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On the normal cycles of subanalytic sets

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Abstract We present a very short and complete proof of the existence of the normal cycle of a subanalytic set. The approach is a blend of Morse theory and geometric integration theory and relies heavily on techniques from o-minimal geometry.

Keywords Normal cycle · Morse theory · o-Minimal geometry · Subanalytic sets · Currents · Slicing

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1 Introduction

The normal and conormal cycles of a (reasonably behaved) subset X of an oriented Euclidean space $\boldsymbol{V}$ of dimension n are currents that encode subtle topological and geometric features of the set. The normal cycle $\boldsymbol{N}^X$ is a Legendrian cycle contained in the unit sphere bundle $S(T\boldsymbol{V})$ associated to the tangent bundle $T\boldsymbol{V}$, while the conormal cycle is a Lagrangian cycle $\boldsymbol{S}^X$ in the cotangent bundle $T^*\boldsymbol{V}$. The two objects completely determine each other in a canonical fashion. Their precise definitions are rather sophisticated in general, but they can be easily described in many concrete examples.

For example, if X is a compact smooth submanifold of $\boldsymbol{V}$, then $\boldsymbol{N}^X$ can be identified with the integration current defined by the total space of the unit sphere bundle associated to the normal bundle of the embedding $X \hookrightarrow \boldsymbol{V}$, while $\boldsymbol{S}^X$ can be identified with the current of integration defined by the total space of the conormal bundle of the embedding $X \hookrightarrow \boldsymbol{V}$.

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The normal cycle $\boldsymbol{N}^X$ is intimately related to Weyl's [28] celebrated tube formula. More precisely, there exist canonical $SO(\boldsymbol{V})$-invariant forms (see [9, §0.3])

$$\eta_0, \dots, \eta_{n-1} \in \Omega^{n-1}\left(S(T\boldsymbol{V}) \right)$$

such that, for any compact submanifold $X \hookrightarrow \boldsymbol{V}$, the integrals

$$\mu_k(X) := \int_{\boldsymbol{N}^X} \eta_k \tag{1.1}$$

can be expressed as integrals over X of universal polynomials in the curvature of the induced metric on X. For example, if $m = \dim X$, then $\mu_m(X)$ is the m-dimensional volume of X, and $\mu_{m-2}(X)$ coincides (up to a universal multiplicative constant) with the integral over X of the scalar curvature. The quantity $\mu_0(X)$ is the Euler characteristic of X which, according to the Gauss–Bonnet theorem, can be expressed as the integral of a universal polynomial in the curvature of X.

The quantities $\mu_k(X)$ are known as *curvature measures*. They are the key ingredients in the tube formula that states that for any sufficiently small $r > 0$, the volume of a tube of radius r around a compact submanifold $X \hookrightarrow \boldsymbol{V}$ of dimension m is (see [22, §9.3.3.])

$$V_X(r) = \sum_{k=0}^{m} \mu_{m-k}(X)\boldsymbol{\omega}_{n-m+k} r^{n-m+k}, \tag{1.2}$$

where $\boldsymbol{\omega}_p$ denotes the volume of the unit p-dimensional ball.

If X is a bounded domain in $\boldsymbol{V}$ with sufficiently regular boundary ∂X, then we have a unit outer normal vector field

$$\boldsymbol{n} : \partial X \to S(\boldsymbol{V}) := \{v \in \boldsymbol{V} : \ |v| = 1\} \tag{1.3}$$

and the normal cycle $\boldsymbol{N}^X$ is the integration current defined by the graph of the above map; see Example 3.1. In this case, the integrals $\int_{\boldsymbol{N}^X} \eta_k$ can be expressed as integrals over ∂X of universal polynomials in the second fundamental form of the hypersurface ∂X, and they are involved in a tube formula similar to (1.2), [22, §9.3.5]. If additionally X happens to be convex, then the curvature measures $\mu_k(X)$ coincide with the *Quermassintegrale* constructed by Minkowski [18].

In the groundbreaking work [5], Federer has explained how to associate curvature measures to subsets of $\boldsymbol{V}$ of positive reach. This class of subsets contains as subclasses the smooth submanifolds of $\boldsymbol{V}$, the bounded domains with smooth boundary and the convex bodies in $\boldsymbol{V}$, and in these cases Federer's curvature measures specialize to the curvature measures described above.

As explained in [18], the *Quermasseintegrale* can be extended in a canonical fashion to finitely additive measures (valuations) defined on the collection of polyconvex subsets of $\boldsymbol{V}$, i.e., sets that are finite unions of convex bodies. In particular, the quantity $\mu_k(X)$ is well defined for any compact PL subset of $\boldsymbol{V}$. For most PL sets the Gauss map (1.3) is not defined and the above definition of $\boldsymbol{N}^X$ is meaningless.

In the late 70s and early 80s, Wintgen [29] and Cheeger et al. [3] have explained how to associate to an arbitrary compact PL subset $X \subset \boldsymbol{V}$ a Legendrian cycle N^X contained in $S(T\boldsymbol{V})$ such that

$$\mu_k(X) = \int_{\boldsymbol{N}^X} \eta_k.$$

Their elementary construction is very intuitive and is based on elementary Morse theory on PL-spaces. Moreover, the correspondence $X \mapsto \boldsymbol{N}^X$ from the collection of compact PL subsets of $\boldsymbol{V}$ to the Abelian group of Legendrian cycles in $S(T\boldsymbol{V})$ is a finitely additive measure, i.e.,

$$\boldsymbol{N}^{X\cup Y} = \boldsymbol{N}^X + \boldsymbol{N}^Y - \boldsymbol{N}^{X\cap Y}, \tag{1.4}$$

for any PL sets X, Y. This article [3] includes the first formal definition of the normal cycle. Roughly speaking, the normal cycle $\boldsymbol{N}^X$ is designed to be an ingenious catalogue of the Morse theoretic behavior of the restrictions to X of "typical" linear functions on $\boldsymbol{V}$.

A few years after [3], Kashiwara and Schapira [17] have shown how to associate a normal cycle to any bounded subanalytic subset of $\boldsymbol{V}$. Although the Morse theoretic point of view is still in the background, their approach is sheaf theoretic and geared towards topological applications. Their proof is quite sophisticated as it relies on highly nontrivial results about the derived categories of sheaves. Schürmann [26] has proposed a simpler sheaf theoretic construction of the normal cycle, but this too requires a good familiarity with stratified spaces and the basic operations in the derived category of sheaves.

Almost immediately following the work of Kashiwara and Schapira, Fu [9] gave another construction of the normal cycle of a subanalytic set using methods of geometric measure theory. His proof is technically very demanding, and the complete details are spread over several papers and more than a hundred pages.

Very recently, Berning [1] has proposed a very ingenious and elegant elementary construction of the normal cycle of a subanalytic set using the recent advances in o-minimal topology and basic facts about currents. Unfortunately there is a flaw in a key existence result, [1, Lemma 6.4]; see Remark 4.1(a) for more details. The present paper grew out of our attempts to fix that flaw.

The main goal of this article is to describe a very short complete proof of the existence of the normal cycle of a bounded subanalytic set by relying on techniques and ideas from o-minimal topology. We rely on several fundamental facts about currents (compactness, slicing), but our consistent usage of recent developments in o-minimal topology drastically reduces the analytical technicalities, making the core geometric ideas much more transparent. The construction has a Morse theoretic flavor, and it is based on two key principles.

- *A uniqueness result closely related to the uniqueness results of Fu [9, Theorem 3.2].* Loosely speaking, this uniqueness result states that there exists a unique Legendrian cycle in $\Sigma^\vee \times \boldsymbol{V}$ that catalogs in a certain explicit fashion (see Remark 1.1(a)) the Morse theoretic properties of the restrictions to X of generic linear functions on $\boldsymbol{V}$. When it exists, this unique cycle is called the *normal cycle* of X, and we denote it by $\boldsymbol{N}^X$.
- *An approximation process pioneered by Fu [9].* More precisely, we show that for any compact subanalytic set X we can find a family of bounded domains $(X_\varepsilon)_{\varepsilon>0}$ with C^3-boundaries such that $X = \cap_{\varepsilon>0} X_\varepsilon$ and the normal cycles $\boldsymbol{N}^{X_\varepsilon}$ converge in the sense of currents to a subanalytic current satisfying the requirements of the uniqueness theorem. Thus, the limit cycle must be the normal cycle of X.

The resulting correspondence $X \mapsto \boldsymbol{N}^X$, X bounded subanalytic set, satisfies the inclusion-exclusion principle (1.4) and for PL sets it coincides with the normal cycle constructed in [3]. Here is a more technical description of our main results.

Let $\boldsymbol{V}$ be an oriented real Euclidean vector space of dimension n. Denote by $\boldsymbol{V}^\vee$ its dual, and by $\Sigma^\vee$ the unit sphere in $\boldsymbol{V}^\vee$. We identify the cotangent bundle $T^*\boldsymbol{V}$ with the product $\boldsymbol{V}^\vee \times \boldsymbol{V}$. We have two canonical projections

$$p : \boldsymbol{V}^\vee \times \boldsymbol{V} \to \boldsymbol{V}^\vee,\ \pi : \boldsymbol{V}^\vee \times \boldsymbol{V} \to \boldsymbol{V}.$$

Let $\langle -, - \rangle : \boldsymbol{V}^\vee \times \boldsymbol{V} \to \mathbb{R}$ denote the canonical pairing

$$\boldsymbol{V}^\vee \times \boldsymbol{V} \ni (\xi, x) \mapsto \langle \xi, x \rangle := \xi(x) \in \mathbb{R}.$$

The Euclidean metric $(-, -)$ on $\boldsymbol{V}$ defines isometries (the classical lowering/raising the indices operations)

$$\boldsymbol{V} \ni x \mapsto x_\dagger \in \boldsymbol{V}^\vee, \quad \boldsymbol{V}^\vee \ni \xi \mapsto \xi^\dagger \in \boldsymbol{V},$$
$$\langle x_\dagger, y \rangle = (x, y), \ \langle \xi, y \rangle = (\xi^\dagger, y), \ \forall x, y \in \boldsymbol{V}, \ \xi \in \boldsymbol{V}^\vee.$$

Let $\alpha \in \Omega^1(T^*\boldsymbol{V})$ denote the canonical 1-form on the cotangent bundle. More explicitly, if $x^1, \dots, x^n$ are Euclidean coordinates on $\boldsymbol{V}$, and $\xi_1, \dots, \xi_n$ denote the induced Euclidean coordinates on $\boldsymbol{V}^\vee$, then

$$\alpha = \sum_i \xi_i \mathrm{d}x^i.$$

We denote by $\omega \in \Omega^2(T^*\boldsymbol{V})$ the associated symplectic form

$$\omega = -d\alpha = \sum_i \mathrm{d}x^i \wedge \mathrm{d}\xi_i.$$

In this article, we will work extensively with subanalytic objects. Our subanalytic sets are the sets in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$ as defined Appendix A to which we refer for more details.

We will work with special classes of currents. For the reader's convenience, we have gathered in Appendix B, the basic notations and facts involving currents that we use throughout this article. For any closed subanalytic subset $X \subset \boldsymbol{V}^\vee \times \boldsymbol{V}$, we denote by $\mathcal{C}_k(X)$ the Abelian group of subanalytic, k-dimensional currents with support on X. More precisely, $\mathcal{C}_k(X)$ is the Abelian subgroup of $\Omega_k(\boldsymbol{V}^\vee \times \boldsymbol{V})$ spanned by the currents of integration over oriented k-dimensional subanalytic submanifolds contained in X. If $S \in \mathcal{C}_k(\Sigma^\vee \times \boldsymbol{V})$, and $\xi \in \Sigma^\vee$, we denote by S_ξ the p-slice of S over ξ,

$$S_\xi := \langle S, p, \xi \rangle \in \mathcal{C}_{k-\dim \Sigma^\vee}(\Sigma^\vee \times \boldsymbol{V})$$

If S is the current of integration along an oriented k-dimensional manifold, then for generic ξ the slice S_ξ is the current of integration along the fiber $S \cap p^{-1}(\xi)$ equipped with a canonical orientation. In general, the slice gives a precise meaning as a current to the intersection of S with the fiber $p^{-1}(\xi)$, provided that this intersection has the "correct" dimension, $\dim S - \dim \Sigma^\vee$.

If $X \subset \boldsymbol{V}$ is a compact subanalytic set, $\xi \in \Sigma^\vee$, and $x \in X$ we set

$$X_{\xi > \xi(x)} := \{ y \in X; \quad \xi(y) > \xi(x) \},$$
$$i_X(\xi, x) := 1 - \lim_{r \searrow 0} \chi \left(B_r(x) \cap X_{\xi > \xi(x)} \right),$$

where χ denotes the Euler characteristic of a topological space. If $x \in \boldsymbol{V} \setminus X$, we set $i_X(\xi, x) := 0$.

The integer $i_X(\xi, x)$ can be interpreted as a Morse index of the function $-\xi : X \to \mathbb{R}$ at x; see [17, §9.5] or Appendix C. For generic $\xi \in \Sigma^\vee$, we have $i_X(\xi, x) = 0$, for all but finitely many points $x \in X$.

The first goal of this article is to give a very short proof of the following uniqueness result closely related to the uniqueness result of Fu [9, Theorem 3.2].

Theorem 1.1 (Uniqueness) *Let X be a compact subanalytic subset of $\boldsymbol{V}$. Then there exists at most one subanalytic current $N \in \mathcal{C}_{n-1}(\Sigma^\vee \times \boldsymbol{V})$ satisfying the following conditions.*

i. *The current N is a cycle, i.e., $\partial N = 0$.*
ii. *The current N has compact support.*
iii. *The current N is Legendrian, i.e.,*

$$\langle \alpha \cup \eta, N \rangle = 0, \quad \forall \eta \in \Omega^{n-2}\left(\Sigma^\vee \times \boldsymbol{V}\right).$$

iv. *For any smooth function $\varphi \in C^\infty(\Sigma^\vee \times \boldsymbol{V})$ we have*

$$\langle \varphi \mathrm{d}V_{\Sigma^\vee}, N \rangle = \int_{\Sigma^\vee} \left(\sum_{x \in X} \varphi(\xi, x) i_X(\xi, x) \right) \mathrm{d}V_{\Sigma^\vee}. \tag{1.5}$$

Remark 1.1 (a) Using [6, Theorem 4.3.2.(1)] we deduce that the equality (iv) is equivalent with the condition

$$N_\xi = \sum_{x \in X} i_X(\xi, x) \delta_{(\xi, x)}, \quad \text{for almost all } \xi \in \Sigma^\vee, \tag{$*$}$$

where $\delta_{(\xi,x)}$ denotes the canonical 0-dimensional current determined by the point (ξ, x). The points x for which $i(x, \xi) \neq 0$ should be viewed as critical points of the function $-\xi : X \to \mathbb{R}$; see [17, §5.4] or Appendix C. Thus, the slice N_ξ records both the collection of critical points of $-\xi|_X$ and their Morse indices.

(b) Our sign conventions are different from the ones used in [1], [9], but they coincide with the conventions in [3], [11].

When a cycle as in Theorem 1.1 exists, it is called the *normal cycle* of X, and we will denote it by $\boldsymbol{N}^X$. In the remarkable paper [9], Fu proved the following result.

Theorem 1.2 (Existence) *Every compact subanalytic set $X \subset \boldsymbol{V}$ has a normal cycle $\boldsymbol{N}^X$.*

Remark 1.2 The *conormal cycle* $\boldsymbol{S}^X \in \mathcal{C}_n(T^*\boldsymbol{V})$ of X constructed by Kashiwara and Schapira [17] can be obtained from normal cycle $\boldsymbol{N}^X$ using a coning procedure described explicitly in (2.1). The equality ($*$) is then a special of the micro-local index theorem [11], [17, Theorem 9.5.6].

The second goal of this article is to show that Theorem 1.2 is a consequence of Theorem 1.1. The proof takes full advantage of the subanalytic context of the problem which prohibits many of the possible pathologies in geometric measure theory. In particular, the geometry of the arguments is much more transparent in this context. Using Theorem 1.2 it is now easy to give a correct proof of the existence part of [1, Theorem 6.2]; see Corollary 1.

Remark 1.3 (a) A sheaf-theoretic approach to the existence of normal cycles based on a conceptually similar approximation method can be found in [26, §5.2.2]. The concept of limit of currents is replaced by the concept of specialization, while the uniqueness theorem is replaced by the injectivity of a certain morphism in Borel–Moore homology, [26, Eq. (5.19)]. This injectivity is ultimately based on a special property of Verdier stratifications, [17, Corollary 8.3.23].

(b) We want to point out a *key* technical difference between the approach in this article and the approach in [9]. The uniqueness theorem [9, Theorem 3.2] is formulated in terms of an integral cycle $\mathcal{I}(\boldsymbol{N}, \xi, t) \in \mathcal{C}_{n-1}(\Sigma^\vee \times \boldsymbol{V})$ defined for *any* Legendrian cycle

$N \in \mathcal{C}_{n-1}(\Sigma^\vee \times V)$ and for almost all $(\xi, t) \in \Sigma^\vee \times \mathbb{R}$; see [9, Definition 3.1]. The construction of the cycle $\mathcal{I}(N, \xi, t)$ is quite involved, but it has a nice payoff because it shows that $\mathcal{I}(N, \xi, t)$ depends continuously on N. This fact is very useful in approximation problems. Our approach skips the construction of $\mathcal{I}(N, \xi, t)$, but we have to pay a small technical price since the proofs of our convergence results are a bit more involved.

(c) We believe that the arguments used in this article can yield an existence result for normal cycles of compact sets in an o-minimal category. We chose to work in the subanalytic category only because of a lack of adequate references for a theory of slicing of o-minimal currents. Hardt's subanalytic work [14], [15] ought to extend with minor changes to an o-minimal context.

Here is a brief outline of this article. We prove Theorem 1.1 in Sect. 2 relying on an o-minimal implementation of the strategy in [7] that affords considerable simplifications. We construct the normal cycle in Sect. 3 via an approximation method. Section 4 describes how a combination of facts proved in this article and in [1] leads to an alternate proof of the existence part of [1, Theorem 6.2]. In Sect. 5, we give an alternate proof to the main convergence theorem in [8]. We conclude with a mostly expository Sect. 6 where we outline a construction of the conormal cycle of a subanalytic subset of a smooth, subanalytic manifold.

We have included two appendices A and B that survey basic facts about subanalytic sets and currents used throughout this article. In Appendix C, we present short o-minimal proofs of some basic facts of singular Morse theory that we use in this article and in our opinion are not widely known. In particular, we have included a very short proof of Kashiwara's non-characteristic deformation lemma in an o-minimal setting.

2 Uniqueness

We will prove Theorem 1.1 following a strategy that is inspired from [7]. We first give a direct and very short proof of a subanalytic version of the general uniqueness theorem [7, Theorem 1.1]. Then, arguing as in the proof of [7, Theorem 4.1], we show that this theorem implies Theorem 1.1.

Theorem 2.1 *Suppose $S \in \mathcal{C}_n(V \times V)$ is a subanalytic n-dimensional current satisfying the following conditions.*

i. The current S is a cycle, $\partial S = 0$,

ii. The current S is lagrangian, i.e., $\omega \cap S = 0$.

iii. The current S is conical, i.e.,

$$(\mu_\lambda)_* S = S, \quad \forall \lambda > 0,$$

where $\mu_\lambda : V^\vee \times V \to \mu_\lambda : V^\vee \times V$ is the rescaling $\mu_\lambda(\xi, x) = (\lambda\xi, x)$.

iv. If $|S|$ denotes the support of S, then the induced map $p : |S| \to V^\vee$ is proper.

v. The set $p(|S|)$ is a conical subanalytic subset of $V^\vee$ of dimension $< n = \dim V^\vee$.

Then $S = 0$.

Proof We argue by contradiction. Let $m := \dim p(|S|)$ so that $m < n$. If $m = 0$, we deduce that $|S| \subset V = p^{-1}(0)$. The *cycle* S is subanalytic, and of dimension $n = V$ so that $S = k[V]$ for some integer k. On the other hand, conditions (iii) and (iv) imply that $\pi(|S|)$ is compact, so that $k = 0$, and therefore $S = 0$ by the constancy theorem.

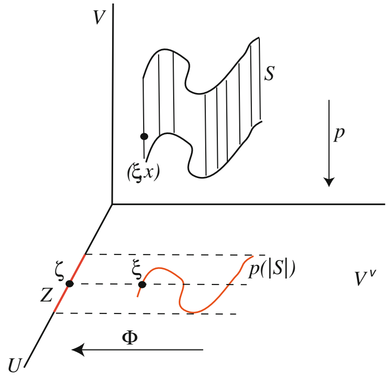

Fig. 1 A rendition of $p(|S|)$. The picture is not entirely accurate since $p(|S|)$ must be a conical subset of $\boldsymbol{V}^\vee$

Suppose $m > 0$. Then we can find a m-dimensional subspace $\boldsymbol{U} \subset \boldsymbol{V}^\vee$ so that if $\Phi : \boldsymbol{V}^\vee \to \boldsymbol{U}$ denotes the orthogonal projection onto $\boldsymbol{U}$, then $Z := \Phi \circ p(|S|)$ is an m-dimensional tame subset of $\boldsymbol{U}$. Moreover, most of the fibers of the induced map $\Phi_S := \Phi|_{p(|S|)} : p(|S|) \to Z$ are zero-dimensional. Set $\Psi := \Phi_S \circ p$; see Fig. 1.

From the properties of subanalytic sets, or more generally, sets in an o-minimal category, [20, §4] or [27], we deduce that there exists a subanalytic subset $Z' \subset Z$ whose complement has dimension $< m$ such that following hold.

(c1) Z' is a C^2-manifold.
(c2) The induced map $\Psi : Y' := \Psi^{-1}(Z') \to Z'$ is a locally (definably) trivial fibration with $(n-m)$-dimensional fibers. Set ${Y'}_\zeta := \Psi^{-1}(\zeta) \subset |S|$, $\zeta \in Z'$.
(c3) If $w \in Y'$ and near w the set Y' is a C^2 manifold, then the differential of Ψ at w is a surjection $\Psi : T_w Y' \to \boldsymbol{U}$.
(c4) The set $\Xi_{Z'} := \Phi^{-1}(Z') \cap p(|S|) \subset \boldsymbol{V}^\vee$ is a C^2-manifold and the induced map $\Xi_{Z'} \xrightarrow{\Phi} Z'$ is a submersion.
(c5) For any $\zeta \in Z'$ the set $\Xi_\zeta := \Phi^{-1}(\zeta) \cap p(|S|)$ is finite.[1] In particular, for any $\zeta \in Z'$ the fiber ${Y'}_\zeta$ is contained in the finite union of planes $\Xi_\zeta \times \boldsymbol{V}$.
(c6) For any $\zeta \in Z'$ the slice $\langle S, \Psi, \zeta\rangle$ is well defined. It is an $(n-m)$-cycle with support $\boldsymbol{cl}(Y_\zeta{}')$.

There exists a subanalytic set $Y'' \subset Y'$ of dimension $< n$ such any $w = \xi \oplus x$ in $Y' \setminus Y''$ belongs both to the C^2-locus of Y', and to the C^2-locus of the fiber ${Y'}_{\zeta=\Phi(\xi)}$ that contains w.

Consider an arbitrary point $w = \xi \oplus x \in Y' \setminus Y''$ and then choose a vector $\dot{w}_1 = \dot{\xi}_1 \oplus \dot{x}_1$ tangent at w to the fiber ${Y'}_\zeta$, $\zeta = \Phi(\xi)$. The condition (c5) shows that ${Y'}_\zeta$ is contained in the finite union of planes $\Xi_\zeta \times \boldsymbol{V}$. This implies that $\dot{\xi}_1 = 0$.

Using the fact that S is a lagrangian current, i.e., $\omega \cap S = 0$, we deduce that for any $\dot{w}_2 = \dot{\xi}_2 \oplus \dot{x}_2 \in T_w Y'$ we have

$$0 = \omega(\dot{w}_1, \dot{w}_2) = \langle \dot{\xi}_2, \dot{x}_1\rangle - \langle \dot{\xi}_1, \dot{x}_2\rangle = \langle \dot{\xi}_2, \dot{x}_1\rangle.$$

If we denote by $\dot{x}_1^* \in \boldsymbol{V}^\vee$, the covector dual to $\dot{x}_1$, then we deduce from the above that $\dot{x}_1^*$ is perpendicular to $p(T_w|S|)$. This is an m-dimensional subspace of $\boldsymbol{V}^\vee$. At the point $w = \xi \oplus x$, the linear map $p : T_w T' \to T_\xi \Xi_{Z'}$ must be a surjection. Thus $\dot{x}_1^* \perp T_\xi \Xi_{Z'}$. We deduce that the tangent plane to ${Y'}_\zeta$ at $\xi \oplus x$ coincides with the plane $\boldsymbol{T}_\xi$,

[1] The definability of the Euler characteristic [27, §4.2] implies that the cardinality of Ξ_ζ is bounded from above by a constant independent of ζ.

$$\boldsymbol{T}_{\xi} := \left\{ x \in \boldsymbol{V};\ \langle \dot{\xi}, x \rangle = 0,\ \forall \dot{\xi} \in T_{\xi} \Xi_{Z'} \right\}.$$

As we already know, Y'_{ζ} is contained in the finite union of planes $\Xi_{\zeta} \times \boldsymbol{V}$. The above remarks show that for any $\xi \in \Xi_{\zeta}$, and any C^2-point w of the component of Y'_{ζ} contained in $\{\xi\} \times \boldsymbol{V}$ the tangent space $T_w Y'_{\zeta}$ coincides with $\boldsymbol{T}_{\xi}$. In other words, the Gauss map of the C^2-locus of Y'_{ζ} has finite range $\left\{ \boldsymbol{T}_{\xi};\ \xi \in \Xi_{\zeta} \right\}$. This shows that the support of the slice $\langle S, \Psi, \zeta \rangle$ is contained in a finite number of $(n-m)$-dimensional planes. The slice $\langle S, \Psi, \zeta \rangle$ is a $(n-m)$-dimensional cycle with compact support. The constancy theorem shows that it must be trivial. This implies that $\dim Y_{\zeta}{}' < (n-m)$. This contradicts (c2) and thus completes the proof of Theorem 2.1. □

Remark 2.1 If we denote by $d\xi \in \Omega^n(\boldsymbol{V}^{\vee})$ the Euclidean volume form on $\boldsymbol{V}^{\vee}$, we see that for a *subanalytic* current $S \in \mathcal{C}_n(\boldsymbol{V}^{\vee} \times \boldsymbol{V})$ the condition (v) of Theorem 2.1 is equivalent to the condition

$$(p^* d\xi) \cap S = 0 \tag{v$'$}$$

employed in [7, Theorem 1.1]. Indeed, clearly (v) ⇒ (v′). The implication (v′) ⇒ (v) follows from Sard's theorem and the fact that outside a subanalytic subset of dimension $\leq (n-1)$ the support $|S|$ can be identified with a real analytic manifold.

Proof of Theorem 1.1 Suppose $N_0, N_1 \in \mathcal{C}_{n-1}(\Sigma^{\vee} \times \boldsymbol{V})$ are two subanalytic cycles satisfying the condition (i),(ii), (iii), (iv) of the theorem. Then the subanalytic cycles $\pi_* N_i \in \mathcal{C}_{n-1}(\boldsymbol{V})$, $i = 0, 1$, have compact support. Since the reduced homology of $\boldsymbol{V}$ is trivial we deduce from [15] that there exist subanalytic currents $D_i \in \mathcal{C}_n(\boldsymbol{V})$ such that

$$\partial D_i = \pi_*(N_i), \quad i = 0, 1.$$

The constancy theorem (Theorem B.1) shows that the currents D_i are uniquely determined by the above equality.

Let $z : \boldsymbol{V} \to \boldsymbol{V}^{\vee} \times \boldsymbol{V} = T^*\boldsymbol{V}$ denote the zero section of $T^*\boldsymbol{V}$, i.e., $z(x) = (0, x),\ \forall x \in \boldsymbol{V}$. Consider the rescaling map

$$\mu : [0, \infty) \times \Sigma^{\vee} \times \boldsymbol{V} \to \boldsymbol{V}^{\vee} \times \boldsymbol{V}, \quad (\lambda, \xi, x) \mapsto (\lambda\xi, x),$$

and, as in [1, Proposition 4.8], we form the currents

$$S_i := \mu_* \left([0, \infty) \times N_i \right) + z_*(D_i), \quad i = 0, 1. \tag{2.1}$$

As explained in [1, Proposition 4.8], the current $S = S_1 - S_0$ satisfies the assumptions (i)–(iv) of Theorem 2.1 and also the condition (v′). Using the Remark 2.1 and Theorem 2.1, we conclude that $S = 0$. □

Remark 2.2 Let us observe that the condition (v) in Theorem 2.1 is equivalent to the condition that the slices S_{ξ} are trivial for almost all ξ in $\boldsymbol{V}^{\vee}$.

3 Existence

We want to show that Theorem 1.1 ⇒ Theorem 1.2. We start by describing a simple well known class of compact subanalytic sets that admit normal cycles.

Example 3.1 [*Normal cycles of regular domains*] Suppose X is a compact subanalytic domain in $\boldsymbol{V}$ with C^2-boundary. Consider the oriented Gauss map $\boldsymbol{\gamma} : \partial X \to \Sigma^\vee$ that associates to each $x \in \partial X$ the unit covector $\boldsymbol{\gamma}(x)$ which is dual to the unit *outer* normal vector at x. We get an embedding

$$\Gamma : \partial X \to \Sigma^\vee \times \boldsymbol{V}, \quad x \mapsto (\boldsymbol{\gamma}(x), x),$$

whose image coincides with the graph of the Gauss map. Denote by $[\partial X]$ the integration current defined by ∂X equipped with the induced[2] boundary orientation. Then the cycle $\Gamma_*[\partial X]$ supported by the graph of the Gauss map is the normal cycle of X.

Indeed, it is obviously a subanalytic cycle, and supp $\Gamma_*[X]$ is compact since X is compact. The Legendrian condition is simply a rephrasing of the fact that for any $x \in \partial X$ the covector $\boldsymbol{\gamma}(x)$ is conormal to $T_x\partial X$.

To verify (iv) we first observe that

$$i_X(\xi, x) = 0, \quad \forall \xi \in \Sigma^\vee, \; x \in X \setminus \partial X.$$

For $x \in \partial X$ denote by $\boldsymbol{II}_x$ the second fundamental form of ∂X at x. The equality (1.5) is a consequence of the following facts. Fix a regular value ξ of $\boldsymbol{\gamma}$, and a point $x \in \boldsymbol{\gamma}^{-1}(\xi)$. Then,

(f1) the local degree of $\boldsymbol{\gamma}$ at x is equal to the sign of the determinant of $-\boldsymbol{II}_x$;
(f2) $i(\xi, x) = (-1)^{\nu_+}$, where ν_+ is the number of positive eigenvalues of $\boldsymbol{II}_x$.

For a proof of (f1) we refer to [22, §9.2.3]. To prove (f2) we can assume that $x = 0$, and near x the hypersurface ∂X is the graph of a quadratic form

$$x^n = q(x^1, \dots, x^{n-1}) = \sum_{i=1}^{\nu_+} (x^i)^2 - \sum_{j=\nu_+ + 1}^{n-1} (x^j)^2,$$

while the interior of X is, locally, the region below the graph. Then $\boldsymbol{II}_x = q$ and (f2) now follows from standard facts of Morse theory.

Suppose now that X is a compact subanalytic set. Fix an integer $p > 2n = 2\dim \boldsymbol{V}$. Then there exists a subanalytic C^p-function $f : \boldsymbol{V} \to \mathbb{R}$ such that $f^{-1}(0) = X$; see [20, Theorem C.11]. Set $g := f^2$ so that g is C^p, nonnegative, subanalytic and $g^{-1}(0) = X$. The following result should be obvious.

Lemma 3.1 *Fix $R > 0$ sufficiently large so that X is contained in the open ball $B_R(0)$. Then there exists $c = c_R > 0$ such that, for any $t \in (0, c_R)$ the level set $g^{-1}(t)$ does not intersect the sphere $\partial B_R(0)$.*

Denote by g_R, the restriction of g to the ball $B_R(0)$ in the above lemma. The set Δ_g of critical values of g_R is a subanalytic 0-dimensional subanalytic subset of $\mathbb{R}$, and thus it consists of a finite number of points.

Fix $c_0 \in (0, c_R)$ so that the interval $(0, c_0)$ consists only of regular values of g_R. Then for any $\varepsilon \in (0, c_0)$ the set

$$X_\varepsilon := \{x \in B_R(0); \; g(x) \leq \varepsilon\}$$

[2] We use the outer-normal-first convention to orient the boundary.

is a compact subanalytic domain with C^2-boundary. Therefore it has a normal cycle $\boldsymbol{N}^\varepsilon = \boldsymbol{N}_{X_\varepsilon}$. The collection $(X_\varepsilon)_{\varepsilon\in(0,c_0)}$ is an increasing subanalytic family of compact subanalytic sets such that

$$X = \bigcap_{0<\varepsilon<c_0} X_\varepsilon.$$

The collection $(\operatorname{supp} \boldsymbol{N}^\varepsilon)_{\varepsilon\in(0,c_0)}$ is a definable collection of compact, subanalytic manifolds of class C^p contained in a common compact subset of $T^*\boldsymbol{V}$. We deduce that their volumes are bounded from above. This shows that the family of currents $(\boldsymbol{N}^\varepsilon)_{\varepsilon\in(0,c_0)}$ is bounded in the mass norm. The compactness theorem [6, Theorem 4.2.17] implies that there exists a subsequence $\varepsilon_\nu \searrow 0$ such that the currents $\boldsymbol{N}^{\varepsilon_\nu}$ converge in the flat metric to a integral Legendrian cycle $\boldsymbol{N} \in \Omega_{n-1}(\Sigma^\vee \times \boldsymbol{V})$.

To prove that $\boldsymbol{N}$ is a *subanalytic* current it suffices to show that its support is contained in a subanalytic set of dimension $\leq (n-1)$. To see this we consider the subanalytic set

$$\begin{aligned}\mathcal{Z} &= \left\{(\xi, x) \in \Sigma^\vee \times \boldsymbol{V};\ \exists 0 < \varepsilon \leq \frac{c_0}{2} :\ x \in \partial X_\varepsilon,\ \xi = \boldsymbol{\gamma}(x)\right\}\\ &= \bigcup_{0<\varepsilon\leq\frac{c_0}{2}} \operatorname{supp} \boldsymbol{N}_\varepsilon.\end{aligned}$$

Then $\dim \mathcal{Z} = n$ and $\boldsymbol{cl}(\mathcal{Z}) \setminus \mathcal{Z}$ is a subanalytic set of dimension $< n$ containing $\operatorname{supp} \boldsymbol{N}$.

We want to show that $\boldsymbol{N}$ satisfies (1.5). Since

$$\langle \varphi d V_{\Sigma^\vee}, \boldsymbol{N}\rangle = \lim_{\nu\to\infty} \langle \varphi d V_{\Sigma^\vee}, \boldsymbol{N}^{\varepsilon_\nu}\rangle, \quad \forall \varphi \in C_0^\infty(\Sigma^\vee \times \boldsymbol{V}),$$

it suffices to show that $\forall \varphi \in C_0^\infty(\Sigma^\vee \times \boldsymbol{V})$ we have

$$\lim_{\varepsilon\searrow 0} \int_{\Sigma^\vee} \left(\sum_x \varphi(\xi, x) i_{X_\varepsilon}(\xi, x)\right) \mathrm{d}V_{\Sigma^\vee} = \int_{\Sigma^\vee} \left(\sum_x \varphi(\xi, x) i_X(\xi, x)\right) \mathrm{d}V_{\Sigma^\vee}. \tag{3.1}$$

To do this, we will need to use some of the topological facts and terminology presented in Appendix C.

Observe that $\operatorname{supp} \boldsymbol{N}^\varepsilon$ is a C^p-manifold of dimension $n-1$, and its projection onto $\boldsymbol{V}$ is ∂X_ε. This shows that the definable set

$$\mathcal{N} = \bigcup_{0<\varepsilon<c_0} \operatorname{supp} \boldsymbol{N}^\varepsilon.$$

has dimension n. Hence, there exists a subanalytic set $\Sigma_0^\vee \subset \Sigma^\vee$ such that $\dim(\Sigma^\vee \setminus \Sigma_0^\vee) < \dim \Sigma^\vee$ and for any $\xi \in \Sigma_0^\vee$ the set $p^{-1}(\xi) \cap \mathcal{N}$ has dimension 1. This implies that for any $\xi \in \Sigma_0^\vee$ there exists $c_\xi > 0$ such that, and any $\varepsilon \in (0, c_\xi)$ we have

$$\dim p^{-1}(\xi) \cap \operatorname{supp} \boldsymbol{N}^\varepsilon \leq 0.$$

Thus, for $\xi \in \Sigma_0^\vee$ and $\varepsilon \in (0, c_\xi)$, the slice $\boldsymbol{N}_\xi^\varepsilon$ is well defined. Let us point out that the set of homological critical points of $-\xi : X_\varepsilon \to \mathbb{R}$ is contained in the projection on $\boldsymbol{V}$ of $\operatorname{supp} \boldsymbol{N}_\xi^\varepsilon$. In particular, if $\xi \in \Sigma_0^\vee$, and $0 < \varepsilon < c_\xi$, this set of homological critical points is finite.

As explained in Appendix C, there exists a subanalytic subset $\Sigma_1^\vee \subset \Sigma^\vee$ such that $\dim(\Sigma^\vee \setminus \Sigma_1^\vee) < \dim \Sigma^\vee$ and for any $\xi \in \Sigma_1^\vee$ the set of homological critical points of $-\xi : X \to \mathbb{R}$ is finite. Set $\Sigma_* = \Sigma_0^\vee \cap \Sigma_1^\vee$ so that $\dim \Sigma^\vee \setminus \Sigma_* < \dim \Sigma^\vee$. We have the following fundamental equality

$$\lim_{\varepsilon\to 0}\sum_{x\in X}\varphi(\xi,x)\left(i_{X_\varepsilon}(\xi,x)-i_X(\xi,x)\right)=0,\quad \varphi\in C_0^\infty(\Sigma^\vee\times \boldsymbol{V}),\quad \xi\in\Sigma_*. \tag{3.2}$$

The choice $\xi\in\Sigma_*$ guarantees that for any $0<\varepsilon<c_\xi$ the above sum consists of finitely many terms.

Let us first show that (3.2) implies (3.1). We will achieve this using the Lebesgue dominated convergence theorem so it suffices to show that there exists a constant $C>0$ so that for any $\varepsilon>0$ there exists a subanalytic set $\Delta_\varepsilon\subset\Sigma^\vee$ such that $\dim\Delta_\varepsilon<\dim\Sigma^\vee$ and

$$\sum_{x\in X_\varepsilon}|i_{X_\varepsilon}(\xi,x)|<C,\quad \forall\xi\in\Sigma^\vee\setminus\Delta_\varepsilon. \tag{3.3}$$

For every $\varepsilon\in(0,c_0)$, we denote by $\Delta_\varepsilon\subset\Sigma^\vee$ the set

$$\Delta_\varepsilon:=\left\{\xi\in\Sigma^\vee;\ p^{-1}(\xi)\text{ does }\underline{\text{not}}\text{ intersect }\operatorname{supp}\boldsymbol{N}^\varepsilon\text{ transversally}\right\}.$$

We deduce that $\dim\Delta_\varepsilon<\dim\Sigma^\vee$. Note that for any $\xi\in\Sigma^\vee\setminus S_\varepsilon$, we have

$$p^{-1}(\xi)\cap\operatorname{supp}\boldsymbol{N}^\varepsilon=\operatorname{supp}\boldsymbol{N}^\varepsilon_\xi$$

and

$$|i_{X_\varepsilon}(\xi,\varepsilon)|=1,\quad\forall(\xi,x)\in\operatorname{supp}\boldsymbol{N}^\varepsilon_\xi. \tag{3.4}$$

The set

$$\mathcal{X}:=\left\{(\xi,\varepsilon)\in\Sigma^\vee\times(0,c_0);\ \xi\in\Sigma^\vee\setminus S_\varepsilon\right\}.$$

is definable, and for any $(\xi,\varepsilon)\in\mathcal{X})$ the set $\operatorname{supp}\boldsymbol{N}^\varepsilon_\xi$ is finite. We have thus obtained a definable collection of finite sets $\left(\operatorname{supp}N^\varepsilon_\xi\right)_{(\xi,\varepsilon)\in\mathcal{X}}$, and we conclude that there exists an integer $K>0$ such that

$$\#\operatorname{supp}\boldsymbol{N}^\varepsilon_\xi<K,\quad\forall(\xi,\varepsilon)\in\mathcal{X}. \tag{3.5}$$

Let $\varepsilon\in(0,c_0)$. Then, for any $\xi\in\Sigma^\vee\setminus\Delta_\varepsilon$, we have

$$\sum_{x\in X}|i_{X_\varepsilon}(\xi,x)|=\sum_{x,\ (\xi,x)\in\operatorname{supp}\boldsymbol{N}^\varepsilon_\xi}|i_{X_\varepsilon}(\xi,x)|\overset{(3.4)}{=}\#\operatorname{supp}\boldsymbol{N}^\varepsilon_\xi\overset{(3.5)}{\le}K.$$

This proves (3.3).

Let us prove (3.2). Fix $\varphi\in C_0^\infty(\Sigma^\vee\times\boldsymbol{V})$ and $\xi\in\Sigma_*$. Define

$$\mathbf{Cr}_\varepsilon:=\left\{x\in\boldsymbol{V};\ (\xi,x)\in\operatorname{supp}\boldsymbol{N}^\varepsilon_\xi\right\}.$$

Using the terminology in Appendix C, we see that $\mathbf{Cr}_\varepsilon$ is the set of numerically critical points of the function $-\xi$ on X_ε. Consider the $\mathbb{R}_{\mathrm{an}}$-definable set

$$\widetilde{\mathbf{Cr}}:=\left\{(t,x)\in[0,c_\xi)\times\boldsymbol{V};\ x\in\mathbf{Cr}_t\right\},$$

and denote by τ the natural projection $\widetilde{\mathbf{Cr}}\to[0,c_\xi)$. Then there exists $\delta>0$ such that over the interval $(0,\delta)$ the map τ is a locally trivial fibration. Its fibers will consists of the same number ℓ of points. Thus, we can find $\mathbb{R}_{\mathrm{an}}$-definable continuous maps

$$x_1,\dots,x_\ell:(0,\delta)\to\boldsymbol{V}$$

such that

$$\mathbf{Cr}_t = \{x_1(t), \dots, x_\ell(t)\}, \quad x_i(t) \neq x_j(t),\ \forall t \in (0,\delta),\ i \neq j.$$

Set $x_i(0) := \lim_{t\searrow 0} x_i(t)$. The above limits exist since the functions $x_i(t)$ are definable and X is compact. For $x \in X$ we define

$$I_x := \{i;\ x_i(0) = x\} \subset \{1, \dots, \ell\}.$$

The equality (3.2) is then a consequence of the following result.

Lemma 3.2

$$i_X(\xi, x) = \lim_{t\to 0} \sum_{i\in I_x} i_{X_t}(\xi, x_i(t)), \quad \forall x \in X. \tag{3.6}$$

In particular, if $I_x = \emptyset$, then the above limit is zero.

Proof Set $c = \xi(x)$. For $r > 0$ and $t \in (0,\delta)$, we set

$$X_r := X \cap B_r(x), \quad X_{t,r} := X_t \cap B_r(x).$$

Choose $r > 0$ and $t_0 > 0$ sufficiently small such that the following hold.

- The topological type of $X_\rho \cap \{\xi > c\}$ is independent of $\rho \in (0, r]$.
- The set X_ρ is contractible for any $\rho \leq r$, and the set $X_r \setminus \{x\}$ contains no homological critical points of $-\xi : X \to \mathbb{R}$.
- $x_i(t) \in B_r(x)$ and $x_j(t) \notin B_r(x)$ $\forall i \in I_x,\ j \notin I_x,\ t \in (0, t_0)$.

Fix $c' < c$ so that the interval $[c', c)$ contains no critical values of $\xi|_{X_r}$. We deduce from (C.7) and (C.9)

$$i_X(\xi, x) = \chi(X_r) - \chi\left(X_r \cap \{\xi > c'\}\right) = \chi(X_r) - \chi\left(X_r \cap \{\xi \geq c'\}\right).$$

The only (numerically) critical points of $-\xi$ on $X_{t,r}$ are $\{x_i(t);\ i \in I_x\}$. Choose $t_1 < t_0$ such that

$$\langle \xi, x_i(t) \rangle > c', \quad \forall i \in I_x, \quad 0 < t < t_1.$$

Using again (C.7) and (C.9) for the function $-\xi$ on $X_{t,r},\ t < t_1$, we deduce

$$\sum_{i\in I_x} i_{X_r}(\xi, x) = \chi(X_{t,r}) - \chi\left(X_{t,r} \cap \{\xi > c'\}\right) = \chi(X_{t,r}) - \chi\left(X_{t,r} \cap \{\xi \geq c'\}\right).$$

The equality (3.6) is obtained by observing that

$$\lim_{t\searrow 0} \chi(X_{t,r}) = \chi(X_r) \text{ and } \lim_{t\searrow 0} \chi\left(X_{t,r} \cap \{\xi \geq c'\}\right) = \chi\left(X_r \cap \{\xi \geq c'\}\right).$$

□

The equality (3.2) is immediate. Set $\mathbf{Cr}_0 := \{x \in \boldsymbol{V};\ I_x \neq \emptyset\}$. Then

$$\begin{aligned}
&\sum_{x\in \boldsymbol{V}} \varphi(\xi, x) i_X(\xi, x) - \sum_{y\in \boldsymbol{V}} \varphi(\xi, y) i_{X_\varepsilon}(\xi, y) \\
&\quad = \sum_{x\in \mathbf{Cr}_0} \varphi(\xi, x) i_X(\xi, x) - \sum_{y\in \mathbf{Cr}_\varepsilon} \varphi(\xi, y) i_{X_\varepsilon}(\xi, y) \\
&\quad = \sum_{x\in \mathbf{Cr}_0} \left(\varphi(\xi, x) i_X(\xi, x) - \sum_{i\in I_x} \varphi(\xi, x_i(\varepsilon)) i_{X_\varepsilon}(\xi, x_i(\varepsilon)) \right).
\end{aligned}$$

The equality (3.6) implies that the last term goes to zero as $\varepsilon \searrow 0$. This proves (3.2) and (3.1).

From Theorem 1.1, we deduce that $\boldsymbol{N}$ must be the normal cycle of X and that $\boldsymbol{N}_{X_\varepsilon}$ converges in the flat metric to $\boldsymbol{N}_X$ as $\varepsilon \searrow 0$. □

4 Normal cycles of constructible functions

Let us explain how the results proved so far lead to an alternate approach to the existence statement in [1, Theorem 6.1]. We need to introduce some terminology.

A function $f : \boldsymbol{V} \to \mathbb{Z}$ is called *constructible* if its range is finite and for any $n \in \mathbb{Z}$ the level set $f^{-1}(n)$ is subanalytic. If we let I_S denote the characteristic function of a set $S \subset \boldsymbol{V}$, then for any constructible function f, we can write

$$f = \sum_{n\in\mathbb{Z}} n I_{f^{-1}(n)}.$$

We denote by $\boldsymbol{C}(\boldsymbol{V})$ the Abelian group of constructible functions and by $\boldsymbol{C}_0(\boldsymbol{V})$ the Abelian group of constructible functions with compact support. The triangulability theorem [27, Theorem 8.2.9] implies that this group is generated by the characteristic functions of compact subanalytic sets.

Observe that for any compact subanalytic sets $X, Y \subset \boldsymbol{V}$ and any $x \in \boldsymbol{V}$ we have

$$i_{X\cup Y}(\xi, x) = i_X(\xi, x) + i_Y(\xi, x) - i_{X\cap Y}(\xi, x),$$

for almost all $\xi \in \Sigma^\vee$. The uniqueness theorem this implies that

$$\boldsymbol{N}^{X\cup Y} = \boldsymbol{N}^X + \boldsymbol{N}^Y - \boldsymbol{N}^{X\cap Y}.$$

From Groemer's extension theorem [18, Corollary 2.2.2] we deduce that the correspondence

$$\text{compact subanalytic subset } X \to \text{normal cycle } \boldsymbol{N}_X$$

extends to a group morphism

$$\boldsymbol{C}_0(\boldsymbol{V}) \ni f \mapsto \boldsymbol{N}^f \in \mathcal{C}_{n-1}(\Sigma^\vee \times \boldsymbol{V}).$$

The normal cycle $\boldsymbol{N}^f$ of a compactly supported constructible function f is a compactly supported, subanalytic legendrian cycle.

In the remainder of this section, we want to compare this construction of the normal cycle of a constructible function to the one proposed in [1].

We will need to use the o-minimal Euler characteristic function χ_o as defined in o-minimal topology; see Appendix A and [27]. We will denote by χ_{top} the usual topological Euler characteristic. These two notions coincide on compact subanalytic sets, but they could be quite different on non-compact ones. For example, if B^k is an *open* k-dimensional ball, then

$$\chi_o(B^k) = (-1)^k, \quad \chi_{top}(B^k) = 1.$$

More generally, if X is locally compact, then $\chi_o(X)$ can be identified with the Euler characteristic of the Borel-Moore homology of X.

Each of these two notions of Euler characteristic has its own advantages. The o-minimal Euler characteristic χ_o is *fully additive* additive, i.e., for <u>any</u> subanalytic sets X and Y we have

$$\chi_o(X \cup Y) = \chi_o(X) + \chi_o(Y) - \chi_o(X \cap Y). \tag{4.1}$$

On the other hand, it is not a homotopy invariant as its topological cousin χ_{top}.

The additivity condition (4.1) and the Groemer extension theorem implies that χ_{o} defines a linear map $\boldsymbol{C}(\boldsymbol{V}) \to \mathbb{Z}$, called the *integral with respect to the Euler characteristic*, and denoted by $\int d\chi$.

Suppose X is a compact subanalytic set, and $\xi \in \Sigma^{\vee}$ is such that the induced function $-\xi : X \to \mathbb{R}$ is a nice in the sense of Appendix C to which we refer for notations. Using (C.7) we deduce that for every $t \in \mathbb{R}$, and every sufficiently small $\varepsilon > 0$ we have

$$\begin{aligned}\sum_{x \in X_{\xi=t}} i_X(\xi, x) &= \chi_{\mathrm{top}}(X_{\xi \geq t}) - \chi_{\mathrm{top}}(X_{\xi > t}) = \chi_{\mathrm{top}}(X_{\xi \geq t}) - \chi_{\mathrm{top}}(X_{\xi > t-\varepsilon}) \\ &= \chi_{\mathrm{top}}(X_{\xi \geq t}) - \chi_{\mathrm{top}}(X_{\xi \geq t-\varepsilon}) = \chi_{\mathrm{o}}(X_{\xi \geq t}) - \chi_{\mathrm{o}}(X_{\xi \geq t-\varepsilon}) \\ &\overset{(4.1)}{=} \chi_{\mathrm{o}}(X_{t-\varepsilon < \xi \leq t}) \overset{(4.1)}{=} \chi_{\mathrm{o}}(X_{t-\varepsilon < \xi < t}) + \chi_{\mathrm{o}}(X_{\xi=t}).\end{aligned}$$

For $\varepsilon > 0$ sufficiently small the induced map $\xi : X_{t-\varepsilon<\xi<t} \to (t-\varepsilon, t)$ is a locally trivial fibration. We denote its fiber by $X_{\xi=t-0}$. Since the o-minimal Euler characteristic of an open interval is -1, we deduce

$$\chi_{\mathrm{o}}(X_{t-\varepsilon<\xi<t}) = \chi_{\mathrm{o}}\left(X_{\xi=t-0} \times (t-\varepsilon, t)\right) = -\chi_{\mathrm{o}}(X_{\xi=t-0}).$$

We conclude that

$$\sum_{x \in X_{\xi=t}} i_X(\xi, x) = \chi_{\mathrm{o}}(X_{\xi=t}) - \chi_{\mathrm{o}}(X_{\xi=t-0}). \tag{4.2}$$

Following [1], we associate to any constructible function $f \in \mathcal{C}_0(\boldsymbol{V})$ and any $\xi \in \boldsymbol{V}^{\vee}$ the integral 0-dimensional (jump) current $J_f(\xi) \in \Omega_0(\mathbb{R})$ given by

$$J_f(\xi) := \sum_{t \in \mathbb{R}} m_f(\xi, t)\delta_t,$$

where $m_f(\xi, t)$ is the integer

$$\begin{aligned}m_f(\xi, t) &:= \lim_{\varepsilon \searrow 0}\left(\int_{\xi=t} f d\chi - \int_{\xi=t-\varepsilon} f d\chi\right) \\ &= \lim_{\varepsilon \searrow 0} \int f \cdot \left(I_{\xi=t} - I_{\xi=t-\varepsilon}\right) d\chi.\end{aligned}$$

Let us point out that when f is the characteristic function of a bounded subanalytic set X, then

$$m_f(x, t) = \lim_{\varepsilon \searrow 0}\left(\chi_{\mathrm{o}}(X_{\xi=t}) - \chi_{\mathrm{o}}(X_{\xi=t-\varepsilon})\right).$$

If we denote by $\boldsymbol{I}_0(\mathbb{R})$ the Abelian group of integral 0-dimensional currents on $\mathbb{R}$, then we can organize the above construction as a *jump* map

$$J_f : \boldsymbol{V}^{\vee} \to \boldsymbol{I}_0(\mathbb{R}).$$

This map is homogeneous and, as proved[3] in [1], it is also Lipschitz with respect to the flat metric on $\boldsymbol{I}_0(\mathbb{R})$. It is also constructible in the sense that the function $m_f : \boldsymbol{V}^{\vee} \times \mathbb{R} \to \mathbb{Z}$ is

[3] Our sign conventions are a bit different from the ones in [1]. More precisely if h_f is the support map as defined in [1], then $h_f(\xi) = J_f(-\xi)$. The “culprit” for this discrepancy is the outer-normal convention used in Example 3.1.

constructible. Let us observe that if $f = I_X$, where X compact subanalytic set with normal cycle $\boldsymbol{N}^X$, then the equality (4.2) implies that for almost any $\xi \in \Sigma^\vee$ we have

$$J_{I_X}(\xi) = \beta_* \boldsymbol{N}^X_\xi, \tag{4.3}$$

where $\beta : \boldsymbol{V}^\vee \times \boldsymbol{V} \to \mathbb{R}$ is the bilinear map $\beta(\xi, x) = \xi(x)$.

We denote by $\mathfrak{X}_{\boldsymbol{V}}$, the Abelian group of constructible, homogeneous, Lipschitz continuous maps $\boldsymbol{V}^\vee \to \boldsymbol{I}_0(\mathbb{R})$, so that the jump construction gives a morphism of Abelian groups

$$J : \boldsymbol{C}_0(\boldsymbol{V}) \to \mathfrak{X}_{\boldsymbol{V}}, \quad \boldsymbol{C}_0(\boldsymbol{V}) \ni f \mapsto J_f \in \mathfrak{X}_{\boldsymbol{V}}.$$

According to [1, Theorem 3.1], this morphism is an *isomorphism* of Abelian groups. The injectivity of J is a consequence of the injectivity of the “motivic” Radon transform of Schapira [25]. The surjectivity is more subtle, and we refer to [1] for details.

Denote by $\mathfrak{L}_{\boldsymbol{V}}$ the Abelian group of subanalytic, conical lagragian cycles $S \in \mathfrak{C}_n(\boldsymbol{V}^\vee \times \boldsymbol{V})$, such that the restriction to supp S of the projection $\pi : \boldsymbol{V}^\vee \times \boldsymbol{V} \to \boldsymbol{V}$ is proper. To any compact subanalytic set X we denote by $\boldsymbol{S}^X \in \mathfrak{L}_{\boldsymbol{V}}$ the conormal cycle constructed as in (2.1),

$$\boldsymbol{S}^X := \mu_* \Big([0, \infty) \times \boldsymbol{N}^X \Big) + z_*(X),$$

where $z : \boldsymbol{V} \to T^*\boldsymbol{V}$ is the zero section, and $\mu : [0, \infty) \times S(T^*\boldsymbol{V}) \to T^*\boldsymbol{V}$ is the multiplication map

$$[0, \infty) \times S(T^*\boldsymbol{V}) \ni (t, \xi, v) \mapsto (t\xi, v) \in T^*\boldsymbol{V}.$$

The resulting correspondence $X \mapsto \boldsymbol{S}^X$ satisfies the inclusion–exclusion identity, i.e.,

$$\boldsymbol{S}^{X \cup Y} = \boldsymbol{S}^X + \boldsymbol{S}^Y - \boldsymbol{S}^{X \cap Y},$$

for any compact subanalytic sets X, Y. Invoking the Groemer extension theorem again we obtain a group morphism $\boldsymbol{S} : \boldsymbol{C}_0(\boldsymbol{V}) \to \mathfrak{L}_{\boldsymbol{V}}$.

Given $S \in \mathfrak{L}_{\boldsymbol{V}}$, the slice $S_\xi = \langle S, p, \xi \rangle$ is defined for almost every $\xi \in \boldsymbol{V}^\vee$. In [1, §6] it is shown that the almost everywhere defined function

$$\boldsymbol{V}^\vee \setminus \text{negligible set} \ni \xi \mapsto \beta_* S_\xi \in \boldsymbol{I}_0(\boldsymbol{R})$$

is the restriction of a function $\sigma_S \in \mathfrak{X}_{\boldsymbol{V}}$. We have thus obtained a morphism of Abelian groups

$$\sigma : \mathfrak{L}_{\boldsymbol{V}} \to \mathfrak{X}_{\boldsymbol{V}}, \ S \mapsto \sigma_S, \ \sigma_S(\xi) = \beta_* S_\xi, \quad \text{for almost any } \xi \in \boldsymbol{V}^\vee.$$

We have the following fundamental result, [1, p. 403].

Theorem 4.1 *The morphism σ is injective. More precisely, if $S \in \mathfrak{L}_{\boldsymbol{V}}$ and*

$$\beta_* S_\xi = 0, \quad \textit{for almost all } \xi \in \boldsymbol{V}^\vee, \tag{v$''$}$$

then $S = 0$.

Proof For the reader’s convenience we decided to include a proof of this result. What follows is a slightly different incarnation of the strategy employed in [1, p.403]. We will show that (v$''$) implies that $S_\xi = 0$ for almost any ξ. We then conclude using Remark 2.2 and Theorem 2.1.

First of all, let us observe that since S is conical and $\omega \cap S = 0$, then $\alpha \cap S = 0$, where we recall that $\alpha \in \Omega^1(T^*\boldsymbol{V})$ is the canonical 1-form and $\omega = -d\alpha$ is the canonical symplectic form. For any subset $\sigma \subset \Sigma^\vee$ we set

$$\Xi_\sigma := \left\{ \xi \in \boldsymbol{V}^\vee \setminus 0;\ \frac{1}{|\xi|}\xi \in \sigma \right\}.$$

Since S is conical and subanalytic we can find a definable triangulation $\mathcal{K}$ of $\Sigma^\vee$ such that for any top dimensional (open) face σ of there exists

– a finite collection $\mathcal{F}_\sigma = \{f_1, \dots, f_{\nu(\sigma)}\}$ of subanalytic, C^1-maps $f : \Xi_\sigma \to \boldsymbol{V}$ and
– a multiplicity map $m_\sigma : \mathcal{F}_\sigma \to \mathbb{Z}$

with the following properties

i. For any $\xi \in \Xi_\sigma$, the map $\mathcal{F}_\sigma \ni f \mapsto f(\xi) \in \boldsymbol{V}$ is injective.
ii. Any $f \in \mathcal{F}_\sigma$ is homogeneous of degree 0.
iii. For any $\xi \in \Xi_\sigma$ the slice S_ξ is well defined and it is described by

$$S_\xi = \sum_{f \in \mathcal{F}_\sigma} m_\sigma(f)\delta_{(\xi, f(\xi))}.$$

Then

$$\beta_* S_\xi = \sum_{f \in \mathcal{F}_\sigma} m_\sigma(f)\delta_{\langle \xi, f(\xi)\rangle} = 0 \in I_0(\mathbb{R}), \quad \forall \xi \in \Xi_\sigma. \tag{4.4}$$

We fix a top dimensional face σ, and we want to prove that $m_\sigma(f) = 0$, $\forall f \in \mathcal{F}_\sigma$. We argue by contradiction so we assume that there exists a function $f_0 \in \mathcal{F}_\sigma$ such that $m_\sigma(f_0) \neq 0$. Then $\mathcal{F}_\sigma{}' = \mathcal{F}_\sigma \setminus \{f_0\} \neq \emptyset$, and we deduce from (4.4) that for any $\xi \in \Xi_\sigma$ the set

$$G_\xi := \left\{ g \in \mathcal{F}'_\sigma;\ \langle \xi, g(\xi)\rangle = \langle \xi, f_0(\xi)\rangle \right\}$$

is non-empty. The collection $(G_\xi)_{\xi \in \Xi_\sigma}$ is a definable collection of subsets of the finite set $\mathcal{F}_\sigma{}'$. From the definable selection theorem we deduce that there exists a definable map

$$\gamma : \Xi_\sigma \to \mathcal{F}'_\sigma, \quad \xi \mapsto \gamma_\xi,$$

such that $\gamma_\xi(\xi) \in G_\xi$, $\forall \xi$. Since γ is definable, there exits a definable set $\Delta \subset \Xi_\sigma$ with the following properties.

– $\dim \Delta < \dim \Xi_\sigma = n$.
– Δ is closed in Ξ_σ.
– For any connected component C of $\Xi_\sigma \setminus \Delta$ the resulting map $\mathcal{C} \ni \xi \mapsto \gamma_\xi \in \mathcal{F}_\sigma{}'$. is constant.

We will refer to the connected components of $\Xi_\sigma \setminus \Delta$ as *chambers*. Let $\xi_0 \in \Xi_\sigma \setminus \Delta$, and denote by $\mathcal{C}_0$ the chamber containing ξ_0. Let $g_0 \in \mathcal{F}_\sigma{}'$ be the constant value of γ on $\mathcal{C}_0$. Set

$$u := f_0(\xi_0) - g_0(\xi_0) \in \boldsymbol{V},$$

and denote by $u_\dagger \in \boldsymbol{V}^\vee$ the dual covector. Since $\mathcal{C}_0$ is open, we deduce that

$$\xi_t = \xi_0 + t u_\dagger \in \mathcal{C}_0 \text{ if } |t| \text{ is sufficiently small.}$$

Hence

$$\langle \xi_t, f_0(\xi_t) - g_0(\xi_t) \rangle = 0, \quad \forall |t| \ll 1.$$

Derivating the above equality at $t = 0$, we deduce

$$0 = \left\langle \dot{\xi}_0, f_0(\xi_0) - g_0(\xi_0) \right\rangle + \frac{d}{dt}|_{t=0} \langle \xi_0, f_0(\xi_t) \rangle - \frac{d}{dt}|_{t=0} \langle \xi_0, g_0(\xi_t) \rangle . \tag{4.5}$$

The C^1-paths $t \mapsto p_t = \xi_t \oplus f_0(\xi_t) \in T^*\boldsymbol{V}$, $t \mapsto q_t = \xi_t \oplus g_0(\xi_t) \in T^*\boldsymbol{V}$ are contained in the C^1-locus of the support of the current S. Since $\alpha \cap S = 0$ we deduce

$$0 = \alpha(\dot{p}_0) = \frac{d}{dt}|_{t=0} \langle \xi_0, f_0(\xi_t) \rangle, \quad 0 = \alpha(\dot{q}_0) = \frac{d}{dt}|_{t=0} \langle \xi_0, g_0(\xi_t) \rangle.$$

Using this in (4.5), and observing that $\dot{\xi}_0 = u_\dagger$, we conclude that $0 = \langle u_\dagger, u \rangle = |u|^2$. Hence $f_0(\xi_0) = g(\xi_0)$. This contradicts the injectivity assumption (i).

Corollary 1 *The morphism σ is bijective.*

Proof We already know that σ is injective. Theorem 1.1 and (4.3) imply that we have a commutative diagram

$$\begin{tikzcd} \boldsymbol{C}_0(\boldsymbol{V}) \arrow[rr, "S"] \arrow[dr, "J"'] & & \mathfrak{L}_{\boldsymbol{V}} \arrow[dl, "\sigma"] \\ & \boldsymbol{I}_0(\mathbb{R}) & \end{tikzcd}$$

Since J is an isomorphism, we deduce that the morphism σ must also be surjective. □

Remark 4.1 (a) The surjectivity of σ is also claimed in [1, §6]. However, the proof is incorrect due to a glitch in the proof of [1, Lemma 6.4]. That lemma is an existence result having to do with a certain cycle S with support contained in the closure of a cell Γ, and such that supp $\partial S \subset \partial\Gamma := \boldsymbol{cl}(\Gamma) \setminus \Gamma$. Loosely speaking, the lemma states that the relative cycle determined by S in the homology of the pair $(\boldsymbol{cl}(\Gamma), \partial\Gamma)$ is trivial. The bordism proving the vanishing of this relative homology class is a cone on S constructed using a certain homotopy. That homotopy is defined *only* on Γ, *not* on $\boldsymbol{cl}(\Gamma)$, so the homotopy formula as stated in [1, Theorem 4.3] cannot be applied.

(b) From the above discussion it follows that the morphism $\boldsymbol{S}$ is also bijective. Its inverse can be described in terms of the *local Euler obstruction*, [17, §IX.7], [24].

5 Approximation

We have now developed enough technology to provide an alternate proof to the convergence theorem in [8]. We denote by χ_{top} the topological Euler characteristic as defined in Appendix C. We use the 'top'-subscript to differentiate it from the o-minimal Euler characteristic χ_o used in [1] and the previous section. As explained Appendix A, these two notions coincide on compact subanalytic sets.

Theorem 5.1 *Suppose that $(X_k)_{k \geq 0}$ and X are compact subanalytic subsets of $\boldsymbol{V}$ satisfying the following conditions.*

(a) *There exists* $R > 0$ *such that*

$$X, X_k \subset \{x \in \boldsymbol{V};\ |x| \leq R\}, \quad \forall k.$$

(b) *The sequence of currents* $\boldsymbol{N}^k := \boldsymbol{N}^{X_k} \in \mathcal{C}_{n-1}(\Sigma^\vee \times \boldsymbol{V})$ *is bounded in the mass norm.*
(c) *For any* $c \in \mathbb{R}$ *and almost any* $\xi \in \Sigma^\vee$ *we have*

$$\lim_{k\to\infty} \chi_{\mathrm{top}}\,(X_k \cap \{\xi \geq c\}) = \chi_{\mathrm{top}}\,(X \cap \{\xi \geq c\}).$$

Then the sequence of currents $\boldsymbol{N}^{X_k}$ *converges in the flat metric to* $\boldsymbol{N} = \boldsymbol{N}^X$.

Proof It suffices to show that any subsequence of $(\boldsymbol{N}^k)$ contains a sub-subsequence that converges in the flat metric to $\boldsymbol{N}$. To keep the notations at bay, we will denote by $(\boldsymbol{N}^k)$ the various intervening subsequences of $(\boldsymbol{N}^k)$.

The conditions (a) and (b) imply via the compactness theorem for integral currents that $(\boldsymbol{N}^k)$ contains a subsequence convergent in the flat metric to a subanalytic cycle $\boldsymbol{N}'$. To prove that $\boldsymbol{N}' = \boldsymbol{N}$ we will invoke Theorem 4.1, so we have to show that

$$\beta_* \boldsymbol{N}_\xi = \beta_* \boldsymbol{N}_\xi{}', \quad \text{for almost all } \xi \in \Sigma^\vee. \tag{5.1}$$

Denote by $\|-\|_\flat$ the flat norm. Using the slicing lemma [19, Lemma 8.1.16], we deduce that

$$\lim_{k\to\infty} \|\boldsymbol{N}^k_\xi - \boldsymbol{N}'_\xi\|_\flat = 0, \quad \text{for almost all } \xi \in \Sigma^\vee. \tag{5.2}$$

Thus, to prove (5.1) is suffices to show that $\beta_* \boldsymbol{N}^k_\xi$ converges weakly to $\beta_* \boldsymbol{N}_\xi$ for almost any $\xi \in \Sigma^\vee$.

We think of the integral currents $\beta_* \boldsymbol{N}^k_\xi$ and $\beta_* \boldsymbol{N}_\xi$ as (signed) Borel measures on the real axis concentrated on finite sets. If $I_{[c,c')}$ denotes the characteristic function of the interval $[c,c')$, $c < c'$, then we deduce from (C.9) that for almost any $\xi \in \Sigma^\vee$ and any $c < c'$ we have

$$\begin{aligned}\langle I_{[c,c')}, \beta_* \boldsymbol{N}_\xi\rangle &= \chi_{\mathrm{top}}\left(X \cap \{c' \leq \xi < c\}\right),\\ \langle I_{[c,\infty)}, \beta_* \boldsymbol{N}^k_\xi\rangle &= \chi_{\mathrm{top}}\,(X_k \cap \{\xi \geq c\}), \quad \forall k \geq 0.\end{aligned}$$

Using (c) and (C.9), we deduce

$$\lim_{k\to\infty} \langle I_{[c,\infty)}, \beta_* \boldsymbol{N}^k_\xi\rangle = \langle I_{[c,\infty)}, \beta_* \boldsymbol{N}_\xi\rangle,\ \forall c,\ \text{for almost any } \xi \in \Sigma^\vee.$$

Since $I_{[c,c')} = I_{[c,\infty)} - I_{[c',\infty)}$ we conclude

$$\lim_{k\to\infty} \langle I_{[c,c')}, \beta_* \boldsymbol{N}^k_\xi\rangle = \langle I_{[c,c')}, \beta_* \boldsymbol{N}_\xi\rangle,\ \forall c < c',\ \text{for almost any } \xi \in \Sigma^\vee. \tag{5.3}$$

Let us show that the above equality implies that $\beta_* \boldsymbol{N}^k_\xi$ converges weakly to $\beta_* \boldsymbol{N}_\xi$.

For $k \geq 0$, we define

$$h_k : \Sigma^\vee \to [0,\infty], \quad h_k(\xi) = \sum_{x\in X} |i_{X_k}(\xi, x)|.$$

Observe that if the slice $\boldsymbol{N}^k_\xi$ is defined, then $h_k(\xi) = \mathrm{mass}(\beta_* \boldsymbol{N}^k_\xi)$. If $\mu := \sup_k \mathrm{mass}\,(\boldsymbol{N}^k)$, then

$$\int_{\Sigma^\vee} h_k(\xi)\,|d\xi| \leq \mu, \quad \forall k.$$

Hence,

$$\mathrm{vol}\left\{\xi \in \Sigma^\vee;\ h_k(\xi) > t\right\} \le \frac{\mu}{t}.$$

We set $h_\infty(\xi) := \sup_k h_k(\xi)$, and we deduce from the above inequality that for almost any $\xi \in \Sigma^\vee$ we have $h_\infty(\xi) < \infty$.

Let $\varphi \in C_0^\infty(\mathbb{R})$. Denote by L the Lipschitz constant of φ and fix a very small $\varepsilon > 0$. Define

$$\varphi_\varepsilon : \mathbb{R} \to \mathbb{R},\ \varphi(t) = \varphi(n\varepsilon), \quad \text{if } n\varepsilon \le t < (n+1)\varepsilon,\ n \in \mathbb{Z}.$$

Using (5.3), we deduce

$$\lim_{k\to\infty} \langle \varphi_\varepsilon, \beta_* \boldsymbol{N}_\xi^k\rangle = \langle \varphi_\varepsilon, \beta_* \boldsymbol{N}_\xi\rangle.$$

Choose $k = k(\varepsilon) > 0$ such that

$$\left|\langle \varphi_\varepsilon, \beta_* \boldsymbol{N}_\xi^k - \beta_* \boldsymbol{N}_\xi\rangle\right| < \varepsilon, \quad \forall k \ge k(\varepsilon).$$

Now observe that $\|\varphi - \varphi_\varepsilon\|_\infty \le L\varepsilon$. We conclude

$$\left|\langle \varphi_\varepsilon - \varphi, \beta_* \boldsymbol{N}_\xi\rangle\right| \le \mathrm{mass}\,(\beta_* \boldsymbol{N}_\xi)L\varepsilon,\ \left|\langle \varphi_\varepsilon - \varphi, \beta_* \boldsymbol{N}_\xi^k\rangle\right| \le h_\infty(\xi)L\varepsilon,\ \forall k.$$

Hence,

$$\left|\langle \varphi, \beta_* \boldsymbol{N}_\xi^k - \beta_* \boldsymbol{N}_\xi\rangle\right| \le L\left(h_\infty(\xi) + \mathrm{mass}\,(\beta_* \boldsymbol{N}_\xi) + 1\right)\varepsilon, \quad \forall k \ge k(\varepsilon).$$

We deduce that

$$\lim_{k\to\infty} \langle \varphi, \beta_* \boldsymbol{N}_\xi^k\rangle = \langle \varphi, \beta_* \boldsymbol{N}_\xi\rangle, \quad \forall \varphi \in C_0^\infty(\mathbb{R}),$$

i.e., $\beta_* \boldsymbol{N}_\xi^k$ converges weakly to $\beta_* \boldsymbol{N}_\xi$ for almost any $\xi \in \Sigma^\vee$. □

6 Conormal cycles of subanalytic subsets of smooth subanalytic manifolds

In this mostly expository section, we want to outline a construction of the conormal cycle of subanalytic subset of a subanalytic manifold. Fix an ambient space, i.e., a smooth, connected, oriented subanalytic manifold $\mathcal{X}$ of dimension n. This means that $\mathcal{X}$ is a manifold definable in the category $\mathbb{R}_{\mathrm{an}}$ in the precise sense described in [27, Chap. 10]. The total space of the cotangent bundle $T^*\mathcal{X}$ is equipped with a natural symplectic structure, and a canonical action of the multiplicative group $\mathbb{R}_{>0}$. In particular, for any $t > 0$, we have a rescaling-along-fibers diffeomorphism $\mu^t : T^*\mathcal{X} \to T^*\mathcal{X}$. We denote by π the canonical projection $\pi : T^*\mathcal{X} \to \mathcal{X}$.

A current $S \in \Omega_n(T^*\mathcal{X})$ is called *lagrangian* if $\omega \cap S = 0$. It is called *conic* if $\mu^t_* S = S,\ \forall t > 0$. It is called *relatively proper* if $\pi(\mathrm{supp}\, S)$ is a compact subset of $\mathcal{X}$.

The *conormal cycle* of a compact subanalytic subset $X \subset \mathcal{X}$ is a subanalytic, lagrangian, conical, relatively proper current S^X which is a cycle in the sense of currents. The support of a subanalytic, lagrangian conic current is a subanalytic, conic lagrangian subset of $T^*\mathcal{X}$ in the sense of [17, Definition 8.3.9]. Such sets were characterized in [17, Proposition 8.3.10].

Suppose now that k is a positive integer and $\mathcal{F}$ is a finite dimensional subspace of $C^\infty(\mathcal{X})$ consisting of subanalytic functions. We also assume that $\mathcal{F}$ is $\mathbb{R}_{\mathrm{an}}$-definable, i.e., it is $\mathbb{R}_{\mathrm{an}}$-definable as a family of subanalytic function. We say that $\mathcal{F}$ is *k-ample* if it satisfies the following condition.

($\mathbf{A}_k$) *For any $x \in \mathcal{X}$, and any k-jet Ξ at x, there exists a function $f \in \mathcal{F}$ whose k-jet at x is Ξ.*

We will refer to such spaces as k-ample *sample* spaces. Fix an 1-ample sample space $\mathcal{F}$. For any $f \in \mathcal{F}$, the graph of the differential of f is a subanalytic oriented Lagrangian submanifold $\Lambda_{\mathrm{d}f}$ of $T^*\mathcal{X}$. It carries a natural orientation induced from the orientation of $\mathcal{X}$ and thus defines a subanalytic lagrangian current that we denote by $[\Lambda_{\mathrm{d}f}]$. As in [17, Proposition 8.3.27], we can prove that if $\Lambda \subset T^*\mathcal{X}$ is conic lagrangian subanalytic set such that $\pi(\Lambda)$ is compact, then for all $f \in \mathcal{F}$ outside a codimension 1 subanalytic subset $\Delta_\Lambda \subset \mathcal{F}$ the lagrangian $\Lambda_{\mathrm{d}f}$ intersects Λ transversally along a finite subset. In particular, if S is a conic, lagrangian, relatively proper subanalytic cycle, then the intersection of the currents S and $[\Lambda_{\mathrm{d}f}]$ is well defined for almost all $f \in \mathcal{F}$. We have the following counterpart to Theorem 2.1.

Theorem 6.1 *Suppose $\mathcal{X}$ is as above and $\mathcal{F}$ is a 2-ample sample space. Assume that $S \in \Omega_n(T^*\mathcal{X})$ is a subanalytic, conic, lagrangian current such that, there exists an open and dense subset $\mathcal{O} \subset \mathcal{F}$ so that, for any $f \in \mathcal{O}$ we have $\Lambda_{\mathrm{d}f} \pitchfork |S|$, and the intersection current $[\Lambda_{\mathrm{d}f}] \cdot S$ is trivial. Then the current S is trivial.*

Remark 6.1 We have to comment on the assumptions and the conclusions of this theorem. First of all, we have a very stringent assumption, namely the 2-ampleness of the sample space. For example if $\mathcal{X}$ is an Euclidean space and $\mathcal{X} = \boldsymbol{V}$, then the sample space used in Theorem 2.1,

$$\mathcal{F} = \boldsymbol{V}^\vee \oplus \{\text{constant functions}\}$$

is 1-ample, but obviously it is not 2-ample. On the other hand, unlike Theorem 2.1, we do not require that S be a cycle, yet we reach the same conclusion, $S = 0$.

To see why this is possible, we argue by contradiction and we assume that $S \neq 0$. We can then find a nonempty open subset $|S|_0 \subset |S|$ such that $|S|_*$ is a subanalytic, conic, lagrangian C^2 manifold and $\dim |S| \setminus |S|_* < \dim |S| = n$.

If $s_0 \in |S|_*$, then the 2-ampleness condition together with the arguments in the proof of [13, Proposition IV.5.2, p.156] show that we can find $f_0 \in \mathcal{F}$ such that $\Lambda_{\mathrm{d}f_0}$ intersects $|S|_*$ transversally at s_0. For any neighborhood U of $s_0 \in |S|_*$ we can find a small perturbation of $f_1 \in \mathcal{F}$ of f_0 such that Λ_{df_1} intersects $|S|_*$ transversally, and moreover, Λ_{df_1} intersects U at precisely one point. This point will have a nontrivial contribution to the intersection current $[\Lambda_{df_0}] \cdot S$.

If X is a compact subanalytic subset, then we define its conormal cycle to be a conic, lagrangian subanalytic cycle $\boldsymbol{S} = \boldsymbol{S}^X \in \Omega_n(T^*\mathcal{X})$ with the following properties.

There exists an open and dense subanalytic subset $\mathcal{O}_X \subset \mathcal{F}$ such that for any $f \in \mathcal{O}_X$ we have:

(CC_1) The Morse index $m(-f, x)$ (see (C.4)) is trivial for all but finitely many $x \in X$.
(CC_2) $\dim \Lambda_{\mathrm{d}f} \cap |\boldsymbol{S}| = 0$.
(CC_3) The intersection current $[\Lambda_{\mathrm{d}f}] \cdot \boldsymbol{S}$ is given by

$$[\Lambda_{\mathrm{d}f}] \cdot \boldsymbol{S} = \sum_{x \in X} m(-f, x)\delta_{(x,\mathrm{d}f(x))} \in \Omega_0(T^*\mathcal{X}).$$

Theorem 6.1 implies that, if the conormal cycle of X exists, then it is unique. Here is a large class of subsets whose conormal cycles exits and have simple descriptions.

Suppose that X is a compact set of the form

$$X = \{x \in \mathcal{X};\ f(x) \leq 0\},$$

where $f : \mathcal{X} \to \mathbb{R}$ is a C^4-subanalytic function such that 0 is a regular value. Define $\boldsymbol{S}^X$ to be the current of integration over the lagrangian subanalytic variety of $T^*\mathcal{X}$

$$z(X) \cup \big\{ \, (x, tdf(x)\,) \in T^*\mathcal{X};\;\; f(x) = 0,\; t \geq 0 \big\},$$

where $z : \mathcal{X} \to T^*\mathcal{X}$ denotes the zero section. Elementary Morse theoretic arguments imply that $\boldsymbol{S}^X$ satisfies the conormal cycle conditions, $CC_1 - CC_3$.

Employing an approximation argument similar to the one used in the proof of Theorem 1.2 one can prove that any compact subanalytic subset of $\mathcal{X}$ admits a conormal cycle.

The above construction may suggest that the conormal cycle depends on the choice of sample space $\mathcal{F}$ so it would be appropriate to denote it by $\boldsymbol{S}^X_{\mathcal{F}}$. Note that the uniqueness theorem implies that if $\mathcal{F}_0, \mathcal{F}_1$ are 2-ample sample spaces such that $\mathcal{F}_0 \subset \mathcal{F}_1$ then

$$\boldsymbol{S}^X_{\mathcal{F}_0} = \boldsymbol{S}^X_{\mathcal{F}_1}.$$

This proves that the conormal cycle is independent of the choice of 2-ample sample space.

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Appendix A: Fast introduction to *o*-minimal topology

Since the subject of tame geometry is not very familiar to many geometers we devote this section to a brief introduction to this topic. Unavoidably, we will have to omit many interesting details and contributions, but we refer to [4], [20], [27] for more systematic presentations. For every set X, we will denote by $\mathcal{P}(X)$ the collection of all subsets of X

An $\mathbb{R}$-*structure*[4] is a collection $\mathcal{S} = \{\, \mathcal{S}^n \,\}_{n \geq 1}\,,\; \mathcal{S}^n \subset \mathcal{P}(\mathbb{R}^n)$, with the following properties.

- $\mathbf{E}_1$. $\mathcal{S}^n$ contains all the real algebraic subvarieties of $\mathbb{R}^n$, i.e., the zero sets of finite collections of polynomial in n real variables.
- $\mathbf{E}_2$. For every linear map $L : \mathbb{R}^n \to \mathbb{R}$, the half-plane $\{\mathbf{x} \in \mathbb{R}^n;\;\; L(x) \geq 0\}$ belongs to $\mathcal{S}^n$.
- $\mathbf{P}_1$. For every $n \geq 1$, the family $\mathcal{S}^n$ is closed under boolean operations, $\cup$, $\cap$ and complement.
- $\mathbf{P}_2$. If $A \in \mathcal{S}^m$, and $B \in \mathcal{S}^n$, then $A \times B \in \mathcal{S}^{m+n}$.
- $\mathbf{P}_3$. If $A \in \mathcal{S}^m$, and $T : \mathbb{R}^m \to \mathbb{R}^n$ is an affine map, then $T(A) \in \mathcal{S}^n$.

Example A.1 (Semialgebraic sets) Denote by $\mathbb{R}_{alg}$ the collection of real semialgebraic sets. Thus, $A \in \mathbb{R}^n_{\mathrm{alg}}$ if and only if A is a finite union of sets, each of which is described by finitely many polynomial equalities and inequalities. The celebrated Tarski–Seidenberg theorem states that $\mathcal{S}_{\mathrm{alg}}$ is a structure.

Let $\mathcal{S}$ be an $\mathbb{R}$-structure. Then a set that belongs to one of the $\mathcal{S}^n$-s is called $\mathcal{S}$-*definable*. If A, B are $\mathcal{S}$-definable, then a function $f : A \to B$ is called $\mathcal{S}$-*definable* if its graph $\Gamma_f := \{\, (a, b) \in A \times B;\;\; b = f(a)\,\}$ is $\mathcal{S}$-definable.

Given a collection $\mathcal{A} = (\mathcal{A}_n)_{n \geq 1}$, $\mathcal{A}_n \subset \mathcal{P}(\mathbb{R}^n)$, we can form a new structure $\mathcal{S}(\mathcal{A})$, which is the smallest structure containing $\mathcal{S}$ and the sets in $\mathcal{A}_n$. We say that $\mathcal{S}(\mathcal{A})$ is obtained from $\mathcal{S}$ by *adjoining the collection* $\mathcal{A}$.

[4] This is a highly condensed and special version of the traditional definition of structure. The model theoretic definition allows for ordered fields, other than $\mathbb{R}$, such as extensions of $\mathbb{R}$ by "infinitesimals". This can come in handy even if one is interested only in the field $\mathbb{R}$.

Definition 1 An $\mathbb{R}$-structure is called *o-minimal* (order minimal) or *tame* if it satisfies the property

T. Any set $A \in \mathcal{S}^1$ is a *finite* union of open intervals (a, b), $-\infty \leq a < b \leq \infty$, and singletons $\{r\}$.

Example A.2 (a) (Tarski–Seidenberg) The collection $\mathbb{R}_{\mathrm{alg}}$ of real semiebraic sets is a tame structure.

(b) (Gabrielov, Hardt, Hironaka, [10,15,16]) A *restricted* real analytic function is a function $f : \mathbb{R}^n \to \mathbb{R}$ with the property that there exists a real analytic function $\tilde{f}$ defined in an open neighborhood U of the cube $C_n := [-1, 1]^n$ such that

$$f(x) = \begin{cases} \tilde{f}(x) & x \in C_n \\ 0 & x \in \mathbb{R}^n \setminus C_n. \end{cases}$$

we denote by $\mathbb{R}_{\mathrm{an}}$ the structure obtained from $\mathcal{S}_{\mathrm{alg}}$ by adjoining the graphs of all the restricted real analytic functions. Then $\mathbb{R}_{\mathrm{an}}$ is a tame structure, and the $\mathbb{R}_{\mathrm{an}}$-definable sets are called *(globally) subanalytic sets*.

The definable sets and functions of a tame structure have rather remarkable *tame* behavior which prohibits many pathologies. It is perhaps instructive to give an example of function which is not definable in any tame structure. For example, the function $x \mapsto \sin x$ is not definable in a tame structure because the intersection of its graph with the horizontal axis is the countable set $\pi\mathbb{Z}$ which violates the tameness condition **T**.

We list below some of the nice properties of the sets and function definable in a fixed tame structure $\mathcal{S}$. Their proofs can be found in [4], [27]. We will interchangeably refer to sets or functions definable in a given tame structure $\mathcal{S}$ as *definable*, *constructible* or *tame*.

- (*Piecewise smoothness of tame functions*) Suppose A is a definable set, p is a positive integer, and $f : A \to \mathbb{R}$ is a definable function. Then A can be partitioned into finitely many definable sets $S_1, \ldots, S_k$, such that each S_i is a C^p-manifold, and each of the restrictions $f|_{S_i}$ is a C^p-function.
- (*Triangulability*) For every compact definable set A, and any finite collection of definable subsets $\{S_1, \ldots, S_k\}$, there exists a compact simplicial complex K, and a definable homeomorphism $\Phi : |K| \to A$ such that all the sets $\Phi^{-1}(S_i)$ are unions of relative interiors of faces of K.
- (*Dimension*) The dimension of a definable set $A \subset \mathbb{R}^n$ is the supremum over all the nonnegative integers d such that there exists a C^1 submanifold of $\mathbb{R}^n$ of dimension d contained in A. Then $\dim A < \infty$, and $\dim(\boldsymbol{cl}(A) \setminus A) < \dim A$.
- (*Definable selection*) Any tame map $f : A \to B$ (not necessarily continuous) admits a tame section, i.e., a tame map $s : B \to A$ such that $s(b) \in f^{-1}(b)$, $\forall b \in B$.
- (*Local triviality of tame maps*) If $f : A \to B$ is a tame continuous map, then there exists a tame triangulation of B such that over the relative interior of any face the map f is a locally trivial fibration.
- (*The o-minimal Euler characteristic*) There exists a function $\chi_o : \mathcal{S} \to \mathbb{Z}$ uniquely characterized by the following conditions.
  - $\chi_o(X \cup Y) = \chi_o(X) + \chi_o(Y) - \chi_o(X \cap Y)$, $\forall X, Y \in \mathcal{S}$.
  - If $X \in \mathcal{S}$ is compact, then $\chi_o(X)$ is the usual Euler characteristic of X.
- (*The scissor principle*) Suppose A and B are two tame sets. Then the following are equivalent

- The sets A and B have the same o-minimal Euler characteristic and dimension.
- There exists a tame bijection $f : A \to B$. (The map f *need not be continuous*.)

- (*Crofton formula*, [2], [6, Theorem 2.10.15, 3.2.26].) Suppose E is an Euclidean space, and denote by $\mathbf{Graff}^k(E)$ the Grassmannian of affine subspaces of codimension k in E. Fix an invariant measure μ on $\mathbf{Graff}^k(E)$.[5] Denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. Then there exists a constant $C > 0$, depending only on μ, such that for every compact, k-dimensional tame subset $S \subset E$, we have

$$\mathcal{H}^k(S) = C \int_{\mathbf{Graff}^k(E)} \chi(L \cap S) d\mu(L).$$

- (*Finite volume.*) Any compact k-dimensional tame set has finite k-dimensional Hausdorff measure.
- (*Uniform volume bounds*) If $f : A \to B$ is a proper, continuous definable map such that all the fibers have dimensions $\leq k$, then there exists $C > 0$ such that

$$\mathcal{H}^k\left(f^{-1}(b)\right) < C, \quad \forall b \in B.$$

Appendix B: Subanalytic currents

In this appendix, we gather without proofs a few facts about the subanalytic currents introduced by Hardt in [14], [15]. Our terminology concerning currents closely follows that of Federer [6] (see also the more accessible [19], [21]). However, we changed some notations to better resemble notations used in algebraic topology.

Suppose X is a C^2, oriented Riemann manifold of dimension n. We denote by $\Omega_k(X)$ the space of k-dimensional currents in X, i.e., the topological dual space of the space $\Omega^k_{\mathrm{cpt}}(X)$ of smooth, compactly supported k-forms on X. We will denote by

$$\langle \bullet, \bullet \rangle : \Omega^k_{\mathrm{cpt}}(X) \times \Omega_k(X) \to \mathbb{R}$$

the natural pairing. The boundary of a current $T \in \Omega_k(X)$ is the $(k-1)$-current defined via the Stokes formula

$$\langle \alpha, \partial T \rangle := \langle d\alpha, T \rangle, \quad \forall \alpha \in \Omega^{k-1}_{\mathrm{cpt}}(X).$$

For every $\alpha \in \Omega^k(X),\ T \in \Omega_m(X),\ k \leq m$ define $\alpha \cap T \in \Omega_{m-k}(X)$ by

$$\langle \beta, \alpha \cap T \rangle = \langle \alpha \wedge \beta, T \rangle, \quad \forall \beta \in \Omega^{n-m+k}_{\mathrm{cpt}}(X).$$

We have

$$\begin{aligned} \langle \beta, \partial(\alpha \cap T) \rangle &= \langle d\beta, (\alpha \cap T), \rangle = \langle \alpha \wedge d\beta, T \rangle \\ &= (-1)^k \langle d(\alpha \wedge \beta) - d\alpha \wedge \beta, T \rangle = (-1)^k \langle \beta, \alpha \cap \partial T \rangle + (-1)^{k+1} \langle \beta, d\alpha \cap T \rangle \end{aligned}$$

which yields the *homotopy formula*

$$\partial(\alpha \cap T) = (-1)^{\deg \alpha} \left(\alpha \cap \partial T - (d\alpha) \cap T \right). \tag{B.1}$$

We have the following important result [6, §4.1.7].

[5] The measure μ is unique up to a multiplicative constant.

Theorem B.1 (Constancy theorem) *Suppose $S \in \Omega_k(\boldsymbol{V}^\vee \times \boldsymbol{V})$ is a k-dimensional cycle whose support is contained in a k-dimensional affine subspace $\boldsymbol{U} \subset \boldsymbol{V}^\vee \times \boldsymbol{V}$. Then there exists an orientation* **or** *on $\boldsymbol{U}$ and an integer ℓ such that $S = \ell[\boldsymbol{U}, \mathbf{or}]$ where $[\boldsymbol{U}, \mathbf{or}]$ is the current of integration along the oriented affine plane $\boldsymbol{U}$. In particular, if* supp S *is compact, then $S = 0$.*

We say that a set $S \subset \mathbb{R}^n$ is *locally subanalytic* if for any $p \in \mathbb{R}^n$ we can find an open ball B centered at p such that $B \cap S$ is globally subanalytic.

Remark B.1 There is a rather subtle distinction between globally subanalytic and locally subanalytic sets. For example, the graph of the function $y = \sin(x)$ is a locally subanalytic subset of $\mathbb{R}^2$, but it is not a globally subanalytic set. Note that a compact, locally subanalytic set is globally subanalytic.

If $S \subset \mathbb{R}^n$ is an orientable, locally subanalytic, C^1 submanifold of $\mathbb{R}^n$ of dimension k, then any orientation $\boldsymbol{or}_S$ on S determines a k-dimensional current $[S, \boldsymbol{or}_S]$ via the equality

$$\langle \alpha, [S, \boldsymbol{or}_S] \rangle := \int_S \alpha, \quad \forall \alpha \in \Omega^k_{\mathrm{cpt}}(\mathbb{R}^n).$$

The integral in the right-hand side is well defined because any bounded, k-dimensional globally subanalytic set has finite k-dimensional Hausdorff measure. For any open, locally subanalytic subset $U \subset \mathbb{R}^n$, we denote by $[S, \boldsymbol{or}_S] \cap U$ the current $[S \cap U, \boldsymbol{or}_S]$.

For any locally subanalytic subset $X \subset \mathbb{R}^n$, we denote by $\mathcal{C}_k(X)$ the Abelian subgroup of $\Omega_k(\mathbb{R}^n)$ generated by currents of the form $[S, \boldsymbol{or}_S]$, as above, where $\boldsymbol{cl}(S) \subset X$. The above operation $[S, \boldsymbol{or}_S] \cap U$, U open subanalytic extends to a morphism of Abelian groups

$$\mathcal{C}_k(X) \ni T \mapsto T \cap U \in \mathcal{C}_k(X \cap U).$$

We will refer to the elements of $\mathcal{C}_k(X)$ as *subanalytic (integral) k-chains* in X.

Given compact subanalytic sets $A \subset X \subset \mathbb{R}^n$ we set

$$\mathcal{Z}_k(X, A) = \big\{ T \in \mathcal{C}_k(\mathbb{R}^n);\ \operatorname{supp} T \subset X,\ \operatorname{supp} \partial T \subset A \big\},$$

and

$$\mathcal{B}_k(X, A) = \{ \partial T + S;\ T \in \mathcal{Z}_{k+1}(X, A)),\ S \in \mathcal{Z}_k(A) \}.$$

We set

$$\mathcal{H}_k(X, A) := \mathcal{Z}_k(X, A)/\mathcal{B}_k(X, A).$$

Hardt has proved in [15] that the assignment

$$(X, A) \longmapsto \mathcal{H}_\bullet(X, A)$$

satisfies the Eilenberg–Steenrod homology axioms with $\mathbb{Z}$-coefficients. This implies that $\mathcal{H}_\bullet(X, A)$ is naturally isomorphic with the integral homology of the pair.

To describe the intersection theory of subanalytic chains we need to recall a fundamental result of R. Hardt, [14, Theorem 4.3]. Suppose E_0, E_1 are two oriented real Euclidean spaces of dimensions n_0 and respectively n_1, $f : E_0 \to E_1$ is a real analytic map, and $T \in \mathcal{C}_{n_0 - c}(E_0)$ a subanalytic current of codimension c. If y is a regular value of f, then the fiber $f^{-1}(y)$ is a submanifold equipped with a natural coorientation and thus defines a subanalytic current $[f^{-1}(y)]$ in E_0 of codimension n_1, i.e., $[f^{-1}(y)] \in \mathcal{C}_{d_0 - d_1}(E_0)$. We would like to define the intersection of T and $[f^{-1}(y)]$ as a subanalytic current $\langle T, f, y \rangle \in \mathcal{C}_{n_0 - c - n_1}(E_0)$. It turns out that this is possibly quite often, even in cases when y is not a regular value.

Theorem B.2 (Slicing Theorem) *Let E_0, E_1, T and f be as above, denote by dV_{E_1} the Euclidean volume form on E_1, by $\boldsymbol{\omega}_{n_1}$ the volume of the unit ball in E_1, and set*

$$\mathcal{R}_f(T) := \big\{ y \in E_1;\ \operatorname{codim}(\operatorname{supp} T) \cap f^{-1}(y) \geq c + n_1,\ \operatorname{codim}(\operatorname{supp} \partial T) \cap f^{-1}(y) \\ \geq c + n_1 + 1 \big\}.$$

For every $\varepsilon > 0$ and $y \in E_1$ we define $T \bullet_\varepsilon f^{-1}(y) \in \Omega_{n_0-c-n_1}(E_0)$ by

$$\big\langle \alpha, T \bullet_\varepsilon f^{-1}(y) \big\rangle := \frac{1}{\boldsymbol{\omega}_{n_1} \varepsilon^{n_1}} \big\langle (f^* dV_{E_1}) \wedge \alpha, T \cap \big(f^{-1}(B_\varepsilon(y)) \big) \big\rangle, \quad \forall \alpha \in \Omega_{cpt}^{n_0-c-n_1}(E_0).$$

Then for every $y \in \mathcal{R}_f(T)$, the currents $T \bullet_\varepsilon f^{-1}(y)$ converge weakly as $\varepsilon > 0$ to a subanalytic current $\langle T, f, y \rangle \in \mathcal{C}_{n_0-c-n_1}(E_0)$ called the f-slice of T over y. Moreover, the map

$$\mathcal{R}_f \ni y \mapsto \langle T, f, y \rangle \in \mathcal{C}_{d_0-c-d_1}(\mathbb{R}^n)$$

is continuous in the locally flat topology.

Appendix C: Elementary Morse theory on singular spaces

Throughout this appendix, we fix an *o*-minimal category of sets and we will refer to the sets and maps in this category as *tame* or *definable*. For a topological space Z, we denote by $H^\bullet(Z)$ the (Čech) cohomology with real coefficients, and we define its topological Euler characteristic to be the integer

$$\chi_{\text{top}}(Z) = \sum_{k \geq 0} (-1)^k \dim H^k(Z),$$

whenever the sum in the right-hand side is well defined.

Suppose X is a locally closed tame subset of $\boldsymbol{V}$, and S is a closed tame subset of X. We define the local cohomology of X along S (with real coefficients) to be

$$H_S^\bullet(X) := H^\bullet(X, X \setminus S).$$

We can now define the *local cohomology sheaves* $\mathcal{H}_S^\bullet = \mathcal{H}_{X/S}^\bullet$ to be the sheaves on X associated to the presheaves $U \longmapsto H^\bullet_{S \cap U}(U)$.

If $x \in X$ and $U_n(x)$ denotes the open ball of radius $1/n$ centered at x, then for every $m \leq n$ we have morphisms $H^\bullet_{S\cap U_m}(U_m) \to H^\bullet_{S \cap U_n}(U_n)$, and then the stalk of $\mathcal{H}_S^p$ at x is the inductive limit $\mathcal{H}_S^\bullet(x) := \lim_{n\to\infty} H^\bullet_{S\cap U_n}(U_n)$. Observe that since X is locally conical we have

$$\mathcal{H}_S^\bullet(x) = 0 \quad \text{for every } x \in (X \setminus S). \tag{C.1}$$

We set

$$\chi_S(X) := \sum_k (-1)^k \dim H_S^k(X), \qquad \chi_S(x) := \sum_{k \geq 0} (-1)^k \dim \mathcal{H}_S^k(x).$$

We have a Grothendieck spectral sequence converging to $H_S^\bullet(X)$ whose E_2 term is $E_2^{p,q} = H^p(X, \mathcal{H}_S^q)$.

If it happens that the local cohomology sheaves are supported by finite sets then

$$H^{p,q}(X, \mathcal{H}_S^q) = 0, \quad \forall p > 0,$$

so that the spectral sequence degenerates at the E_2-terms. In this case we have

$$H^q_S(X) \cong H^0\big(X, \mathcal{H}^q_S\big) \cong \bigoplus_{x\in X} \mathcal{H}^q_S(x). \tag{C.2}$$

In particular

$$\chi_S(X) = \sum_{x\in X} \chi_S(x). \tag{C.3}$$

Suppose now that X is a compact connected tame subset of $\boldsymbol{V}$ and $f : X \to \mathbb{R}$ is a definable, continuous function. We will write

$$X_{f\geq c} := \{\, x \in X;\ f(x) \geq c \,\}, \qquad X_{f\leq c} := \{\, x \in X;\ f(x) \leq c \,\}, \text{ etc.}$$

A real number c is said to be a *regular* value of f if there exists $\varepsilon > 0$ such that the induced map

$$f : \{\, |f - c| < \varepsilon \,\} \subset X \to (c - \varepsilon, c + \varepsilon)$$

is a locally, definably trivial fibration. A real number c is said to be a *critical value* of f if it is not a critical value. The local triviality of tame maps implies that the set of critical values of f is finite.

Fix a real number c, and consider the sheaves of local cohomology $\mathcal{H}^\bullet_{f\geq c}$ associated to the closed subset set $S = X_{f\geq c} \subset X$. Note that if $f(x) = c$, and c is a regular value, then $\mathcal{H}^\bullet_{f\geq c}(x) = 0$. Any number outside the range of f is automatically a regular value. A point $x \in X$ is called a *homological* critical point of f if

$$\mathcal{H}^\bullet_{f\geq c}(x) \neq 0, \quad \text{where } c = f(x).$$

Note that in this case c must be a critical value of f. We denote by $\mathbf{Cr}_f$ the set of homological critical points of f. We say that f is *nice* if $\mathbf{Cr}_f$ is finite.

We define the *index* of f at a point $x \in \boldsymbol{V}$ to be the integer

$$\boldsymbol{m}(f, x) = \boldsymbol{m}_X(f, x) := \chi(\mathcal{H}^\bullet_{f\geq c}(x)) = \lim_{r\searrow 0} \chi\left(H^\bullet\left(B_r(x) \cap X, B_r(x) \cap X_{f<c}\right)\right), \tag{C.4}$$

where $c = f(x)$ and $B_r(x)$ denotes the *open* ball in $\boldsymbol{V}$ of radius r, centered at x. Note that $\boldsymbol{m}(f, x) = 0$ if $x \notin X$. Due to the local conical structure of X we have

$$\boldsymbol{m}(f, x) = 1 - \lim_{r\searrow 0} \chi\left(H^\bullet(B_r(x) \cap X_{f<c})\right). \tag{C.5}$$

A point $x \in X$ is called a *numerically critical point* of f if $\boldsymbol{m}(f, x) \neq 0$. We denote by $\mathbf{Cr}^\#_f$ the set of numerically critical points of f. Observe that $\mathbf{Cr}^\#_f \subset \mathbf{Cr}_f$.

One can ask the following natural question. Given a compact tame set X, do there exist nice continuous tame functions $f : X \to \mathbb{R}$? The answer is, yes, plenty of them. More precisely one can show (see [12], [23]) that there exists a subset $\Delta_X \subset \Sigma^\vee$, such that $\dim \Delta_X < \dim \Sigma^\vee = (n - 1)$ and for any $\xi \in \Sigma^\vee \setminus \Delta_X$, the induced function $\xi : X \to \mathbb{R}$ has only a finite number of homological critical points.

Example C.1 (a) Note that if X is a compact C^2-submanifold of $\boldsymbol{V}$, f is a Morse function on X, and x is a critical point of f with Morse index λ, then $\boldsymbol{m}(f, x) = (-1)^\lambda$. In this case $\mathbf{Cr}_f = \mathbf{Cr}^\#_f$.

(b) If X is a compact, convex, subanalytic subset of $\boldsymbol{V}$ and $\xi : \boldsymbol{V} \to \mathbb{R}$ is a linear map, then a point$x \in X$ is critical for the restriction of ξ to X if and only if x is a minimum point for ξ. In this case we have $\boldsymbol{m}(\xi, x) = 1$.

Lemma C.1 *[Kashiwara] Suppose that X is a compact connected tame subset of $\boldsymbol{V}$ and $f : X \to \mathbb{R}$ is a nice continuous tame function that contains no homological critical points on the level set $\{f = c\}$. Then the inclusion induced morphism $H^\bullet(X_{f<c+\varepsilon}) \to H^\bullet(X_{f<c})$ is an isomorphism for all $\varepsilon > 0$ sufficiently small.*

Proof We first prove that the morphism $H^\bullet(X_{f\leq c}) \to H^\bullet(X_{f<c})$. is an isomorphism. Indeed, it suffices to show that the local cohomology of $Z = X_{f\leq c}$ along $S = \{f \geq c\}$ is trivial. This follows from (C.2) by observing that the local cohomology sheaves $\mathcal{H}^\bullet_{Z/S}(x)$ are trivial.

Next observe that for some $\varepsilon_0 > 0$, the induced map $f : \{c < f < c + \varepsilon_0\} \to (c, c + \varepsilon_0)$ is a locally trivial fibration. This implies that for any $\varepsilon' < \varepsilon'' < \varepsilon_0$ the induced morphism

$$H^\bullet(X_{f<c+\varepsilon''}) \to H^\bullet(X_{f<c+\varepsilon''})$$

is an isomorphism. We conclude by observing that $H^\bullet(X_{f\leq c}) = \varinjlim_{t\searrow 0} H^\bullet(X_{f<c+t})$. □

Remark B.2 Kashiwara's lemma is valid in a much more general context, [17, Proposition 2.7.2]. The proof in the general case is much more involved.

Suppose is a compact connected tame set subset of $\boldsymbol{V}$ and $f : X \to \mathbb{R}$ is a nice, continuous tame function. Fix $c \in f(X)$. We have $H^\bullet_{X_{f\geq c}}(X_{f<c+\varepsilon}) = H^\bullet(X_{f<c+\varepsilon}, X_{f<c})$. From the equality

$$\bigcap_{\varepsilon>0} X_{f<c+\varepsilon} = X_{f\leq c}$$

we deduce

$$\varinjlim_{\varepsilon} H^\bullet_{X_{f\geq c}}(X_{f<c+\varepsilon}) = H^\bullet_{X_{f\geq c}}(X_{f\leq c}) = H^\bullet_{X_{f=c}}(X_{f\leq c}) = H^\bullet(X_{f\leq c}, X_{f<c}). \quad \text{(C.6)}$$

Using (C.1) and (C.3), we deduce that for any sufficiently small $\varepsilon > 0$, we have

$$\sum_{\substack{x\in \mathbf{Cr}^\#_f,\\ f(x)=c}} \boldsymbol{m}(f, x) = \chi_{\text{top}}\left(X_{f\leq c}\right) - \chi_{\text{top}}\left(X_{f<c}\right) = \chi_{\text{top}}\left(X_{f<c+\varepsilon}\right) - \chi_{\text{top}}\left(X_{f<c}\right). \quad \text{(C.7)}$$

Suppose now that $c', c \in f(X)$, $c' < c$ and the interval $(c, {}'c)$ contains no critical values of f. Then,

$$\chi_{\text{top}}(X_{f<c'}) = \chi_{\text{top}}(X_{f<c}). \quad \text{(C.8)}$$

Iterating (C.7) and (C.8), we deduce that for any $c, c' \in f(X)$, $c' < c$, we have

$$\chi_{\text{top}}(X_{f\leq c}) - \chi_{\text{top}}(X_{f\leq c'}) = \sum_{\substack{x\in \mathbf{Cr}^\#_f\\ c'<f(x)\leq c}} \boldsymbol{m}(f, x). \quad \text{(C.9)}$$

References

1. Bernig, A.: The normal cycle of compact definable sets. Isr. J. Math. **159**, 373–411 (2007).
2. Bröcker, L., Kuppe, M.: Integral geometry of tame sets. Geom. Ded. **82**, 285–323 (2000)
3. Cheeger, J., Müller, W., Schrader, R.: Kinematic and tube fromulas for piecewise linear spaces. Indian Univ. Math. J. **35**, 737–754 (1986)

4. Coste, M.: An Introduction to *o*-minimal Geometry, Real Algebraic and Analytic Geometry Network. http://www.ihp-raag.org/publications.php
5. Federer, H.: Curvature measures. Trans. Am. Math. Soc. **93**, 418–491 (1959)
6. Federer, H.: Geometric Measure Theory. Springer Verlag, Berlin (1969)
7. Fu, J.: Monge-Ampère functions I. Indiana Univ. Math. J. **38**, 745–771 (1989)
8. Fu, J.: Convergence of curvatures in secant approximations. J. Diff. Geom. **37**, 117–190 (1993)
9. Fu, J.: Curvature measures of subanalytic sets. Am. J. Math. **116**, 819–890 (1994)
10. Gabrielov, A.M.: Projections of semianalytic sets. Funct. Anal. Appl. **2**, 282–291 (1968)
11. Grinberg, M., MacPherson, R.: Euler characteristics and lagrangian intersections in the volume symplectic geometry and topology. In: Eliashberg, Y., Trynor, L., (eds.) IAS/Park City Mathematics Series. American Mathematical Society, Providence (1999)
12. Goresky, M., MacPherson, R.: Stratified Morse Theory, Ergebnisse der Mathematik, vol 14. Springer Verlag, Berlin (1988)
13. Guillemin, V., Sternberg, S.: Geometric Asymptotics. Revised Edition. Mathematics Surveys and Monographs, vol. 14. American Mathematical Society, Providence (1997)
14. Hardt, R.: Slicing and intersection theory for chains associated with real analytic varieties. Acta Math. **129**, 57–136 (1971)
15. Hardt, R.: Topological properties of subanalytic sets. Trans. Amer. Math. Soc. **211**, 57–70 (1975)
16. Hironaka, H.: Subanalytic sets, Number Theory, Algebraic Geometry, and. Commutative Algebra, in Honor of Yasuo Akizuki, pp.453–493. Kinokunya, Tokyo (1973)
17. Kashiwara, M., Schapira, P.: Sheaves on Manifolds, Gründlehren der mathematischen Wissenschaften, vol. 292. Springer Verlag, Berlin (1990)
18. Klain, D.A., Rota, G.-C.: Introduction to Geometric Probability. Cambridge University Press, Cambridge (1997)
19. Krantz, S.G., Parks, H.R.: Geometric Integration Theory. Birkhäuser, Switzerland (2008)
20. Miller, C., van den Dries, L.: Geometric categories and *o*-minimal structures. Duke Math. J. **84**, 497–540 (1996)
21. Morgan, F.: Geometric Measure Theory. A Beginner's Guide. 3rd edn. Academic Press, San Diego (2000)
22. Nicolaescu, L.I.: Lectures on the Geometry of Manifolds. 2nd edn. World Scientific, Singapore (2007)
23. Pignoni, R.: Density and stability of Morse functions on a stratified space, series 4. Ann. Scuola Norm. Sup. Pisa **6**, 593–608 (1979)
24. Schapira, P.: Constructible functions, Lagrangian cycles and computational geometry. In: The Gelfand Mathematical Seminars, 1990–1992, pp. 189–202. Birkäuser, Boston (1993)
25. Schapira, P.: Tomography of constructible functions.In: Applied Algebra, Algebraic Algorithms And Error-Correcting Codes, Paris, 1995. Lecture Notes in Computer Science, vol. 948, pp. 427–435. Springer, Berlin, (1995). http://people.math.jussieu.fr/~schapira/respapers/TomoLN.pdf
26. Schürmann, J.: Topology of Singular Spaces and Constructible Sheaves, Monografie Matematyczne, vol. 63. Birkhäuser Verlag, Switzerland (2003)
27. van den Dries, L.: Tame topology and *o*-minimal Structures. In: London Mathematical Society Lectures Notes Series, vol. 248. Cambridge University Press, Cambridge (1998)
28. Weyl, H.: On the volume of tubes. Am. J. Math **61**, 461–472 (1939)
29. Wintgen, P.: Normal cycle and integral curvature for polyhedra in Riemannian manifolds. Colloquia Mathematica Societatis Janos Bolyai, vol. 30 (1979). In: Soos, G., Szente, J. (eds.) Differential Geomatery. North Holland, Amsterdam (1982)