)].$$

Denote by $\eta_{[S] \cap f_*[R]}$ the Poincaré dual of this cycle, i.e.,

$$\int_M \omega \wedge \eta_{[S] \cap f_*[R]} = \int_{f^{-1}(S)} f^*\omega, \;\; \forall \omega \in H^{r-k}(M). \tag{3.1}$$

Theorem 3.4. *Let M, R, S and f as above. Then*

$$f^*\eta_S^M = \eta^R_{f^{-1}(S)}. \tag{3.2}$$

$$\eta_{[S]\cap f_*[R]} = \eta_S \wedge \eta_{f_*[R]}. \tag{3.3}$$

Proof. Let first observe that (3.3) is an immediate consequence of (3.2). Indeed, for any $\omega \in H^{r-k}(M)$ we have

$$\int_M \omega\wedge\eta_S\wedge\eta_{f_*[R]} = \int_R f^*(\omega\wedge\eta_S) = \int_R f^*\omega\wedge f^*\eta_S$$

$$\stackrel{(3.2)}{=} \int_R f^*\omega\wedge\eta^R_{f^{-1}(S)} = \int_{f^{-1}(S)} f^*\omega \stackrel{(3.1)}{=} \int_M \omega\wedge\eta_{[S]\cap f_*[R]}.$$

To prove (3.2) we will rely on Corollary 2.4. Fix a Riemann metric h on M and a metric g on R. For simplicity we set $X = f^{-1}(S) \subset R$. Form the tubular neighborhood $\mathcal{N}_h^\varepsilon$ of S in M and $\mathcal{N}_g^\rho$ of X in R as explained in Remark 1.4. For $x \in X$ we denote by $\mathcal{N}_g^\rho(x)$ the fiber over x of the natural projection $\mathcal{N}_g^\rho \to X$ and by $\mathcal{N}_h^\varepsilon(y)$, $y = f(x)$ the fiber of the natural projection $\mathcal{N}_h^\varepsilon \to S$. Observe that

$$\mathcal{N}_g^\rho(x) = \bigl\{ z\in R;\ \operatorname{dist}_g\bigl(z, X\bigr) = \operatorname{dist}_g(z,x) < \rho \bigr\}.$$

Lemma 3.5. *There exists $\rho_0 > 0$ sufficiently small such that the restriction of f to $\mathcal{N}_g^{\rho_0}(x)$ is an embedding for any $x \in X$.*

Proof. Denote by $\dot{f}_x$ the differential of f at x. It is a linear operator $\dot{f}_x : T_xR \to T_yM$, whose kernel is T_xX. Thus, its restriction to $T_x\mathcal{N}_g^\rho(x) = (T_xX)^\perp$ is injective. We can then find a neighborhood U_x of $x \in S$ and $\rho(x) > 0$ such that for any $x' \in U$ the restriction of f to $\mathcal{N}_g^{\rho(x)}(x')$ is an embedding. Now cover X with finitely many neighborhood $U_{x_1}, \dots, U_{x_\nu}$ and set

$$\rho_0 = \min\bigl\{\rho(x_1), \dots, \rho(x_\nu)\bigr\}.$$

□

For $\rho < \rho_0$ we set

$$L_x^\rho := f\bigl(\mathcal{N}_g^\varepsilon(x)\bigr).$$

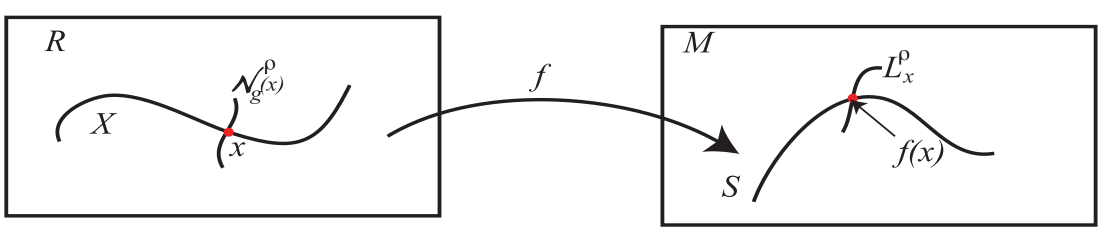

FIGURE 2. *Mapping the normal slice $\mathcal{N}_g^\rho(x)$.*

Lemma 3.6. *Fix $\rho < \rho_0$. There exists $\varepsilon = \varepsilon(\rho) > 0$ such that for any $x \in X$ the intersection $L_x^\rho \cap \mathcal{N}_h^\varepsilon$ is closed in $\mathcal{N}_h^\varepsilon$.*

Proof. Set
$$\Sigma_\rho := \big\{ z \in R;\ \ \mathrm{dist}_g\big(z, X\big) = \rho \big\}.$$
The set $f(\Sigma_\rho) \subset M$ is compact and disjoint from S and thus
$$d(\rho) := \mathrm{dist}_h\big(f(\Sigma_\rho), S\big) > 0.$$
Let $\varepsilon = \frac{1}{2}d(\rho)$. We claim that that $L_x^\rho \cap \mathcal{N}_h^\varepsilon$ is closed in $\mathcal{N}_h^\varepsilon$. Suppose that (y_n) is a sequence in $L_x^\rho \cap \mathcal{N}_h^\varepsilon$ that converges to a point $y_* \in \mathcal{N}_h^\varepsilon$. We need to show that $y_* \in L_x^\rho$.

By construction, there exist points $p_n \in \mathcal{N}_g^\rho(x)$ such that $f(p_n) = x_n$. The sequence (p_n) has a subsequence that converges to a point $p_* \in \mathcal{N}_g(x)$ such that $\mathrm{dist}\,(p_*, x) \leq r$. Then $f(p_*) = y_*$ and
$$\mathrm{dist}(y_*, S) < \varepsilon < d(\rho).$$
This implies that $p_* \not\in \Sigma_\rho$ because $\mathrm{dist}(f(\Sigma_\rho), S) > d(\rho)$. Hence $\mathrm{dist}(p_*, x) = \mathrm{dist}(p_*, S) < \rho$ so that $p_* \in \mathcal{N}_g^\rho(x)$ and thus $y_* \in L_x^\rho$. □

Let $\varepsilon = \varepsilon(\rho)$ as in the above Lemma. Denote by η_S^ε a closed form representing the Poincaré dual of S having support in $\mathcal{N}_h^\varepsilon$. According to Corollary 2.4 the differential form $f^*\eta_S^\varepsilon$ represents the Poincaré dual of $X = f^{-1}(S)$ in R if
$$\int_{\mathcal{N}_g^\rho(x)} f^*\eta_S^\varepsilon = 1,\ \ \forall x \in X.$$
We have
$$\int_{\mathcal{N}_g^\rho(x)} f^*\eta_S^\varepsilon = \int_{L_X^\rho} \eta_S^\varepsilon = \int_{L_x^\rho \cap \mathcal{N}_h^\varepsilon} \eta_S^\varepsilon,$$
where L_x^ρ is equipped with the orientation induced by the diffeomorphism $f : \mathcal{N}_g^\rho(x) \to L_x^\rho$. The orientation on the normal slice $\mathcal{N}_g^\rho(x)$ is the co-orientation of X which is determined by the co-orientation of S. Hence the tangent space of L_x^ρ at x is equipped with the coorientation of S at x. Our choice of ϵ guarantees that $L_x^\rho \cap \mathcal{N}_h^\varepsilon$ is closed in $\mathcal{N}_h^\varepsilon$. Theorem 2.6 now implies that
$$\int_{L_x^\rho \cap \mathcal{N}_h^\varepsilon} \eta_S^\varepsilon = [S] \bullet [L_x^\rho] = 1.$$
□

In the special case when $f : R \hookrightarrow M$ is an embedding, the equality (3.2) can be reformulated as follows.

Corollary 3.7. *Suppose that R, S are compact oriented submanifolds of the oriented finite type manifold M and $S \pitchfork R$. Then $S \cap R$ is a smooth submanifold of R. We orient $S \cap R$ using the natural isomorphism*
$$T_{S\cap R}R \cong (T_S M)|_{S\cap R}.$$
Then the Poincaré dual of $S \cap R$ in R is equal to the restriction to R of the Poincaré dual of S in M. □

We want to define a new operation on smooth cycles. Suppose that M_0, M_1 are smooth oriented, finite type manifolds of dimensions m_0 and respectively m_1. Let $f_i : R_i \to M_i$, $i = 0, 1$ be smooth maps, where R_i is a smooth compact manifold of dimension r_i. These maps define smooth cycles $(f_i)_*[R_i]$, $i = 0, 1$, with Poincaré duals
$$\eta_{(f_i)_*[R_i]} \in H^{m_i - r_i}(M_i).$$

We have a natural bilinear map

$$\boxtimes : H^*(M_0) \times H^*(M_1) \to H^*(M_0 \times M_1),\ \ \alpha_0 \boxtimes \alpha_1 = (\pi_0^*\alpha_0) \wedge (\pi_1^*\alpha_1),$$

where $\pi_i : M_0 \times M_1 \to M_i$, $i = 0, 1$, denotes the canonical projection. We have a smooth map

$$f_0 \times f_1 : R_0 \times R_1 \to M_0 \times M_1.$$

Equip $R_0 \times R_1$ and $M_0 \times M_1$ with the product orientations. We obtain a smooth cycle

$$(f_0 \times f_1)_*[R_0 \times R_1]$$

with Poincaré dual

$$\eta_{(f_0\times f_1)_*[R_0\times R_1]} \in H^{(m_0+m_1)-(r_0+r_1)}(M_0 \times M_1).$$

Proposition 3.8.

$$\eta_{(f_0\times f_1)_*[R_0\times R_1]} = (-1)^{r_1(m_0-r_0)}\eta_{(f_0)_*[R_0]} \boxtimes \eta_{(f_1)_*[R_1]}. \tag{3.4}$$

Proof. We have

$$\int_{M_0\times M_1} \alpha_0 \boxtimes \alpha_1 \wedge \eta_{(f_0\times f_1)_*[R_0\times R_1]} = \int_{R_0\times R_1} (f_0 \times f_1)^*(\alpha_0 \boxtimes \alpha_1) = \int_{R_0\times R_1} f_0^*\alpha_0 \boxtimes f_1^*\alpha_1$$
$$= \left(\int_{R_0} f_0^*\alpha_0\right)\left(\int_{R_1} f_0^*\alpha_1\right).$$

On the other hand

$$\int_{M_0\times M_1}(\alpha_0 \boxtimes \alpha_1) \wedge (\eta_{(f_0)_*[R_0]} \boxtimes \eta_{(f_1)_*[R_1]}) = \int_{M_0\times M_1} \pi_0^*\alpha_0 \wedge \pi_1^*\alpha_1 \wedge \pi_0^*\eta_{(f_0)^*[R_0]} \wedge \pi_1^*\eta_{(f_1)_*[R_1]}$$
$$= (-1)^{r_1(m_0-r_0)} \int_{M_0\times M_1} \pi_0^*(\alpha_0 \wedge \eta_{(f_0)_*[R_0]}) \wedge \pi_1^*(\alpha_1 \wedge \eta_{(f_1)_*[R_1]})$$
$$= (-1)^{r_1(m_0-r_0)}\left(\int_{M_0} \alpha_0 \wedge \eta_{(f_0)_*[R_0]}\right)\left(\int_{M_1} \alpha_0 \wedge \eta_{(f_1)_*[R_1]}\right)$$
$$= (-1)^{r_1(m_0-r_0)}\left(\int_{R_0} f_0^*\alpha_0\right)\left(\int_{R_1} f_0^*\alpha_1\right).$$

Künneth theorem implies that any cohomology class in $M_0 \times M_1$ is a linear combination of cohomology classes of the form $\alpha_0 \boxtimes \alpha_1$. The equality (3.4) is now obvious.

□

Example 3.9. Suppose that M is a smooth compact oriented manifold of dimension M and $f : M \to M$ is a smooth map. Then the graph of f is a smooth submanifold $\Gamma_f \subset M \times M$ equipped with a natural orientation induced by the diffeomorphism

$$\delta_f : M \to \Gamma_f,\ \ x \mapsto (x, f(x)).$$

We want to offer a description of of the Poincaré dual $\eta_{\Gamma_f} \in H^m(M \times M)$.

Fix a basis $(\omega_j)_{1\leq i\leq b(M)}$ of $H^*(M)$, $b(M) = \sum_k b_k(M)$, where ω_i denotes a closed from of degree $|\omega_i|$. Denote by $(\omega^j)_{1\leq j\leq N}$ the Poincaré dual basis of $H^*(M)$, i.e., the forms ω^j satisfy the equalities[2]

$$\int_M \omega^i \wedge \omega_j = \delta^i_j := \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Above ω^j is a form of degree $|\omega^j| = n - |\omega_j|$.

[2]Throughout we use the convention that $\int_M \alpha = 0$ if α is a differential form such that $\deg \alpha \neq m$.

Kunneth's theorem then implies that the collection $\Omega^i_j := \omega^i \times \omega_j \in H^*(M\times M), 1 \leq i,j \leq b(M)$ is a basis of $H^*(M)$. Similarly, the collection

$$\Theta^i_j = \omega_j \times \omega^i$$

is a basis of $H^*(M \times M)$ and therefore, Poincaré dual of Γ_f has the form

$$\eta_{\Gamma_f} = \sum_{i,j} c^j_i \Theta^i_j.$$

There exist real numbers (a^i_j) such that

$$f^*\omega_j = \sum_i a^i_j \omega_i, \quad \forall j.$$

More precisely

$$a^i_j = \int_M \omega^i \wedge f^*\omega_j.$$

We want to express the coefficients c^i_j in terms of the coefficients a^i_j. To achieve this we use the defining property of η_{Γ_f}, i.e.

$$\int_{M\times M} \Omega^k_\ell \wedge \eta_{\Gamma_f} = \int_{\Gamma_f} \Omega^k_\ell = \int_M \delta^*_f \Omega^k_\ell.$$

Observe first that

$$\int_M \delta^*_f \Omega^k_\ell = \int_M \omega^k \wedge f^*\omega_l = \sum_j a^j_\ell \int_M \omega^k \wedge \omega_j = \sum_j a^j_\ell \delta^k_j = a^k_\ell.$$

For $i = 0, 1$ denote by $\pi_i : M \times M \to M$ the projection $(x_0, x_1) \mapsto x_i$.

We have

$$\int_{M\times M} \Omega^k_\ell \wedge \eta_{\Gamma_f} = \sum_{i,j} c^j_i \int_{M\times M} \Omega^k_\ell \wedge \Theta^i_j$$

$$= \sum_{i,j} c^j_i \int_{M\times M} (\pi^*_0\omega^k \wedge \pi^*_1\omega_\ell) \wedge (\pi^*_0\omega_j \wedge \pi^*_1\omega^i)$$

$$= \sum_{i,j} (-1)^{|\omega_\ell|\cdot|\omega_j|} c^j_i \int_{M\times M} \pi^*_0(\omega^k \wedge \omega_j) \wedge \pi^*_1(\omega_\ell \wedge \omega^i)$$

$$= \sum_{i,j} (-1)^{|\omega_\ell|\cdot(|\omega_j|+|\omega^i|)} c^j_i \left(\int_M \omega^k \wedge \omega_j\right) \left(\int_M \omega^i \wedge \omega_\ell\right)$$

$$= \sum_{i,j} (-1)^{|\omega_\ell|\cdot(|\omega_j|+|\omega^i|)} c^j_i \delta^k_j \delta^i_\ell = (-1)^{|\omega_\ell|\cdot(|\omega_k|+|\omega^\ell|)} c^k_\ell.$$

Hence

$$c^k_\ell := (-1)^{|\omega_\ell|\cdot(|\omega_k|+|\omega^\ell|)} a^k_\ell = (-1)^{|\omega_\ell|\cdot(|\omega_k|+|\omega^\ell|)} \int_M \omega^k \wedge f^*\omega_\ell. \tag{3.5}$$

□

4. Applications

Suppose that N is a compact oriented smooth manifold of even dimension $2m$. For any compact oriented submanifold S of middle dimension m we define the self intersection number $[S]\bullet[S]$ to be

$$[S]\bullet[S] := \int_M \eta_S\wedge\eta_S.$$

One can show (see [2, Thm. 14.6]) then there exists a smooth map

$$F:[0,1]\times S\to M,\ \ (t,x)\mapsto F_t(x)$$

such that for any t the map F_t is an embedding, F_0 coincides with the inclusion $i:S\hookrightarrow M$ and for any $t>0$ the map F_t is transversal to S. If we denote by S_t the image of F_t then we deduce that $S_t \pitchfork S$, $\eta_{S_t}=\eta_S$ and thus

$$[S]\bullet[S]=[S_t]\bullet[S]=\sum_{p\in S_t\cap S}\epsilon(S_t,S,p)\ \ \forall t\in(0,1].$$

The computations in Example 3.3 can be reformulated as

$$[H]\bullet[H]=1.$$

Theorem 4.1 (Lefschetz fixed point theorem)**.** *Suppose M is a compact oriented smooth manifold of dimension m and $f:M\to M$ is a smooth map. We define its* Lefschetz *number to be the quantity*

$$L(f)=\sum_{k=0}^m(-1)^k\operatorname{tr}_k f^*,\ \ \operatorname{tr}_k f^*:=\operatorname{tr}\Big(\,f^*:H^k(M)\to H^k(M)\,\Big)$$

Then

$$\int_{M\times M}\eta_{\Gamma_f}\wedge\eta_\Delta=L(f).$$

Hence, if $\Gamma_f\pitchfork\Delta$ then

$$[\Gamma_f]\bullet[\Delta]=L(f).$$

In particular, we deduce that if $L(f)\neq 0$, then the map f must have at least one fixed point.

Proof. We continue to use the notations in Example 3.9. We have

$$\eta_{\Gamma_f}=\sum_{i,j}c_i^j\omega_j\times\omega^i.$$

Then

$$\int_{M\times M}\eta_{\Gamma_f}\wedge\eta_\Delta=\int_\Delta\eta_{\Gamma_f}=\sum_{i,j}c_i^j\int_\Delta\omega_j\times\omega^i=\sum_{i,j}c_i^j\int_M\omega_j\wedge\omega^i$$

$$=\sum_{i,j}(-1)^{|\omega^i|\cdot|\omega_i|}c_i^j\int_M\omega^i\wedge\omega_j=\sum_{i,j}(-1)^{|\omega^i|\cdot|\omega_j|}c_i^j\delta_j^i$$

$$=\sum_i(-1)^{|\omega^i|\cdot|\omega_i|}c_i^i\stackrel{(3.5)}{=}\sum_i(-1)^{|\omega^i|\cdot|\omega_i|+m|\omega_i|}a_i^i=\sum_i(-1)^{|\omega_i|}a_i^i=\sum_k(-1)^k\underbrace{\sum_{|\omega_i|=k}a_i^i}_{=\operatorname{tr}_k f^*}.$$

□

Corollary 4.2. *Suppose that M is a compact oriented manifold of dimension m and Δ denotes the diagonal in $M \times M$, i.e., the submanifold*

$$\Delta = \big\{ (x,x);\ \ x \in M \,\big\}.$$

It is tautologically diffeomorphic to M and thus it carries a natural orientation. Then

$$[\Delta] \bullet [\Delta] = \chi(M) = \textit{the Euler characteristic of } M. \tag{4.1}$$

Proof. This follows from the Lefschetz fixed point theorem by observing that Δ is the graph of the identity map $\mathbb{1}_M$ and that

$$\operatorname{tr}_k(\mathbb{1}_M^*) = b_k(M).$$

□

Definition 4.3. Suppose that $E \to M$ is a smooth, oriented vector bundle of rank r over the compact smooth manifold M. The *Euler class of E* is the cohomology class $\boldsymbol{e}(E) \in H^r(M)$ defined by the equality

$$\boldsymbol{e}(E) = i^*\Phi_E,$$

where $\Phi_E \in H^r_c(E)$ denotes the Thom class of E and $i : M \to E$ denotes the inclusion of M into E as zero section. □

Theorem 4.4. *Let $\pi : E \to M$ be as above.*[3] *If E admits a nowhere vanishing section $s : M \to E$, then $\boldsymbol{e}(E) = 0$.*

Proof. Fix a metric g on E. For any $\varepsilon > 0$ we denote by $D_\varepsilon(E) \subset$ the open subset consisting of the vectors of length $< \varepsilon$ in all the fibers of E. Fix $\varepsilon > 0$ such that

$$|s(x)|_g > \varepsilon,\ \ \forall x \in M.$$

Next choose a representative $\Phi^\varepsilon \in \Omega^r_c(E)$ of Φ_E such that $\operatorname{supp}\Phi^\varepsilon \subset D_\varepsilon(E)$. By construction, the image of s and the support of Φ^ε are disjoint. Hence

$$s^*\Phi^\varepsilon = 0.$$

On the other hand, the form $s^*\Phi^\varepsilon$ represents the cohomology class $\boldsymbol{e}(E)$. Indeed consider

$$H : [0,1] \times M \to E,\ \ (t,x) \mapsto t * s(x),$$

where $t*$ denotes the multiplication by t along the fibers of E. Observe that $H_0 = i$ and $H_1 = s$. Hence the maps i and s are homotopic and therefore the forms $i^*\Phi^\varepsilon$ and $s^*\Phi^\varepsilon$ are cohomologous. □

Remark 4.5. The above theorem implies that if E is a trivial vector bundle, then $\boldsymbol{e}(E) = 0$ because trivial vector bundles admit nowhere vanishing sections. We can turn this result on its head and conclude that if $\boldsymbol{e}(E) \neq 0$, then the bundle E cannot be trivial. In other words, the Euler class can be interpreted as a measure on nontriviality of E. □

Theorem 4.6. *Suppose that $E \to M$ is a smooth, oriented real vector bundle of rank r over the smooth, compact, oriented manifold M of dimension $m \geq r$. Suppose that $s : M \to E$ is a smooth section that is transversal to the zero section $i : M \to E$. Denote by Z the zero set of s. Then Z is a smooth submanifold of M of codimension r and we have a natural bundle isomorphism*

$$T_Z M \cong E|_M.$$

[3] Observe that we impose no orientability assumption on M.

The above isomorphism equips Z with a coorientation and orientation and the Poincaré dual of the cycle $[Z]$ is equal to the Euler class $\boldsymbol{e}(E)$.

Proof. This is a special case of Corollary 3.7 applied to the submanifolds $R = i(M)$ and $S = s(M)$ of the total space of E. □

Theorem 4.7. *Suppose that $E \to M$ is a smooth, oriented real vector bundle of rank m over the smooth, compact, oriented manifold M of dimension m. Suppose that $s : M \to E$ is a smooth section that is transversal to the zero section $i : M \to E$. Then*

$$[M] \bullet [s(M)] = \int_M \boldsymbol{e}(E). \tag{4.2}$$

Proof. Denote by η_M The Poincaré dual of M in E and by $\eta_{s(M]}$ the Poincaré dual of $s(M)$ in E. Then $\eta_M = \Phi_E =$ the Thom class of E. We have

$$[M] \bullet [s(M)] = \int_E \eta_M \wedge \eta_{s^*[M]} = \int_E \Phi_E \wedge \eta_{s^*[M]} = \int_M s^* \Phi_E.$$

As in the proof of Theorem 4.4 we observe that $s^*\Phi_E = i^*\Phi_E = \boldsymbol{e}(E)$ in $H^m(M)$ since the sections s and i are homotopic.

□

Remark 4.8. Observe that the intersection $M \cap s(M) = i(M) \cap s(M)$ is the set of zeros of the section s. The above theorem then states that a certain signed count of zeros of a transversal section is equal to the integral of the Euler class over the base when the dimension of the base is equal to the rank of the bundle.

The sign associated to a transverse zero can be computed as follows. Suppose that U is a coordinate patch diffeomorphci to $\mathbb{R}^n$ with local coordinates $x^1, \dots, x^m$ such that the orientation of M along U is given by $dx^1 \wedge \cdots \wedge dx^m$. Fix an *orientation preserving* trivialization $E|_U \to \mathbb{R}^m_U$. we get m-function $y^1, \dots, y^m : E|_U \to \mathbb{R}$, describing linear coordinates long fibers such that the orientation of each fiber is given by $dy^1 \wedge \cdots \wedge dy^m$. A section of s is described over U as a map

$$U \to \mathbb{R}^m, \;\; y^j = y^j(x^1, \dots, x^m), \;\; j = 1, \dots, m.$$

If $p_0 = (x_0^1, \dots, x_0^m)$ is a zero of s in U, then the intersection of $s(M)$ with M at p_0 is transversal if and only if

$$\det \frac{\partial(y^1, \dots, y^m)}{\partial(x^1, \dots, x^m)}|_{p_0} \neq 0.$$

If this is the case, then zero p_0 is counted with sign

$$\epsilon(p_0) = \operatorname{sign} \det \frac{\partial(y^1, \dots, y^m)}{\partial(x^1, \dots, x^m)}|_{p_0}.$$

One can show that this sign is independent of the various choices. □

Definition 4.9. Let M be a compact oriented smooth manifold of dimension m. Then the *Euler class* of M is the cohomology class $\boldsymbol{e}(M) := \boldsymbol{e}(TM) \in H^m(M)$. □

Theorem 4.10 (Poincaré-Hopf)**.**

$$\int_M \boldsymbol{e}(M) = \chi(M).$$

Proof. Consider the diagonal $\Delta \subset M \times M$ and its normal bundle $E := T_\Delta M \times M$. We identify E with a tubular neighborhood $\mathcal{N}$ of Δ in $M \times M$ and the Thom class Φ_E of E with the Poincaré dual η_Δ of Δ in $\mathcal{N}$, and by extension, with the Poincaré dual of Δ in $M \times M$. We have

$$\int_\Delta \boldsymbol{e}(E) = \int_\Delta \Phi_E = \int_{\mathcal{N}} \eta_\Delta \wedge \eta_\Delta \stackrel{(4.1)}{=} \chi(M).$$

The diagonal Δ is clearly isomorphic to M. To complete the proof the theorem wee need the following result.

Lemma 4.11. *The normal bundle $T_\Delta(M \times M)$ is orientation preservingly isomorphic to $TM = T\Delta$.*

Proof of the Lemma. Note that we have a short exact sequence of bundles over $M = \Delta$,

$$0 \to TM \xrightarrow{\delta} TM \oplus TM \xrightarrow{p} TM \to 0$$

where for any $x \in M$ the map

$$\delta_x : T_x M \to T_x M \oplus T_x M$$

is given by $v \mapsto v \oplus v$, while the map

$$p_x : T_x M \oplus T_x M \to T_x M$$

is given by $p_x(v_0, v_1) = \frac{v_1 - v_0}{2}$. The image of δ coincides with the tangent bundle $T\Delta$ and thus p induces an isomorphism

$$T_\Delta(M \times M) \to TM.$$

This isomorphism is orientation preserving. To see this observe first that the above short exact sequence sequence admits a splitting

$$\sigma : TM \to TM \oplus TM, \;\; p \circ \sigma = \mathbb{1}_{TM},$$

given by

$$\sigma_x(v) = -v \oplus v, \;\; \forall x \in M, \;\; \forall v \in T_x M.$$

To check that the isomorphism $p : T_\Delta(M \times M) \to TM$ is orientation preserving it suffices to check that for any $x \in M$ we have

$$\boldsymbol{or}\, \delta(T_x M) \wedge \boldsymbol{or}\, \sigma(T_x M) = \boldsymbol{or}(T_x M \oplus T_x M).$$

Fix an oriented basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ of $T_x M$. Set

$$\boldsymbol{u}_i := \boldsymbol{e}_i \oplus 0 \in T_x \oplus T_x M, \;\; \boldsymbol{v}_j = 0 \oplus \boldsymbol{e}_j \in T_x M \oplus T_x M.$$

The collection $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m, \boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ is an oriented basis of $T_x M \oplus T_x M$. Observe that

$$\delta(\boldsymbol{e}_i) = \boldsymbol{u}_i + \boldsymbol{v}_i, \;\; \sigma(\boldsymbol{e}_j) = \boldsymbol{v}_j - \boldsymbol{u}_j.$$

Then

$$\boldsymbol{or}\, \delta(T_x M) \wedge \boldsymbol{or}\, \sigma(T_x M) = \boldsymbol{or}\big(\delta(\boldsymbol{e}_1) \wedge \cdots \wedge \delta(\boldsymbol{e}_m) \wedge \sigma(\boldsymbol{e}_i) \wedge \cdots \sigma(\boldsymbol{e}_m) \big).$$

We have

$$\delta(\boldsymbol{e}_i) \wedge \cdots \wedge \delta(\boldsymbol{e}_m) \wedge \sigma(\boldsymbol{e}_i) \wedge \cdots \sigma(\boldsymbol{e}_m) = (-1)^{\frac{m(m-1)}{2}} \delta(\boldsymbol{e}_1) \wedge \sigma(\boldsymbol{e}_1) \wedge \cdots \wedge \delta(\boldsymbol{e}_m) \wedge \sigma(\boldsymbol{e}_m).$$

Now we observe that

$$\delta(\boldsymbol{e}_j) \wedge \sigma(\boldsymbol{e}_j) = 2\boldsymbol{u}_j \wedge \boldsymbol{v}_j$$

and we conclude

$$(-1)^{\frac{m(m-1)}{2}} \delta(\boldsymbol{e}_i) \wedge \cdots \wedge \delta(\boldsymbol{e}_m) \wedge \sigma(\boldsymbol{e}_i) \wedge \cdots \sigma(\boldsymbol{e}_m) = \boldsymbol{u}_1 \wedge \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{u}_m \wedge \boldsymbol{v}_m$$
$$= (-1)^{\frac{m(m-1)}{2}} 2^m \boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_m \wedge \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_m.$$

This completes the proof of the lemma and thus of Theorem 4.10. □

5. The Euler class of an oriented rank two real vector bundle

Suppose that $\pi : E \to M$ is an oriented rank two real vector bundle. We denote by $\Phi_E \in H^2_{cv}(E)$ its Thom class and by $\boldsymbol{e}(E) \in H^2(M)$ its Euler class defined by

$$\boldsymbol{e}(E) = i^*\Phi_E,$$

where $i : M \to E$ denotes the inclusion of M in E as zero section. We want to give an explicit description of Φ_E and $\boldsymbol{e}(E)$. To this aim we fix a metric on E and a good (coordinate cover) $\{U_\alpha\}_{\alpha\in A}$ that trivializes E. Thus E is defined by a gluing cocycle

$$g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{SO}(2).$$

More precisely, we can write

$$g_{\beta\alpha}(x) = \left[\begin{array}{cc} \cos\varphi_{\beta\alpha} & -\sin\varphi_{\beta\alpha} \\ \sin\varphi_{\beta\alpha} & \cos\varphi_{\beta\alpha} \end{array}\right],$$

where $\varphi_{\beta\alpha} : U_{\alpha\beta} \to \mathbb{R}$ is a smooth function. Moreover

$$\varphi_{\beta\alpha} = -\varphi_{\alpha\beta},. \tag{5.1}$$

The cocycle condition

$$g_{\alpha\gamma}(x)\cdot g_{\gamma\beta}(x)\cdot g_{\beta\alpha}(x) = \mathbb{1},\ \ \forall x \in U_{\alpha\beta\gamma}$$

is then equivalent to the condition

$$\varphi_{\alpha\gamma}(x) + \varphi_{\gamma\beta}(x) + \varphi_{\beta}(x) \in 2\pi\mathbb{Z},\ \ \forall x \in U_{\alpha\beta\gamma}.$$

Since $U_{\alpha\beta\gamma}$ is connected we deduce that for any α, β, γ there exists $n_{\gamma\beta\alpha} \in \mathbb{Z}$ such that

$$\varphi_{\alpha\gamma}(x) + \varphi_{\gamma\beta}(x) + \varphi_{\beta\alpha}(x) = 2\pi n_{\gamma\beta\alpha},\ \ \forall x \in U_{\alpha\beta\gamma}. \tag{5.2}$$

the correspondence $(\alpha, \beta, \gamma) \mapsto n_{\gamma\beta\alpha}$ is skew-symmetric in its arguments, i.e.,

$$n_{\gamma\beta\alpha} = -n_{\beta\gamma\alpha} = -n_{\gamma\alpha\beta} = -n_{\alpha\beta\gamma}.$$

The metric and the orientation on E induce polar coordinates $(r_\alpha, \theta_\alpha)$ on $E^0|_{U_\alpha}$ where E^0 denotes complement of the zero section, $E \setminus i(M)$. On the overlap $E^0|_{U_{\alpha\beta}}$ we have $r_\alpha = r_\beta$, but the angular coordinates θ_β and θ_α are related by the equalities

$$\theta_\beta - \theta_\alpha = \pi^*\varphi_{\beta\alpha}. \tag{5.3}$$

Lemma 5.1. *There exist $\xi_\alpha \in \Omega^1(U_\alpha)$ such that for any α, β we have*

$$\frac{1}{2\pi}d\varphi_{\beta\alpha} = \xi_\beta - \xi_\alpha \ \ \text{on } U_{\alpha\beta}. \tag{5.4}$$

Proof. Consider a partition of unity (ρ_γ) subordinated to (U_γ). Now define

$$\xi_\alpha = -\frac{1}{2\pi}\sum_\gamma \rho_\gamma d\varphi_{\gamma\alpha}.$$

Then

$$\xi_\beta - \xi_\alpha = \frac{1}{2\pi}\sum_\gamma \rho_\gamma d(\varphi_{\gamma\alpha} - \varphi_{\gamma\beta})$$

$$\stackrel{(5.1)}{=} -\frac{1}{2\pi}\sum_\gamma \rho_\gamma d(\varphi_{\alpha\gamma} + \varphi_{\gamma\beta}) \stackrel{(5.2)}{=} \frac{1}{2\pi}\sum_\gamma \rho_\gamma d(\varphi_{\beta\alpha} - 2\pi n_{\gamma\beta\alpha}) = \frac{1}{2\pi}d\varphi_{\beta\alpha}.$$

□

From (5.4) we deduce that
$$d\xi_\alpha = d\xi_\beta \ \text{ on } U_{\alpha\beta}.$$
Thus the collection of closed 2-forms $(\omega_\alpha = d\xi_\alpha)$ defines a closed form $\omega \in \Omega^2(M)$.

Proposition 5.2. *The cohomology class determined by ω is the Euler class $\boldsymbol{e}(E)$.*

Proof. The equalities (5.3) and (5.4) show that for any α, β we have the equality
$$\frac{1}{2\pi}d\theta_\alpha - \pi^*\xi_\alpha = \frac{1}{2\pi}d\theta_\beta - \pi^*\xi_\beta, \ \text{ on } E^0|_{U_{\alpha\beta}}\,.$$
We deduce that the collection of 1-forms $\psi_\alpha := \frac{1}{2\pi}d\theta_\alpha - \pi^*\xi_\alpha$ determines a global 1-form $\psi \in \Omega^1(E^0)$ with the property that for any U_α and any $x \in U_\alpha$ the restriction to the fiber E_x^0 coincides with the angular form $\frac{1}{2\pi}d\theta_\alpha$. For this reason we say that ψ is a *global angular form*.

The form ψ is defined outside the zero section. Note that
$$d\psi_\alpha = -\pi^* d\xi_\alpha = -\pi^*\omega_\alpha$$
so that
$$d\psi = -\pi^*\omega,$$
so that $d\psi$ extends to a form on E.

Consider a smooth function $\rho : [0,\infty) \to [-1,0]$ whose graph has the shape depicted in Figure 3

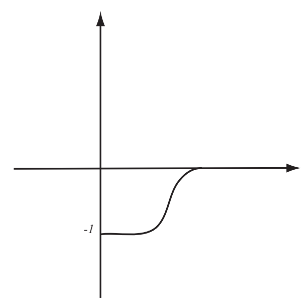

FIGURE 3. *A smooth cut-off function $\rho(r)$.*

Define
$$\Phi = d\big(\rho(r)\psi\big) \in \Omega^2(E^0),$$
where $r : E \to [0,\infty)$ denotes the radial distance function defined by the metric on E. We have
$$\Phi = d\rho(r) \wedge \psi - \rho(r)\pi^*\omega. \tag{5.5}$$
The form $d\rho$ is identically zero near the zero section and we deduce that Φ extends as a closed 2-form on E. From the shape of ρ we deduce that it has compact vertical support. If $x \in M$ then
$$\int_{E_x} \Phi = \int_{E_x} d\rho(r) \wedge \psi.$$
On E_x we have $\psi = \frac{1}{2\pi}d\theta$= the angular form on E_x. We deduce
$$\int_{E_x} \Phi = \frac{1}{2\pi}\int_{\mathbb{R}^2} d\rho(r) \wedge d\theta,$$
where (r,θ) denote the angular coordinates on $\mathbb{R}^2$. We have
$$\frac{1}{2\pi}\int_{\mathbb{R}^2} d\rho(r)\wedge d\theta = \lim_{\varepsilon\searrow 0}\frac{1}{2\pi}\int_{r\geq\varepsilon} d\rho(r)\wedge d\theta$$

(use Stokes formula)

$$= -\lim_{\varepsilon \searrow 0} \frac{1}{2\pi} \int_{r=\varepsilon} \rho(\varepsilon) d\theta = 1$$

since $\rho(\varepsilon) = -1$ if ε is sufficiently small. Thus Φ represents the Thom class of E. We deduce

$$i^*\Phi = -\rho(0)\omega = \omega.$$

The conclusion follows from the definition of the Euler class as the pullback of the Thom class via the zero section. □

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