

Hodge Decomposition

(1)

M - compact, orientable
diff. manifold of dim. n .

$\mathbb{T}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ complexified tangent bundle

\mathcal{A}_M^k = sheaf of diffble sections of $\wedge^k \mathbb{T}^*M$.

de Rham complex

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{A}_M^0 \xrightarrow{d} \mathcal{A}_M^1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{A}_M^n \rightarrow 0$$

de Rham theorem:

$$H^k(M, \mathbb{C}) \cong H^k(\mathcal{A}^\bullet(M), d)$$

let us fix ~~the~~ Hermitian metrics on $\wedge^k \mathbb{T}^*M$.

there is an adjoint operator $d^*: \mathcal{A}_M^k \rightarrow \mathcal{A}_M^{k-1}$

form Laplacian of d : $\Delta = dd^* + d^*d$

harmonic k -forms $\mathcal{H}^k(M) = \{m \in \mathcal{A}^k(M), \Delta m = 0\}$
 $= \{m \in \mathcal{A}^k(M), dm=0, d^*m=0\}$ - closed & co-closed.

$$\mathcal{H}^k(M) \cong H^k(M; \mathbb{C})$$

-operator $$: $\wedge^k T^*M \xrightarrow{\cong} \wedge^{n-k} T^*M$ \mathbb{R} operator, \mathbb{C} -linear

$$*: \wedge^k \mathbb{T}^*M \xrightarrow{\cong} \wedge^{n-k} \mathbb{T}^*M$$

$$d^* = (-1)^k w * d * \quad w = \text{Weil operator } (-1)^{w+1} \text{ on } \wedge^k \mathbb{T}^*M.$$

$$\Delta * = * \Delta.$$

Consequence: Poincaré Duality. the pairing $H^k(M, \mathbb{C}) \times H^{n-k}(M, \mathbb{C}) \rightarrow \mathbb{C}$
 $(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$
 is degenerate.

Pf of Poincaré Duality:

$$\bar{*}: \mathcal{H}^k(M) \xrightarrow{\cong} \mathcal{H}^{n-k}(M)$$

$$(u, u) \longrightarrow \int_M u \wedge \bar{*}u = \|u\|^2 > 0 \text{ if } u \neq 0.$$

$M =$ compact cplx mfd of dim n over \mathbb{C}

$$\Lambda^n T^*M = \bigoplus_{p+q=n} (\Lambda^p T_{\mathbb{C}}^*M \otimes \Lambda^q \overline{T_{\mathbb{C}}^*M})$$

decomposition

$$\begin{aligned} \text{Recall: } T^*M &= (T_{\mathbb{C}}M)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \\ &= T_{\mathbb{C}}^{1,0}M \oplus T_{\mathbb{C}}^{0,1}M \end{aligned}$$

$\mathcal{A}_M^{p,q}$ - sheaf of diff. sect. of $T^{p,q}M$

Ω_M^p = sheaf of holom. forms of deg p .

Dolbeault complex:

$$0 \rightarrow \Omega_M^p \rightarrow \mathcal{A}_M^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}_M^{p,1} \xrightarrow{\bar{\partial}} \dots \mathcal{A}_M^{p,n} \rightarrow 0$$

Dolbeault Theorem: $H_{\mathbb{C}}^{p,q}(M, \Omega_M^p) \simeq H_{\mathbb{C}}^{p,q}(\mathcal{A}_M^{p,\bullet}(M), \bar{\partial})$
" $H^{p,q}(M, \mathbb{C})$

$$\bar{\partial}^*: \mathcal{A}_M^{p,q} \rightarrow \mathcal{A}_M^{p,q-1} \quad (\bar{\partial}^* = - * \partial *)$$

$$\Delta_{\bar{\partial}} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$$

$$\mathcal{H}^{p,q}(M) = \{u \in \mathcal{A}_M^{p,q}(M), \Delta_{\bar{\partial}} u = 0\}$$

$$= \{u \in \mathcal{A}_M^{p,q}(M), \bar{\partial} u = 0, \bar{\partial}^* u = 0\}$$

$$H^{p,q}(M) \simeq H^{p,q}(M, \mathbb{C})$$

SERRE DUALITY:

$$H^{p,q}(M, \mathbb{C}) \times H^{n-p, n-q}(M, \mathbb{C}) \longrightarrow \mathbb{C}$$

$$(u, v) \longrightarrow \int_M u \wedge v$$

is nondegenerate.

Kähler Manifolds

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M ... mfd of dim n over \mathbb{C} ,
endowed w/ Hermitian metric:

$$h = \sum_{i,j=1}^n h_{ij} dz^i \otimes d\bar{z}^j$$

$$\omega = -\frac{1}{2} \operatorname{Im} h = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^n h_{ij} dz^i \wedge d\bar{z}^j$$

associate $(1,1)$ -form to h .

Def: (M, h) is said to be a Kähler if $d\omega = 0$

Equivalent
condition:

$g = \operatorname{Re} h$, Riemannian metric on M .

∇ Levi-Civita connection

J -complex structure on $T_{\mathbb{R}M}$, $J: T_{\mathbb{R}M} \rightarrow T_{\mathbb{R}M}$, $J^2 = -1$.

$$d\omega = 0 \iff \nabla J = 0 \text{ meaning } \nabla_X JY = J \nabla_X Y \quad \forall X, Y.$$

Examples

\mathbb{C}^n w/ Euclidean metric

$$h = \sum_{i,j=1}^n dz^i \otimes d\bar{z}^j$$

$\Lambda \subseteq \mathbb{C}^n$ lattice

$$\Lambda \cong \mathbb{Z}^{2n}$$

\mathbb{C}^n / Λ cpct mfd
called torus

h on \mathbb{C}^n descends to a metric on \mathbb{C}^n / Λ which makes
the torus Kähler.

\mathbb{P}^n complex projective space

equipped with Fubini-Study metric

$$\omega_{FS} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \|z\|^2$$

$$= \frac{1}{4\pi} dd^c \log \|z\|^2$$

$z = (z_0, \dots, z_n)$

$\partial \bar{\partial} \log \|z\|^2$ (1,1)-form on \mathbb{C}^{n+1}

Fact: any submanifold of a Kähler mfd is Kähler,
 so any projective mfd is Kähler
 (although not all Kähler mfd's are projective).

Kähler mfd's have extra structure on the cohomology:

$L: \mathcal{A}_M^{p,q} \longrightarrow \mathcal{A}_M^{p+1, q+1}$ Lefschetz operator

$L(u) = \omega \wedge u$

$L^*: \mathcal{A}_M^{p,q} \longrightarrow \mathcal{A}_M^{p-1, q-1}$

Lemma: $[L, d] = 0$ (we use the Kähler condition)
 $[L, d^*] = d^c = \frac{1}{\sqrt{-1}}(\partial - \bar{\partial})$

Kähler relations

M compact, Kähler. Then $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$

Consequence: $u \in \mathcal{A}^k(M)$, u is harmonic \iff each $u_{p,q}$ is harmonic
 \iff each $u_{p,q}$ is $\bar{\partial}$ -harmonic

$$u = \sum_{p+q=k} u_{p,q}$$

Get: $\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M)$

Note $\mathcal{H}^{p,q} \cong \overline{\mathcal{H}^{q,p}}$

HODGE DECOMPOSITION: M cpct, Kähler then
 $H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M, \mathbb{C})$

$\mathcal{H}^k = \sum_{p+q=k} \mathcal{H}^{p,q}$

$$b_r = \sum_{p+q=r} h^{p,q}$$

$$h^{p,q} = h^{q,p}$$

$\Rightarrow b_r$ is even if r is odd (a top. constraint on Kähler mfd.)

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How to build non-Kähler mfd's

- when 1st Betti number is odd.

eg: Hopf surfaces:

$\mathbb{C}^2 \setminus \{0\} / \mathbb{Z}$ given by action of powers of α , where $\mathbb{C}^2 \setminus \{0\} \xrightarrow{\alpha} \mathbb{C}^2 \setminus \{0\}$

quartic in \mathbb{P}^3

1	0	1
0	20	0
1	0	1

Hodge diamond

$h^{0,0}$	$h^{0,1}$	$h^{0,2}$...	$h^{0,n}$
$h^{1,0}$	$h^{1,1}$	$h^{1,n}$
\vdots	\vdots			\vdots
$h^{n,0}$	$h^{n,1}$	$h^{n,n}$

Lefschetz decomposition

L, L^* , H . mult by $(n-k)$ on k -forms
 operators

Lemma: $[L, L^*] = H$ $[H, L] = 2L$ $[H, L^*] = -2L^*$

- these operators on forms induced decomposition on cohomology.

Obs: $\wedge^k \mathbb{T}_x^* M$ becomes an $sl_2(\mathbb{C})$ -representation

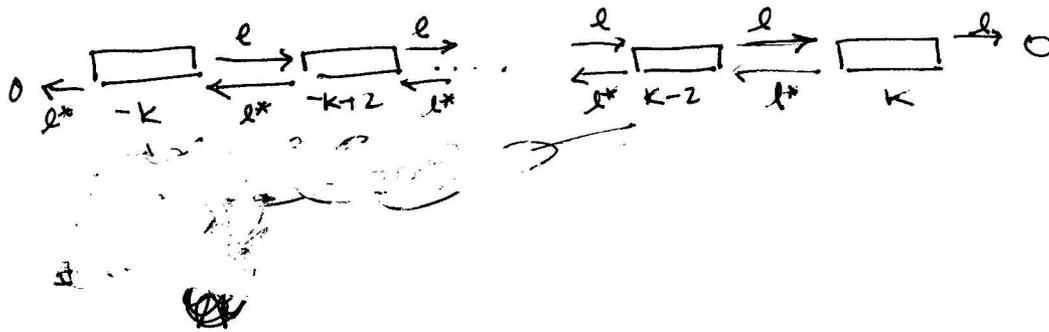
by sending $L = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow L, L^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \rightarrow L^*, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow H.$

$sl_2(\mathbb{C})$ is a semisimple Lie algebra.

\Rightarrow any finite dim rep'n is a direct sum of irreducible rep'ns.

Irreducible Representations of $sl_2(\mathbb{C})$ are of the form $S^k \mathbb{C}^2$

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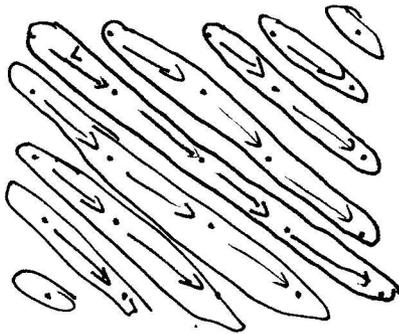


Notice that $\bigoplus_{q-p=S} \bigwedge^q \mathbb{C}^2 \otimes \bigwedge^p \mathbb{C}^2 \otimes M$ is a subrepresentation

So each decomposes as direct sum of irred $sl_2(\mathbb{C})$ -rep's

Example:

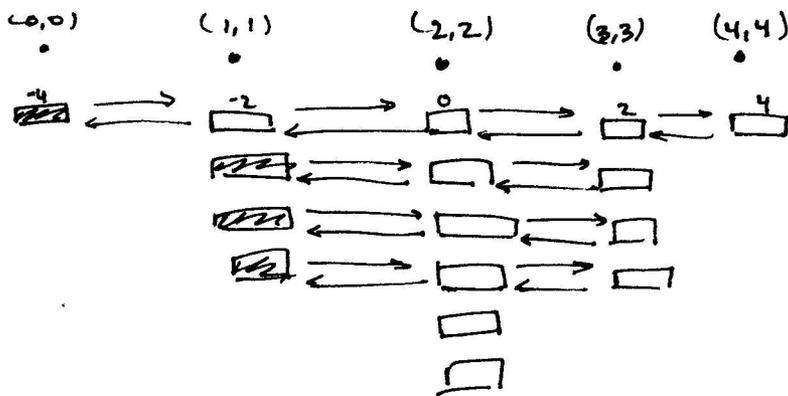
$n=4$



Hodge diamond

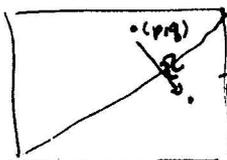
Each circled part is invariant under the action.

Consider main diagonal:



Shaded ones = building blocks of type (p,q)

Observation:



L maps to symmetric pos'n, $(n-q, n-p)$.

$$L^{n-p-q} : \Lambda^{p,q} \Pi_x^* M \xrightarrow{\cong} \Lambda^{n-q, n-p} \rightarrow \Pi_x^* M$$

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Definition: $\varphi \in \Lambda^{p,q} \Pi_x^* M$ is called PRIMITIVE
 if $L^* \varphi = 0 \iff L^{n-p-q+1} \varphi = 0$.

Primitive Decomposition (for forms)

Given $\varphi \in \Lambda^{p,q} \Pi_x^* M$, there is a unique decomposition

$$\varphi = \sum_{s \geq (k-n)^+} L^s \varphi_s, \quad \varphi_s \text{ primitive}$$

$$(k-n)^+ = \max \{0, k-n\}$$

$$k = p+q.$$

$[L, \Delta] = 0$, so what we did for forms, we can do for cohomology:

$$\mathcal{H}_{\text{prim}}^{p,q}(M, \mathbb{C}) = \{ [u], u \in \mathcal{L}^{p,q}(M) \mid L^* u = 0 \iff L^{n-p-q+1} u = 0 \}$$

- Primitive parts are building blocks for cohomology.

Lefschetz decomposition:

$$\text{Take } \varphi \in H^{p,q}(M, \mathbb{C})$$

then there is a unique decomposition

$$\varphi = \sum_{s \geq (n-p-q)^+} L^s \varphi_s, \quad \varphi_s \in H_{\text{prim}}^{p-s, q-s}(M, \mathbb{C})$$

Obs: $h^{p,q} - \binom{n-p-q}{s} = \dim H_{\text{prim}}^{p,q}(M, \mathbb{C})$.