Deligne's Mixed Hodge Structure for Projective Varieties with only Normal Crossing Singularities

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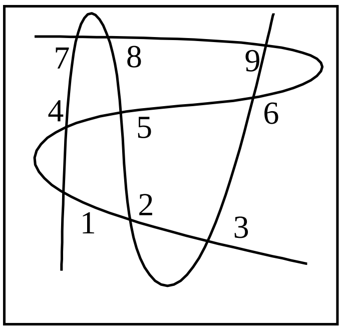
Abstract

Following the survey article by Griffiths and Schmid, I'll talk about the existence of a natural mixed Hodge structure on the cohomology of projective varieties whose irreducible components are smooth and meet at singular subsets that look like coordinate hyperplane intersections. I will compute some simple examples and also use the Mayer-Vietoris sequence to provide motivation for the existence of such a structure.

1 Motivation

Suppose that one wants to compute the cohomology of the subvariety of $\mathbb{P}^2$ consisting of two generic cubic curves (topologically each one is $S^1 \times S^1$). Bézout's theorem says that these curves meet at 9 distinct points. Let $V = V_1 \cup V_2$ be this variety, so that $V_1 \cap V_2 = \{p_1, \ldots, p_9\}, p_i \in \mathbb{P}^2$.



We use the Mayer-Vietoris sequence of the constant sheaf $\mathbb{C}_V$,

$$0 \to \mathbb{C}_V \to i_{1*}\,\mathbb{C}_{V_1} \oplus i_{2*}\,\mathbb{C}_{V_2} \to i_{12*}\,\mathbb{C}_{V_1 \cap V_2} \to 0$$

The sequence is not fine (or flasque, etc..), but it is exact so we get a long exact sequence in cohomology:

$$0 \to H^0(V, \mathbb{C}_V) \to H^0(V, i_{1*}\,\mathbb{C}_{V_1} \oplus i_{2*}\,\mathbb{C}_{V_2}) \to^{\alpha} H^0(V, i_{12*}\,\mathbb{C}_{V_1 \cap V_2}) \to$$
$$H^1(V, \mathbb{C}_V) \to^{\beta} H^1(V, i_{1*}\,\mathbb{C}_{V_1} \oplus i_{2*}\,\mathbb{C}_{V_2}) \to H^1(V, i_{12*}\,\mathbb{C}_{V_1 \cap V_2}) \to \cdots$$

Remark 1.1. Recall that for any closed subvariety $i : Y \hookrightarrow V$ and any sheaf $\mathcal{F}$ on Y we have that: $H^i(V, i_*\mathcal{F}) \cong H^i(Y, \mathcal{F})$. Indeed, we may compute cohomology using a *flasque* resolution on Y. Its direct image is still flasque on V and is still a resolution since the functor is exact on a sequence of flasque sheaves (the higher direct image sheaves vanish) [2] (III.8 Cor. 8.3).

Now we see a short exact sequence that determines $H^1(V, \mathbb{C}_V)$:

$$0 \to \operatorname{coker} \alpha \cong \mathbb{C} \to H^1(V, \mathbb{C}_V) \to H^1(V_1, \mathbb{C}_{V_1}) \oplus H^1(V_2, \mathbb{C}_{V_2}) \to 0$$

One way of interpreting this sequence is that there is an increasing filtration:
$0 \subset W_0 \subset W_1 = H^1(V, \mathbb{C}_V)$ where $W_0 = \operatorname{coker} \alpha$ and so the quotient $W_1/W_0 \cong H^1(V_1, \mathbb{C}_{V_1}) \oplus H^1(V_2, \mathbb{C}_{V_2})$.

The point of this is that we have a filtration on $H^1(V, \mathbb{C}_V)$ whose associated graded spaces are either the 1-st cohomology of a disjoint union of smooth complex projective varieties, or are a quotient of the 0-th cohomology of a disjoint union of smooth (and 0-dimensional) projective varieties, i.e. the points of intersection. Each of these spaces has a *pure hodge structure* of weight 1 and 0 respectively.

Thus we have produced in a rough way a mixed hodge structure on the cohomology of our variety with normal crossings. The goal of these notes is to explain how to do this in general on a nice class of singular projective varieties, and also to do it in a functorial way.

2 Varieties with normal crossings

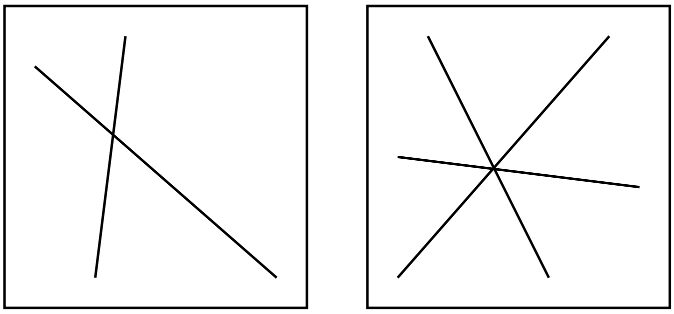
Now we'll construct a functorial mixed hodge structure for arbitrary varieties with normal crossings. But first, what is a "projective variety with only normal crossing singularities"?

Definition 2.1. Call a closed (not neccesarily irreducible) subvariety $V \subset \mathbb{P}^{n+1}$ a **variety with normal crossings** if

1. $V = \cup_{i=1}^{N} V_i$ with V_i smooth closed subvarieties.
2. Locally (in the metric topology) there is a $1 \le k \le n+1$ and an $\epsilon > 0$ such that $V \cong \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} | \, z_1 \cdot z_2 \cdots z_k = 0, \, |z_i| < \epsilon\}$

Condition 2 just says that the every point of V has a neighborhood which looks like some number of coordinate hyperplanes meeting at the origin. If the number is 1 then the point is obviously a smooth point. Otherwise the singularities are fairly mild. Note that the definition allows for at most $(n+1)$ hyperplanes to meet if the dimension of V is n.

Example 2.2. The simplest example of a projective variety with normal crossings is two generic lines in $\mathbb{P}^2$ that meet at a point: $\{[x, y, z] \mid x \cdot y = 0\}$. However the normal crossings condition rules out the variety $\{[x, y, z] \mid x \cdot y \cdot (x+y) = 0\}$ which looks like three lines meeting at the origin of an appropriate affine patch.



One consequence of the definition is that if we take any subset of indices $\{i_1, \ldots i_q\} \subset \{1, \ldots, N\}$ then the intersection $V_{i_1 \cdots i_q} := V_{i_1} \cap \cdots \cap V_{i_q}$ is a smooth and projective variety.

The basic idea of the construction in section 4 is to compute the cohomology of V by slicing it up into the irreducible components V_i whose cohomologies have pure hodge structures. Then we reassemble V keeping careful track of the intersections of the V_i which also have pure hodge structures, being smooth projective varieties. This is accomplished with a spectral sequence which takes the place of the Mayer-Vietoris sequence used in the initial example.

3 Spectral sequence of a double complex revisited

We'll use one of the two spectral sequences associated to a double complex. Let $\{A^{\bullet,\bullet}, d, \delta\}$ be a first quadrant double complex of vector spaces over $\mathbb{C}$ (the convention here is that the first "bullet" is the x-coordinate, second is the y-coordinate so that d does to the right and δ goes upward).

Recall that there is a spectral sequence $\{E_r^{\bullet,\bullet}, d_r\}$ which converges to the cohomology of the total complex: $TA := \oplus_k(\oplus_{p+q=k} A^{p,q})$ with differential $D = d + \delta$). The data for this spectral sequence is the following:

$$
\begin{array}{llll}
(E_0 & = \mathrm{A}, & d_0 & = d) \\
(E_1 & = H_d^*(A), & d_1 & = \delta) \\
(E_2 & = H_\delta^*(H_d^*(A)), & d_2 & = ?)
\end{array}
$$

For this data the choice of filtration on A is by rows [3] (Thm 2.15): $\widetilde{W_r} := \oplus_{i \geq r} A^{*,i}$. This is a decreasing filtration. It induces decreasing and increasing filtrations on the cohomology of the total complex which are denoted by $\widetilde{W}$ and W respectively. The convergence of this spectral sequence is guaranteed (since we're assuming it is bounded) and the limit term is:

$$E_\infty^{p,q} \cong gr_q^{\widetilde{W}} H^{p+q}(TA, D) = \widetilde{W}_q H^{p+q}(TA, D)/\widetilde{W}_{q+1} H^{p+q}(TA, D)$$

The decreasing filtration mentioned on $H^m(TA, D)$ is defined to be $W_r := \widetilde{W}_{m-r}$ so in terms of W the limit term is:

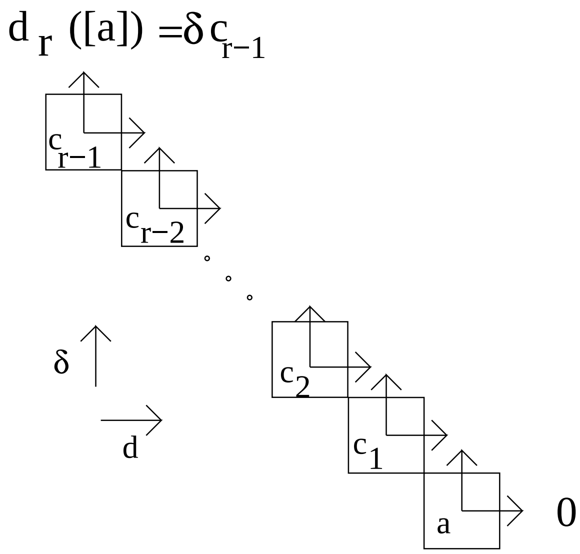
$$E_\infty^{p,q} \cong gr_{p+q-q}^W H^{p+q}(TA, D) \cong gr_p^W H^{p+q}(TA, D) \tag{1}$$

Where now W is an *increasing* filtration. Note that we want the weight filtration of our mixed hodge structure to be increasing.

Now, at several points we want to show that such a spectral sequence degenerates. This will have to be done even in the case where the differentials d_r have non-zero domain and target, which means we need to understand better what the d_r do.

Definition 3.1. ([4] Chapter III section 14) An element $a \in A^{p,q}$ is said to **survive to** E_r if its class in $E_i^{p,q}$ is a non-zero cocycle for d_i for all $i < r$

Recall that each $E_i^{p,q}$ is a sub-quotient of $A^{p,q}$ so the definition makes sense. It can be seen that an element a as above surviving to E_r implies that it can be "extended to a zig-zag":



where $c_i \in A^{p+i,q-i}$ and

$$\begin{aligned} da &= 0 \\ \delta a &= dc_1 \\ \delta c_1 &= dc_2 \\ &\vdots \\ \delta c_{r-2} &= dc_{r-1} \end{aligned}$$

This zig-zag is important because it determines the differential at stage r: $d_r([a]) = [\delta c_{r-1}] \in E_r^{p-(r-1),q+r}$. Most importantly the class of δc_{r-1} in E_r doesn't depend on the choice of c_{r-1} up to the restriction: $[\delta\beta] = [\delta c_{r-1}]$ for any β such that $d\beta = \delta c_{r-2}$. We'll use this fact at the end.

4 The Double Complex for $H^*(V)$

To use the spectral sequence above in the situation of a variety with normal crossings we need to split up V:

Recall the that for any non-empty index set $I = \{i_1, \ldots, i_k\}$ we defined V_I above as a certain k-fold intersection of the V_i. Analogous to the Mayer-Vietoris construction we'll look at differential forms on the disjoin union of these intersections:

$$V^{(q)} := \sqcup_{|I|=q+1} V_I$$

These varieties are smooth, compact, and have canonical Kähler structures, hence canonical pure Hodge structures on their cohomologies.

Example 4.1. In the initial example, $V^{(0)} = V_1 \sqcup V_2$, and $V^{(1)} = \sqcup_{i=1}^{9} \{p_i\}$

Let $\mathcal{A}^{p,q}$ be the sheaf of differential p-forms on $V^{(q)}$. This sheaf of bi-graded vector spaces has two natural differentials: $d : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ which is just exterior differentiation, and δ the "combinatorial differential". For an open set U and $s \in \mathcal{A}^{p,q}(U)$ let s_I be the value of s on V_I. Then:

$$\delta(s)_{\{i_1,\ldots,i_{q+1}\}} = \sum_{l=1}^{q+1} (-1)^l s_{\{i_1,\ldots,\hat{i_l},\ldots,i_{q+1}\}}|_{V_{i_1,\ldots,i_{q+1}}}$$

Taking $A^{p,q} := \mathcal{A}^{p,q}(U)$ and the differentials above we get a double complex. The fact that $d^2 = 0$, $\delta^2 = 0$, and $d\delta + \delta d = 0$ follows from the same arguments used in the Čech-deRham double complex construction [4] (Example 14.16).

5 deRham's theorem for varieties with normal crossings

The first main step in producing a functorial mixed Hodge structure is to show that the cohomology of V can be obtained from the total complex of the double complex described above where $U = V$ i.e. the one for global sections.

Let $\mathcal{A}^k = \oplus_{p+q=k} i_{q*} \mathcal{A}^{p,q}$ where $i_q : V^{(q)} \to V$ is the "gluing" map and $D = d + \delta$. This is a complex of sheaves and there is an obvious sequence:

$$0 \to \mathbb{C}_V \to \mathcal{A}^0 \to \mathcal{A}^1 \to \cdots$$

where the first map $\mathbb{C}_V \to \mathcal{A}^{0,0} = i_{0*}(C^\infty(V_1) \oplus \cdots \oplus C^\infty(V_N))$ is the obvious sum of inclusions and the rest are given by D.

The first goal is to show that this is a fine (and hence Γ-acyclic) resolution of the constant sheaf on V.

Theorem 5.1. *The sequence above is a resolution of $\mathbb{C}_V$ by fine sheaves.*

Proof. The fact that the sheaves are fine follows from the ordinary Poincaré lemma since the sheaves are just sums of sheaves of differential forms on smooth manifolds.

Exactness is a little bit more tricky. We'll follow [1] and use the spectral sequence to prove that the cohomology of this complex vanishes when we look locally. So for the moment let $A^{p,q} := \mathcal{A}^{p,q}(U)$ where U is an open set where V is diffeomorphic to the intersection of k hyperplanes near the origin of $\mathbb{C}^{n+1}$. Thus we have a bounded spectral sequence converging to the cohomology of the total complex which is exactly our sequence above evaluated at U.

On the E_1 page of the spectral sequence we take cohomology with respect to d, the exterior derivative. The Poincaré lemma says that $E_1^{p,q} \cong 0$ when $p > 0$. So on the E_2 page we just need to take cohomology of the first column with respect to the combinatorial differential δ, i.e. $E_2^{0,q} = H^q((E_1^{0,\bullet}, \delta))$. Now the formula for δ and the complex are exactly what one uses to compute the cohomology of a $(k-1)$-simplex. Hence the only non-zero term is $\mathcal{A}^{0,0}(U) \cong E_2^{0,0} \cong \mathbb{C}$. So it is clear that the sequence is exact after taking into account the image of the $\mathbb{C}_V(U)$ term. $\square$

Now the homological machinery tells us that:

Corollary 5.2. *(Normal crossing deRham theorem) $H^*(V, \mathbb{C}_V) \cong H^*(\mathcal{A}^\bullet(V), D)$*

6 Mixed Hodge Structure on $H^*(\mathcal{A}^\bullet(V), D)$

The second main step is to put a functorial mixed Hodge structure on the cohomology of the total complex $(TA = \oplus_k \mathcal{A}^k(V), D)$ above.

So let $A^{p,q} := \mathcal{A}^{p,q}(V)$ be the initial data for our spectral sequence. Recall that we're taking $E_1^{\bullet,\bullet} = H_d^*(A^{\bullet,\bullet})$, and $E_2^{\bullet,\bullet} = H_\delta^*(H_d^*(A^{\bullet,\bullet}))$. So on any column of the E_1 page (p is fixed) we have p-th d-cohomology groups of the varieties $V^{(q)}$ for various q. Thus column p can be equipped with the classical pure Hodge structure of weight p, the filtration for which we'll denote by F^k. There is also the filtration $\widetilde{W}$ of the double complex which induces the important filtration W of $H^m(TA, D)$:

$$\widetilde{W}^r = \oplus_{i \geq r} A^{*,i}$$
$$F^r = \oplus_{i,j} F^r A^{i,j}$$

The key observation is that to go from the E_1 page, which we know has some kind of Hodge structure on its columns, to E_2 we need to take cohomology with respect to the combinatorial differential. This differential consists of linear combinations of the pull-back maps coming from inclusions of closed subvarieties. Thus it is a morphism of pure Hodge structure of weight 0! Hence all kernels and cokernels along a fixed column have induced pure Hodge structure of the same weight as the column number, and hence the cohomology groups do as well.

Here is the final step in the construction:

Theorem 6.1. *([1] Lemma 4.8)*

1. *The spectral sequence under consideration collapses at the E_2 term.*
2. *The filtrations $W_\bullet, F^\bullet$ induce a functorial mixed Hodge structure on $H^m(TA, D)$.*

Proof. We have to show that $d_r = 0$ for $r \geq 2$. The argument for d_2 is given in [1] but it is no harder to give a general one.

Recall from the spectral sequence that if $a \in A^{p,q}$ lives to $E_r^{p,q}$ then there is a zig-zag of elements c_i such that $d_r([a]) = [\delta c_{r-1}]$ where the class is taken in the appropriate group. The goal is to represent this element by forms of different (p, q) type, since the purity of the Hodge structure along each column will then imply that the cohomology class of δc_{r-1} is zero.

The zig-zag implied that $\delta c_{r-2} = dc_{r-1}$ and so δc_{r-2} is an exact form. Now we can assume without loss of generality that c_{r-2} is a form of type (k, l) where $k + l = p - (r - 1)$. Use the $\partial\overline{\partial}$-lemma ([5] Prop. 6.17 or [1] Lemma 2.13) to write δc_{r-2} in two ways:

$$\delta c_{r-2} = d\beta = d\beta'$$

where β has type $(k - 1, l)$ and β' has type $(k, l - 1)$. But now the classes of $\delta\beta$ and $\delta\beta'$ and δc_{r-1} are all the same in $E_r^{p-(r-1),q+r}$ (see section 3) and their cohomology classes have to be zero there since this space has a pure Hodge structure, δ preserves (p, q) type, and β, β' have different types. $\square$

Thus the pure Hodge structure of weight p survives to $E_\infty^{p,q}$. We know that $E_\infty^{p,q} \cong gr_p^W H^{p+q}(TA, D)$ so the following corrolary is clear:

Corollary 6.2. *The filtration induced by $F^\bullet$ is a pure Hodge structure of weight p on $gr_p^W H^{p+q}(TA, D)$.*

There are two further remarks to make:

Remark 6.3. The weight filtration we've constructed on $H^m(V)$ has the form:

$$\{0\} \subset W_0 \subset \cdots \subset W_{m-1} \subset W_m = H^m(V)$$

Remark 6.4. Functoriality of this construction follows from the fact that a map between varieties with normal crossings must map irreducible components to irreducible components, and hence intersections of such to intersections of such, so there are induced morphism of spectral sequences.

7 Example computations

We can carry out this construction now without much trouble in low dimensions (here for curves in $\mathbb{P}^2$). We'll see that the weight filtration and Hodge filtration it produces really coincide with what the Mayer-Vietoris sequence gives in the initial example of section 1.

Example 7.1. $V = V_1 \cup V_2$ where $V_i \cong S^1 \times S^1$. $V^{(0)} = V_1 \sqcup V_2$ and $V^{(1)} = \{p_1, \ldots, p_9\}$. Here are the spectral sequence pages: (places not shown are zero, recall that columns are indexed by p, rows by q)

$E_0^{\bullet,\bullet}$

0	0	0
$\Omega^0(\mathrm{pt})^{\oplus 9}$	$\Omega^1(\mathrm{pt})^{\oplus 9}$	$\Omega^2(\mathrm{pt})^{\oplus 9}$
$\Omega^0(V_1) \oplus \Omega^0(V_2)$	$\Omega^1(V_1) \oplus \Omega^1(V_2)$	$\Omega^2(V_1) \oplus \Omega^2(V_2)$

$E_1^{\bullet,\bullet}$

0	0	0
$H^0(\mathrm{pt})^{\oplus 9}$	0	0
$H^0(V_1) \oplus H^0(V_2)$	$H^1(V_1) \oplus H^1(V_2)$	$H^2(V_1) \oplus H^2(V_2)$

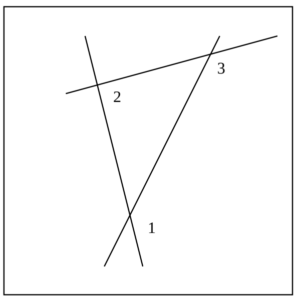
$E_2^{\bullet,\bullet}$

0	0	0
$\mathbb{C}^9 / \mathbb{C}_\Delta \cong \mathbb{C}^8$	0	0
$\mathbb{C}_{-\Delta} \cong \mathbb{C}$	$H^1(V_1) \oplus H^1(V_2)$	$H^2(V_1) \oplus H^2(V_2)$

Now, to find $H^m(V, \mathbb{C}_V)$ we just sum along the appropriate diagonal. For instance $H^1(V, \mathbb{C}_V) \cong (\mathbb{C}^9 / \mathbb{C}_\Delta) \oplus (H^1(V_1) \oplus H^1(V_2))$. To find the induced filtration W_r we just take $\widetilde{W}^r$ (everything on E_2 above and including the r-th row), re-index based on what group H^m we're looking at, and intersect with the m-th diagonal. For instance $W_0 H^1(V) \cong \mathbb{C}^9 / \mathbb{C}_\Delta \cong \mathbb{C}^8$ which is a quotient of $H^0(\mathrm{pt})^{\oplus 9}$ and has the induced pure Hodge structure of weight 0. Also, $W_1 H^1(V) = H^1(V)$ so the associated graded piece is: $gr_1^W H^1(V) \cong H^1(V_1) \oplus H^1(V_2)$ which has the pure Hodge structure of weight 1.

Thus the weight filtration here is just the one coming from the Mayer-Vietoris long exact sequence.

Example 7.2. The second example is that of three lines ($\mathbb{P}^1$'s) which intersect in a triangle pattern in $\mathbb{P}^2$.



Here we have $V = V_1 \cup V_2 \cup V_3$, $V_i \cong \mathbb{P}^1$. Also, $V^{(0)} = V_1 \sqcup V_2 \sqcup V_3$, $V^{(1)} = \{p_1, p_2, p_3\}$, and $V^{(r)} = \phi$ for $r \geq 2$.

$E_0^{\bullet,\bullet}$

0	0	0
$\Omega^0(\mathrm{pt})^{\oplus 3}$	$\Omega^1(\mathrm{pt})^{\oplus 3}$	$\Omega^2(\mathrm{pt})^{\oplus 3}$
$\Omega^0(V_1) \oplus \Omega^0(V_2) \oplus \Omega^0(V_3)$	$\Omega^1(V_1) \oplus \Omega^1(V_2) \oplus \Omega^1(V_3)$	$\Omega^2(V_1) \oplus \Omega^2(V_2) \oplus \Omega^2(V_3)$

$E_1^{\bullet,\bullet}$

0	0	0
$\mathbb{C}^3$	0	0
$\mathbb{C}^3$	0	$\mathbb{C}^3$

Here we've used $H^1(\mathbb{P}^1) \cong 0$, and $H^i(\mathbb{P}^1) \cong \mathbb{C}$ for $i = 0, 2$

The only non-zero combinatorial differential at this point is $\delta : \mathbb{C}^3 \to \mathbb{C}^3$ in the 0-th column. These two spaces and the differential between them is exactly the simplicial complex of the 1 skeleton of a 2-simplex. Hence $\ker \delta \cong \mathbb{C}$ and $\operatorname{coker} \delta \cong \mathbb{C}$.

$E_2^{\bullet,\bullet}$

0	0	0
$\mathbb{C}$	0	0
$\mathbb{C}$	0	$\mathbb{C}^3$

Summing along the diagonals of the E_2 page we see that

$$H^i(V) \cong \begin{cases} \mathbb{C} & i = 0 \\ \mathbb{C} & i = 1 \\ \mathbb{C}^3 & i = 2 \end{cases}$$

It is immediately clear that $H^1(V)$ could not have a pure Hodge structure of weight 1 because its dimension is odd. We see that the weight filtration on $H^1(V)$ has the form: $\{0\} \subset W_0 = W_1 = H^1(V)$ meaning it does have a pure Hodge structure but of weight 0! These 1-cocycles of weight 0 are essentially combinatorial since they come from the "nerve" of the intersection lattice of the lines V_i.

The last statement in example 2 is true in general about the cocycles which live in $W_0 H^m(V)$ where V is a variety with normal crossings: Let $\Gamma(V)$ be the nerve of the intersection lattice of the irreducible components of V (i.e. it is a simplicial complex where 0-simplicies correspond to the V_i, 1-simplicies correspond to non-empty 2-fold intersections, etc..). Then

$$W_0 H^m(V) \cong H^m(\Gamma(V))$$

One can find this statement and many other interesting corrolaries in [6] chapter 4, section 2.8.

References

[1] Griffiths, P. and Schmid, W. *Recent Developments in Hodge Theory: A Discussion of Techniques and Results.*

[2] Hartshorne, R. *Algebraic Geometry.*

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