MIXED HODGE STRUCTURES ON SMOOTH ALGEBRAIC VARIETIES

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ABSTRACT. We discuss some of the details of Deligne's proof on the existence of a functorial mixed Hodge structure on a smooth quasiprojective variety.

CONTENTS

NOTATIONS

• $H^\bullet(S) := H^\bullet(S, \mathbb{C})$.

• If $W_\bullet$ is an increasing filtration on the vector space V then we denote by $W_-^\bullet$ the decreasing filtration defined by

$$W_-^\ell = W_{-\ell}.$$

If $F^\bullet$ is a decreasing filtration then we can associate in a similar fashion the increasing filtration $F_\bullet^-$.

• For an increasing filtration $W_\bullet$ and $k \in \mathbb{Z}$ we define the shifted filtration

$$W[n]_\bullet = W_{n+\bullet}.$$

We define the shifts of decreasing filtrations in a similar way. Note that

$$W[n]_-^\bullet = W_-[-n]^\bullet.$$

• For every graded object $C^\bullet = \oplus_{n\in\mathbb{Z}} C^n$ and every integer k we denote by $C^\bullet[k]$ the graded object defined by

$$C^n[k] := C^{n+k}.$$

• For a bigraded object $C^{\bullet,\bullet}$ and every integers (k, m) the shifted complex $C^{\bullet,\bullet}[k, m]$ is defined in a similar fashion.

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1. Formulation of the problem

We assume X^* is a smooth, complex, n-dimensional algebraic variety. According to Hironaka it admits a smooth compactification X such that the complement $D = X \setminus X^*$ is a normal crossings divisor. This means that every point $p \in D$ there exist an integer $1 \leq k \leq n$ and an open neighborhood $\mathcal{O}$ bi-holomorphic to an open ball B in $\mathbb{C}^n$ centered at 0 such that

$$D \cap \mathcal{O} \cong \{(z_1, \cdots, z_n) \in B;\ \ z_1 \cdots z_k = 0\}.$$

We assume X is projective and we explain how to produce a mixed Hodge structure on X^* using the Hodge structure on X. The mixed Hodge structure thus produced will be independent of the compactification X.

The strategy we will employ is easy to describe. Denote by $j : X^* \hookrightarrow X$ the natural inclusion of X^* as an open subset of X. We observe that

$$H^\bullet(X^*) \cong H^\bullet(X, j_*\underline{\mathbb{C}}).$$

The construction of a mixed Hodge structure on $H^\bullet(X^*, \mathbb{Z})$ is carried on in several steps.

Step 1. We construct a complex of sheaves $\mathcal{S}^\bullet$ on X quasi-isomorphic to $j_*\underline{\mathbb{C}}$. This complex will be equipped with a natural decreasing filtration F^p and a natural increasing filtration W_ℓ. We then produce a hypercohomology spectral sequence $E_r^{\bullet,\bullet}$ associated to the decreasing filtration $W_-^\ell := W_{-\ell}$ and converging to $\mathbb{H}^\bullet(X^*, \mathcal{S}^\bullet) \cong H^\bullet(X^*, \mathbb{C})$. The increasing filtration W_ℓ induces an increasing filtration on $H^\bullet(X^*, \mathbb{C})$ which, up to a shift, will be the weight filtration.

Step 2. We will show that the filtration F induces pure Hodge structures on $E_1^{p,q}$ and the differential d_1 is a morphism of pure Hodge structures of a given bidegree $(0,0)$. In particular, we deduce that E_2 is equipped with a mixed Hodge structure.

Step 3. We will show that for every $r \geq 2$ the differential d_r on E_r vanishes so that

$$\mathbf{Gr}_\ell^W\, H^m(X^*) \cong E_2^{-\ell, m+\ell}$$

is equipped with a natural pure Hodge structure induced by F of weight $m+\ell$. We deduce that the filtrations $(F^\bullet, W[-m]_\bullet)$ define a mixed Hodge structure on $H^m(X^*)$. We have

$$\emptyset = W_{m-1}H^m(X^*) \subset j^*H^m(X) = W_mH^m(X^*) \subset \cdots \subset W_{2m}H^m(X^*) = H^m(X^*).$$

Step 4. We will show that a holomorphic map between smooth quasiprojective varieties induces a morphism of mixed Hodge structures.

The implementation of Step 1 requires the introduction of smooth and holomorphic *log complexes*. Step 2 requires the use of the *Poincaré residue*. Step 3 is based on a clever algebraic argument of P. Deligne known as *"le lemme de deux filtrations"*. The last step makes heavy use of *Hironaka's resolution of singularities theorem.*

2. The logarithmic complexes

For every subset $S \subset X$ we define

$$S^* := S \setminus D.$$

For every integer $m \geq 0$ and every open set $V \subset X$ we denote by $\mathcal{A}_X^m(V, \log D)$ the subspace of $\mathcal{A}_X^m(V^*)$ consisting of smooth, complex valued m-forms on V^* with the propriety that for any coordinate neighborhood $(\,U, (z_j)\,) \subset V$ such that $D \cap U = \{z_1 \cdots z_k = 0\}$ the forms $z_1 \cdots z_k \varphi$ and $z_1 \cdots z_k d\varphi$ on U^* extend to *smooth* forms on U^*. We define

$$\Omega_X^m(V, \log D) = \Omega_X^m(V^*) \cap \mathcal{A}_X^m(V, \log D).$$

The correspondences

$$V \longmapsto \mathcal{A}_X^m(V, \log D), \quad U \longmapsto \Omega_X^m(V, \log D)$$

define sheaves $\mathcal{A}_X^m(\log D)$ and $\Omega_X^m(\log D)$ on X.

From the definition we deduce

$$d\mathcal{A}_X^\bullet(V, \log D) \subset \mathcal{A}_X^\bullet(V, \log D)[1], \quad \partial\Omega_X^\bullet(\log D) \subset \Omega_X^\bullet(\log D)[1],$$

and thus we obtain two complexes of sheaves on X: $(\mathcal{A}_X^\bullet(\log D), d)$ called the *smooth logarithmic complex*, and $(\Omega_X^\bullet(\log D), \partial)$ called the *holomorphic logarithmic complex*.

Denote by j the natural inclusion $j : X^* \hookrightarrow X$. By definition, $(\Omega_X^\bullet(\log D), \partial)$ is a subcomplex of $(\mathcal{A}_X^\bullet(\log D), d)$ which is a subcomplex of $(j_*\mathcal{A}^\bullet, d)$.

Theorem 2.1. *The inclusions*

$$(\Omega_X^\bullet(\log D), \partial) \hookrightarrow (\mathcal{A}_X^\bullet(\log D), d) \tag{2.1}$$

and

$$(\Omega_X^\bullet(\log D), \partial) \hookrightarrow (j_*\mathcal{A}_X^\bullet, d) \tag{2.2}$$

are quasi-isomorphisms of complexes of sheaves. In particular, the inclusion

$$(\mathcal{A}_X^\bullet(\log D), d) \hookrightarrow (j_*\mathcal{A}_X^\bullet, d) \tag{2.3}$$

is also a quasi-isomorphism.

Proof The proof of this result will occupy the remainder of this section. We use a combination of the approaches in [6, §5] and [8, Chap. 8]. We begin by giving an alternate description of $\mathcal{A}_X^\bullet(\log D)$.

Lemma 2.2. *Suppose* $(U, (z_j))$ *is an open coordinate neighborhood on* X *such that*

$$D \cap U = \{z_1 \cdots z_k = 0\}.$$

Then any $\alpha \in \mathcal{A}_X^m(U, \log D)$ *can be written as a combination*

$$\alpha = \alpha_0 + \sum_{j=1}^k \sum_{1 \le i_1 < \cdots \le i_j \le k} \alpha_{i_1 \cdots i_j} \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_j}}{z_{i_j}},$$

where

$$\alpha_0 \in \mathcal{A}_X^m(U), \quad \alpha_{i_1 \cdots i_j} \in \mathcal{A}_X^{m-j}(U).$$

Proof For simplicity we consider only the case $k = 1$ and we write $z = z_1$. Note first that we can write

$$\alpha = \frac{1}{z}\beta, \quad \beta \in \mathcal{A}_X^m(U).$$

We write

$$\beta = \beta_0 + dz \wedge \alpha_1, \quad \alpha_1 \in \mathcal{A}_X^{m-1}(U), \quad \beta_0 \in \mathcal{A}_X^m(U), \quad \frac{\partial}{\partial z} \lrcorner \beta_0 = 0.$$

On the other hand, since $z d\alpha \in \mathcal{A}_X^{m+1}(U)$ we deduce

$$z\Big(-\frac{dz}{z^2} \wedge \beta + \frac{1}{z} d\beta\Big) = -\frac{dz}{z} \wedge \beta + d\beta \in \mathcal{A}_X^{m+1}(U).$$

Hence

$$-\frac{dz}{z} \wedge \beta_0 + d\beta_0 - dz \wedge d\alpha_1 \in \mathcal{A}_X^{m+1}(U)$$

Hence

$$\beta_0 = z\alpha_0, \ \ \alpha_0 \in \mathcal{A}^m_X(U).$$

$\square$

It is convenient to introduce some simplifying notations. If U is a coordinate neighborhood in which D is described by the monomial equation $z_1 \cdots z_k$ then for every multi-index $I = (1 \leq i_1 < \cdots < i_j \leq k)$ we set

$$|I| := j, \ \ (d\log z)^I := \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_j}}{z_{i_j}}$$

and

$$\alpha = \sum_{|I| \geq 0} \alpha_I \wedge (d\log z)^I$$

We will refer to such a representation as a *local logarithmic representation.*

To prove the quasi-isomorphism (2.1) we will show that for every $p \geq 0$ the sequence of sheaves over X

$$0 \to \Omega^{p,0}_X(\log D) \hookrightarrow \mathcal{A}^{p,0}_X(\log D) \xrightarrow{\bar\partial} \mathcal{A}^{p,1}_X(\log D) \xrightarrow{\bar\partial} \cdots \tag{2.4}$$

is exact. To achieve this we will need a $\bar\partial$-version of the Poincaré lemma. We state below a more refined version due to Nickerson, [7]. Denote by $\mathbb{D}^n_r$ the polydisk in $\mathbb{C}^n$ defined by

$$\mathbb{D}^n_r = \{(z_1, \cdots, z_n); \ \ |z_j| < r, \ \ \forall j = 1, \cdots, n\}.$$

Lemma 2.3 (Dolbeault Lemma)**.** *For every integers $n \geq 1$, $0 \leq p \leq n$, $1 \leq q \leq n$ there exists a linear operator*

$$T_0 : \mathcal{A}^{\bullet,\bullet}(\mathbb{D}^n_r) \to \mathcal{A}^{\bullet,\bullet}(\mathbb{D}^n_{r/2})[0,-1]$$

such that $\forall \alpha \in Z^{p,q}(\mathbb{D}^n_r)$ we have

$$\alpha\,|_{\mathbb{D}^n_{r/2}} = \bar\partial T_0\alpha + T_0\bar\partial\alpha.$$

$\square$

For every integers $0 \leq k \leq n$ we denote by $S_k = S_{k,n}$ the normal crossings divisor in $\mathbb{D}^n_r$ defined by the equation $z_1 \cdots z_k = 0$ if $k > 0$, $S_0 = \emptyset$, if $k = 0$. Set

$$Z^{\bullet,\bullet}_k(\mathbb{D}^n_r) := \ker\Big(\bar\partial : \mathcal{A}^{\bullet,\bullet}(\mathbb{D}^n_r, \log S_k) \to \mathcal{A}^{\bullet,\bullet}(\mathbb{D}^n_r, \log S_k)[0,1]\Big).$$

To prove the exactness of the sequence (2.4) it suffices to show the following.

Lemma 2.4. *For every $k \leq n$ and every $r > 0$ there exists a linear map*

$$T_k : Z^{\bullet,\bullet}_k(\mathbb{D}^n_r) \to \mathcal{A}^{\bullet,\bullet}(\mathbb{D}^n_{r/2}, \log S_k)[0,-1],$$

such that

$$\alpha\,|_{\mathbb{D}^n_{r/2}} = \bar\partial T_k\alpha, \ \ \forall \alpha \in Z^{\bullet,\bullet}_k(\mathbb{D}^n_r).$$

Proof We will argue by induction over k.

For $k = 0$ this follows from the Dolbeault lemma. Let us prove the inductive step. Set $z = z_{k+1}$. Suppose $\alpha \in Z^{p,q}_{k+1}(\mathbb{D}^n_r)$. Then Lemma 2.2 implies that the forms

$$\beta = z\partial_z \lrcorner\, \alpha \in \mathcal{A}^{p-1,q}(\mathbb{D}^n_r, \log S_k), \ \ \gamma = \alpha - \frac{dz}{z} \wedge \beta \in \mathcal{A}^{p,q}(\mathbb{D}^n_r, \log S_k)$$

contain no dz. By design, we have

$$\alpha = \frac{dz}{z} \wedge \beta + \gamma.$$

From the equality $\bar{\partial}\alpha = 0$ we deduce

$$\bar{\partial}\gamma - \frac{dz}{z} \wedge \bar{\partial}\beta = 0.$$

Since β and γ contain no dz we deduce

$$\bar{\partial}\gamma = 0 = \bar{\partial}\beta,$$

so that by induction, $T_k\beta$ and $T_k\gamma$ are well defined. Now set

$$T_{k+1}\alpha = -\frac{dz}{z} \wedge T_k\beta + T_k\gamma.$$

Since β and γ depend linearly on α we deduce that T_{k+1} is a linear operator. Clearly

$$dT_{k+1} = \alpha.$$

□

To prove the quasi-isomorphism (2.2) it suffices to show that any point $p \in X$ admits a fundamental system of neighborhoods $\mathcal{U}_p$ such that every neighborhood $U \in \mathcal{U}_p$ contains a neighborhood $U' \in \mathcal{U}_p$ such that the natural map

$$\Omega^p_X(U, \log D), \partial) \to \big(j_*\mathcal{A}^\bullet_X(U'), d \big)$$

is a quasi-isomorphism. To achieve this it suffices to show that for any integers $0 \leq k \leq n$ and every positive real number r the natural map

$$(\Omega^\bullet(\mathbb{D}^n_r, \log S_k), \partial) \to (\mathcal{A}^\bullet(\mathbb{D}^n_{r/2} \setminus S_k), d)$$

is a quasi-isomorphism. Let first observe that the above map induces a surjection in cohomology. Indeed

$$H^\bullet(\mathcal{A}^\bullet(\mathbb{D}^n_{r/2} \setminus S_k), d) \cong H^\bullet(\mathbb{D}^n_{r/2} \setminus S_k) \cong H^\bullet\big((\mathbb{C}^*)^k \times \mathbb{C}^{n-k} \big) \cong H^\bullet\big((\mathbb{C}^*)^k \big).$$

This shows that the cocycles $(d\log z)^I$ of $(\Omega^\bullet(\mathbb{D}^n_r, \log S_k), \partial)$ restrict to a $\mathbb{C}$-basis of the DeRham cohomology groups $H^\bullet(\mathbb{D}^n_{r/2} \setminus S_k, \mathbb{C})$. This proves the claimed surjectivity.

To prove the injectivity we argue by induction over k and show that the restriction to $\mathbb{D}^n_{r/2}$ of a cocycle of $(\Omega^\bullet(\mathbb{D}^n_r, \log S_k), \partial)$ is cohomologous to constant logarithmic form, i.e. linear combination of $(d\log z)^I$ with constant coefficients. We denote by $\sim$ the cohomology equivalence relation.

For $k = 0$ this follows from the Dolbeault Lemma. Suppose $\alpha \in \Omega^p(\mathbb{D}^n_r, \log S_{k+1})$ is such that $\partial\alpha = 0$. Split the coordinates $(z_1, \cdots, z_n)$ into two groups

$$(z_1, \cdots, z_n) = (z_{k+1}; w), \;\; w = (z_1, \cdots, z_{k-1}, z_{k+1}, \cdots, z_n).$$

As in the proof of the exactness of (2.4) we can write

$$\alpha = \frac{dz}{z} \wedge \beta + \gamma,$$

where $\beta \in \Omega^{p-1}(\mathbb{D}^n_r, \log S_k)$,$\gamma \in \Omega^p(\mathbb{D}^n_r, \log S_k)$ contain no dz. We describe the dependence of β on (z, w) as $\beta(z, w)$, we denote by $\mathbb{D}^{n-1}_r$ the corresponding polydisk in the w subspace and we set

$$\beta_0 = \beta(0, w) \in \Omega^{p-1}(\mathbb{D}^{n-1}_r, \log S_k) \subset \Omega^{p-1}(\mathbb{D}^n_r, \log S_k),$$

$$\gamma_1 = \frac{1}{z}(\beta - \beta_0) \in \Omega^{p-1}(\mathbb{D}_r^n, \log S_k).$$

Hence

$$\alpha = \frac{dz}{z} \wedge \beta_0 + dz \wedge \gamma_1 + \gamma, \quad \beta_0, \gamma, \gamma_1 \in \Omega^\bullet(\mathbb{D}_r^n, \log S_k)$$

and the coefficients of β_0 are independent of z. Since $\partial\alpha = 0$

$$\frac{dz}{z} \wedge \partial_w \beta_0 = \partial\gamma - dz \wedge \partial\gamma_1.$$

Since the right hand side does not have poles along $z = 0$ we deduce

$$\frac{dz}{z} \wedge \partial_w \beta_0 = 0 \Longleftrightarrow \partial_w \beta_0 = 0 \text{ and } \partial(\frac{dz}{z} \wedge \beta_0) = 0.$$

Thus β_0 is a cocycle of the complex $(\Omega^\bullet(\mathbb{D}_r^{n-1}, \log S_k), \partial)$ and by induction, its restriction to $\mathbb{D}_{r/2}^{n-1}$ is cohomologous to a constant logarithmic form ω_0 in the variables $d\log w$ so that

$$\frac{dz}{z} \wedge \beta_0 \sim (d\log z) \wedge \omega_0 \text{ on } \mathbb{D}_{r/2}^n.$$

On the other hand,

$$\partial(\frac{dz}{z} \wedge \beta_0) = 0 = \partial\alpha \Longrightarrow \partial(dz \wedge \gamma_1 + \gamma) = 0.$$

By induction $dz \wedge \gamma_1 + \gamma$ is also cohomologous to a constant logarithmic form.

This concludes the proof of the quasi-isomorphism (2.1) and thus the proof of Theorem 2.1.

□

3. The weight filtration and the associated spectral sequence

We say that a section $\alpha \in \mathcal{A}_X^\bullet(V, \log D)$ has *weight* $\leq \ell$, and we write this $w(\alpha) \leq \ell$ if in each coordinate neighborhood it admits a logarithmic representation such that $\alpha_I = 0$ for all $|I| > \ell$. Equivalently, this means that every point $p \in V$ admits a coordinate neighborhood $(U, (z_i))$ such that

$$S_\ell := \{z_1 \cdots z_\ell = 0\} \subset D \cap U$$

and

$$\alpha\,|_U \in \mathcal{A}^\bullet(U, \log S_\ell).$$

We denote by $W_\ell \mathcal{A}_X^\bullet(V, \log D)$ the subspace of $\mathcal{A}_X^\bullet(V, \log D)$ consisting of sections of weight $\leq \ell$. Equivalently, we have

$$W_\ell \mathcal{A}_X^\bullet(V, \log D) = \mathcal{A}_X^\ell(V, \log D) \wedge \mathcal{A}_X^\bullet(V)[-\ell].$$

The correspondence

$$V \longmapsto \mathcal{A}_X^\bullet(V, \log D)$$

defines a subsheaf $W_\ell \mathcal{A}_X^\bullet(\log D)$ of $\mathcal{A}_X^\bullet(\log D)$. The subsheaf $W_\ell \Omega_X^\bullet(\log D)$ of $\Omega_X^\bullet(\log D)$ is defined in a similar fashion. Note that

$$W_\ell \subset W_{\ell+1}, \;\; dW_\ell, \; \bar\partial W_\ell \subset W_\ell, \;\; W_\ell \wedge W_{\ell'} \subset W_{\ell+\ell'},$$

$$W_\ell \mathcal{A}_X^\ell(\log D) = \mathcal{A}_X^\ell(\log D), \;\; W_0 \mathcal{A}_X^\bullet(\log D) = \mathcal{A}_X^\bullet, \;\; W_{-1} = 0.$$

The increasing filtration $W_\ell \mathcal{A}_X^\bullet(\log D)$ is called the *weight filtration.* Note that

$$dW_\ell \mathcal{A}_X^\bullet(\log D) \subset W_\ell \mathcal{A}_X^\bullet(\log D)[1], \;\; \bar\partial W_\ell \mathcal{A}_X^{p,q}(\log D) \subset W_\ell \mathcal{A}_X^{p+1,q}(\log D)$$

Clearly the sheaves $W_\ell \mathcal{A}_X^\bullet(\log D)$ are fine.

Denote by $\{D_\lambda,\ \ \lambda\in\Lambda\}$ the irreducible components of D. Fix a total order on the finite set Λ. For every finite subset $L\subset\Lambda$ and every positive integer ℓ we set

$$D_L:=\bigcap_{\lambda\in L}D_\lambda,\ \ D_\ell=\bigcup_{|L|=\ell}D_L,\ \ \tilde{D}_\ell=\bigsqcup_{|L|=\ell}D_L.$$

Since D is a normal crossings divisor $\tilde{D}_\ell$ is a projective *manifold* of dimension $n-\ell$, $n=\dim_{\mathbb{C}}X$. We denote by a the natural *holomorphic* map $\tilde{D}_\ell\to X$ induced by the inclusions $i:D_L\to X$. Note that

$$a_*\Omega^\bullet_{\tilde{D}_\ell}=\bigoplus_{|L|=\ell}i_*\Omega^\bullet_{D_L},\ \ a_*\mathcal{A}^\bullet_{\tilde{D}_\ell}=\bigoplus_{|L|=\ell}i_*\mathcal{A}^\bullet_{D_L}.$$

Proposition 3.1. *There exist natural quasi-isomorphism of complexes of sheaves*

$$\big(\,\mathbf{Gr}^W_\ell\,\Omega^\bullet_X(\log D),\partial\,\big)\to a_*(\Omega^\bullet_{\tilde{D}_\ell}[-\ell],\partial),\tag{3.1}$$

$$\big(\,\mathbf{Gr}^W_\ell\,\mathcal{A}^{p,\bullet}_X(\log D),\bar\partial\,\big)\to a_*(\mathcal{A}^{p-\ell,\bullet}_{\tilde{D}_\ell},\bar\partial).\tag{3.2}$$

$$\big(\,\mathbf{Gr}^W_\ell\,\mathcal{A}^\bullet_X(\log D),d\,\big)\to a_*(\mathcal{A}^\bullet_{\tilde{D}_\ell}[-\ell],d),\tag{3.3}$$

Proof The above isomorphisms are induced by the *Poincaré residue*

$$\mathrm{Res}^\ell:W_\ell\mathcal{A}^{p,q}_X(\log D)_x\to a_*\mathcal{A}^{p-\ell,\bullet}_{\tilde{D}_\ell}$$

which is defined as follows.

Let $x\in X$. If $x\in X\setminus D_\ell$ then

$$\mathrm{Res}^\ell:W_\ell\mathcal{A}^{p,q}_X(\log D)_x\to(a_*\mathcal{A}^{p-\ell,\bullet}_{\tilde{D}_\ell})_x.$$

is trivial. Suppose $x\in D_\ell$ and $\alpha\in\mathcal{A}^{p,q}_X(\log D)_x$. Then we can find a coordinate neighborhood U of x and $k\geq\ell$ with the following properties.

• The triplet $(U,D\cap U,x)$ is biholomorphic to the triplet $(\mathbb{D}^n_r,S_k,0)$, where we recall that

$$S_k=\{z_1\cdots z_k=0\}.$$

Set $[k]=\{1,\cdots,k\}$, $\zeta_i=z_i\partial_{z_i}$, $\forall i\in[k]$.

• For $i\in[k]$ we denote by D_{λ_i} the component of D such that

$$D_{\lambda_i}\cap U=\{z_i=0\}.$$

Then

$$i<j\Longrightarrow\lambda_i<\lambda_j.$$

• The germ α can be represented by a form

$$\alpha=\sum_{L\subset[k],\,|L|\leq\ell}\alpha_L\wedge(d\log z)^L,\ \ \alpha_L\in\mathcal{A}^{p-|L|,q}_X(U),$$

where the coefficients α_L are uniquely determined by the conditions $\zeta_i\lrcorner\,\alpha_L=0$, $\forall i\in L$.

For $|L|=\ell$ we define

$$\mathrm{Res}^\ell_L\,\alpha:=(2\pi\mathbf{i})^\ell\alpha_L\,|_{D_L}\in a_*\mathcal{A}^{p-\ell,q}_{D_L}(U).$$

To see that this definition is independent of the defining equations $\{z_i = 0\}$ suppose we replace one of them $z = 0$ by an equivalent one $w = 0$, where $w = uz$ and u is a holomorphic function, nowhere vanishing on U, then

$$\frac{dw}{w} = \frac{dz}{z} + \frac{du}{u}$$

and we see that this affects only the terms α_L with $|L| < \ell$. This shows the map Res^ℓ_L is independent of the various choices. Moreover

$$\mathrm{Res}^\ell_L W_{\ell-1}\mathcal{A}^{p;q}_X(\log D) = 0,$$

so that we do have a well defined map

$$\mathrm{Res}^\ell = \prod_{|L|=\ell} \mathrm{Res}^\ell_L : \mathbf{Gr}^W_\ell \mathcal{A}^{p,q}_X(\log D)_x \to a_*\mathcal{A}^{p-\ell,\bullet}_{\tilde{D}_\ell}.$$

This map is surjective, yet is not injective. For example, if $X = \mathbb{C}$, $D = \{0\}$ then $\mathrm{Res}\,\frac{\bar{z}}{z}dz = 0$ yet $\frac{\bar{z}}{z}dz \notin W_0$. However, it induces an *isomorphism* of complexes of sheaves

$$(\mathbf{Gr}_\ell\,\Omega^\bullet_X, \partial) \to a_*(\Omega^\bullet_{\tilde{D}_\ell}[-\ell], \partial).$$

The proof of Lemma 2.4 shows that the inclusion

$$W_\ell\Omega^p_X(\log D) \hookrightarrow (W_\ell\mathcal{A}^{p,\bullet}, \bar{\partial})$$

is a resolution of the sheaf $W_\ell\Omega^p_X(\log D)$. We deduce that

$$\mathbf{Gr}_\ell\,\Omega^p_X(\log D) \hookrightarrow \mathbf{Gr}^W_\ell\,\mathcal{A}^{p,\bullet}, \bar{\partial})$$

is a quasi-isomorphism. In particular we obtain a commutative diagram of complexes of sheaves

$$\begin{tikzcd}
\mathbf{Gr}_\ell\,\Omega^p_X(\log D) \arrow[r, "\mathrm{Res}^\ell_\Omega"] \arrow[d, hook, "f_l"] & a_*\Omega^{p-\ell}_{\tilde{D}_\ell} \arrow[d, hook, "f_r"] \\
(\mathbf{Gr}^W_\ell\,\mathcal{A}^{p,\bullet}, \bar{\partial}) \arrow[r, "\mathrm{Res}^\ell_{\mathcal{A}}"] & a_*(\mathcal{A}^{p-\ell,\bullet}_{\tilde{D}_\ell}, \bar{\partial})
\end{tikzcd}$$

in which the vertical arrows f_l and f_r are quasi-isomorphisms[1] and the top horizontal arrow is an isomorphism. Hence Res^ℓ_A is a quasi-isomorphism which proves (3.2).

The complex $(W_\ell\mathcal{A}^\bullet_X(\log Y), d)$ is the total complex associated to the double complex $(W^\ell\mathcal{A}^{\bullet;\bullet}_X, \partial, \bar{\partial})$ in which the columns are exact. Thus the inclusion

$$(W_\ell\Omega^\bullet_X(\log Y), \partial) \hookrightarrow ((W_\ell\mathcal{A}^\bullet_X(\log Y), d)$$

is a quasi-isomorphism.

The proof of (3.3) now follows using la similar commutative diagram relating via the Poincaré residue the Dolbeault-to-DeRham spectral sequences $(\mathbf{Gr}^W_\ell\,\mathcal{A}^{\bullet;\bullet}_X(\log Y), \partial + \bar{\partial})$ to the Dolbeault-to-DeRham spectral sequence $(\mathcal{A}^{\bullet;\bullet}_{\tilde{D}_\ell}[-\ell], \partial + \bar{\partial})$. The columns of both these double complexes are acyclic.

□

[1] f_r is a quasi-isomorphism since a_* is an exact functor.

The complex of sheaves $\mathcal{A}^\bullet_X(\log D), d)$ is an *fine resolution* of $j_*\underline{\mathbb{C}}$ so that

$$H^\bullet(\mathcal{A}^\bullet_X(X, \log D), d) \cong H^\bullet(X, j_*\underline{C}) \cong H^\bullet(X^*).$$

The decreasing filtration $W^\ell_- = W_{-\ell}$ on $\mathcal{A}^\bullet_X(\log D)$ leads to a spectral sequence

$$E_r^{\bullet,\bullet} = E_r^{\bullet,\bullet}(W_-) \Longrightarrow H^\bullet(X^*).$$

In particular, we obtain an increasing filtration W_ℓ on $H^\bullet(X^*)$ such that

$$W_k = 0, \;\; \forall k < 0$$

and

$$W_0 H^\bullet(X^*) = j^* H^\bullet(X). \tag{3.4}$$

The last equality follows from the definition of $W_0H^\bullet(X^*)$ as the image of the homology of

$$H^\bullet\big(\Gamma(X, W_0\mathcal{A}^\bullet_X(\log D), d\big) = H^\bullet\big(\Gamma(X, \mathcal{A}^\bullet_X), d\big) \cong H^\bullet(X)$$

in

$$H^\bullet(H^\bullet\big(\Gamma(X, \mathcal{A}^\bullet_X(\log D), d\big) \cong H^\bullet(X^*).$$

Since the sheaves $W_\ell\mathcal{A}^p_X(\log D)$ are fine we deduce that the sheaves $\mathbf{Gr}^W_\ell\,\mathcal{A}^p_X(\log D)$ are acyclic and the E_0 term of the above spectral sequence is given by

$$E_0^{-\ell,m} = \mathbf{Gr}^W_\ell\,\Gamma\big(\,X, \mathcal{A}^{m-\ell}_X(\log D)\,\big) \cong \Gamma\big(\,X, \mathbf{Gr}^W_\ell\,\mathcal{A}^{m-\ell}_X(\log D)\,\big), \;\; d_0 = d.$$

Note that $E_0^{k,m} = 0$ for $k > 0$ or $m < 0$. From (3.3) we deduce that the complex of acyclic sheaves $\mathbf{Gr}^W_\ell\,\mathcal{A}^\bullet_X(\log D)$ is quasi-isomorphic to $a_*\mathcal{A}^\bullet_{\tilde{D}_\ell}[-\ell]$. Since the functor a_* is exact and $\mathcal{A}^\bullet_{\tilde{D}_\ell}$ is a resolution of $\underline{\mathbb{C}}_{\tilde{D}_\ell}$ we deduce

$$\begin{aligned} E_1^{-\ell,m} &= H^{m-\ell}\big(\,X, \mathbf{Gr}^W_\ell\,\mathcal{A}^\bullet_X(\log D)\,\big) \cong \mathbb{H}^{m-\ell}\big(\,X, \mathbf{Gr}^W_\ell\,\mathcal{A}^\bullet_X(\log D)\,\big) \\ &\cong \mathbb{H}^{m-l}(\tilde{D}_\ell, \mathcal{A}^\bullet_{\tilde{D}_\ell}[-\ell]) \cong H^{m-2\ell}(\tilde{D}_\ell, \mathbb{C}). \end{aligned}$$

For example, if X is a smooth algebraic surface, and D is a normal crossing divisor then the E_1-page of this spectral sequence has the look

$$\begin{array}{ccccccc}
H^0(\tilde{D}_2) & \xrightarrow{d_1} & H^2(\tilde{D}_1) & \xrightarrow{d_1} & H^4(X) & \xrightarrow{d_1} & 0 \\
0 & \xrightarrow{d_1} & H^1(\tilde{D}_1) & \xrightarrow{d_1} & H^3(X) & \xrightarrow{d_1} & 0 \\
0 & \xrightarrow{d_1} & H^0(\tilde{D}_1) & \xrightarrow{d_1} & H^2(X) & \xrightarrow{d_1} & 0 \\
0 & & 0 & \xrightarrow{d_1} & H^1(X) & \xrightarrow{d_1} & 0 \\
0 & & 0 & \xrightarrow{d_1} & H^0(X) & \xrightarrow{d_1} & 0 \\
 & & & & & & \\
2 \geq W & & 1 \geq W & & 0 \geq W & & -1 \geq W
\end{array}$$

The complex $(\mathcal{A}^\bullet_X(\log D), d)$ is equipped with a natural decreasing filtration $F = F_X$

$$F^p\mathcal{A}^m_X(\log D) = \bigoplus_{k\geq p} \mathcal{A}^{k,m-k}_X(\log D),$$

and the differential d is clearly compatible with this filtration. We will refer to this as the Hodge filtration. Note that we have an isomorphism of complexes of sheaves

$$(\mathbf{Gr}^p_F \mathcal{A}^\bullet_X(\log D), \mathbf{Gr}^p_F d) = (\mathcal{A}^{p,\bullet}_X(\log D), \bar{\partial}).$$

Using (3.2) we deduce that they are complexes of acyclic sheaves. We conclude that the sheaves

$$\mathbf{Gr}_F \mathbf{Gr}^W \cong \mathbf{Gr}^W \mathbf{Gr}_F$$

are acyclic. This implies

$$F\,\mathbf{Gr}^W \Gamma(X, \mathcal{A}^\bullet_X(\log D)) = \Gamma(X, F\,\mathbf{Gr}^W \mathcal{A}^\bullet_X(\log D)$$

and

$$\mathbf{Gr}_F \mathbf{Gr}^W \Gamma(X, \mathcal{A}^\bullet_X(\log D)) = \Gamma(X, \mathbf{Gr}_F \mathbf{Gr}^W \mathcal{A}^\bullet_X(\log D).$$

We obtain a filtration a canonical F_X on $E_0^{\bullet,\bullet}$. The sheaves $\mathcal{A}^k_{D_\ell}$ are also equipped with decreasing filtrations $F = F_D$ defined by

$$F^p\mathcal{A}^m_{\tilde{D}_\ell} = \bigoplus_{k\geq p} \mathcal{A}^{k,m-k}_{\tilde{D}_\ell}$$

and the Poincaré residue isomorphism

$$R_\ell : E_0^{-\ell,m} = \Gamma(X, \mathbf{Gr}^W_\ell \mathcal{A}^{m-\ell}_X(\log D)) \to \Gamma(\tilde{D}_\ell, \mathcal{A}^{m-2\ell}_{\tilde{D}_\ell})$$

satisfies

$$R_\ell F^\bullet_X \subset F_D[-\ell]^\bullet.$$

The filtration F_D induces on $H^{m-2\ell}(\tilde{D}_\ell, \mathbb{C})$ a pure Hodge structure of weight $m - 2\ell$. Using the Poincaré residue isomorphism

$$R_\ell : E_1^{-\ell,m} \to H^{m-2\ell}(\tilde{D}_\ell)$$

we obtain a pure Hodge structure of weight m on $E_1^{-\ell,m}$. We want to prove that d_1 is a morphism of pure Hodge structures of bidegree $(0, 0)$ and so that we have a canonical Hodge structure on E_2. To complete the argument we will show that $d_r = 0$ for all $r \geq 2$.

To show that d_1 is a morphism of Hodge structures we can proceed in two different ways. The first approach is more geometric and is based on an explicit geometric description of d_1 in terms of *Gysin maps*. This description is contained in the Appendix and it is particularly useful in concrete computations. The second approach, due to P. Deligne is more algebraic in nature and is based on a careful analysis of filtered spectral sequences. We follow this method as it will also lead to a very elegant proof of the degeneration of the spectral sequence. We need a brief algebraic interlude.

4. Spectral sequences

We want to first recall a few elementary constructions involving filtered modules.

Suppose R is a commutative ring with 1 and $F^\bullet$ is a decreasing filtration on the R-module A. If B is a submodule of A then F induces a canonical filtration F_B on B such that the inclusion $B \hookrightarrow A$ is strict with respect to the corresponding filtrations. More explicitly,

$$F_B = F \cap B.$$

The quotient A/B has a canonical filtration uniquely determined by the requirement that the natural projection $A \to A/B$ is strict. More precisely

$$F^p(A/B) = \frac{F^p(A) + B}{B}.$$

If C is a submodule of B then the quotient B/C has a canonical filtration uniquely determined by the requirement that both natural morphisms

$$B/C \hookrightarrow A/C, \;\; B \twoheadrightarrow B/C$$

are strict. More precisely

$$F^p(B/C) = \frac{F^p \cap B + C}{C} \cong \frac{F^p \cap B}{F^p \cap C}.$$

We have a canonical morphism $\phi : A/C \to A/B$ completing the commutative diagram

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[dr] & A/C \arrow[d, dashed, two heads, "\phi"] \\ & A/B \end{tikzcd} .$$

Note that $\ker\phi = B/C$ and thus we have a canonical isomorphism

$$\sigma : \frac{A/C}{\ker\phi} = \frac{A/C}{B/C} \to \mathbf{Im}\,\phi = A/B$$

This isomorphism is *strict* with respect to the canonical filtrations induced by F on both sides.

If X, Y are two submodules of A then we have a natural isomorphism

$$\psi : \frac{X}{X \cap Y} \to \frac{X+Y}{Y}, \tag{4.1}$$

which completes the commutative diagram

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[dr] & \frac{X}{X\cap Y} \arrow[d, dashed, two heads, "\phi"] \\ & \frac{X+Y}{Y} \end{tikzcd}$$

ψ is compatible with the filtrations induced by F, *but not necessarily strictly.* More precisely, ψ is strict if and only if the following condition is satisfied

$$F^p \cap (X+Y) \subset F^p \cap X + F^p \cap Y, \;\; \forall p. \tag{4.2}$$

Suppose (K,d) is a cochain complex of R-module. For simplicity, we do not include grading of the complex in our notations. Then any decreasing filtration W on $K^\bullet$ compatible with the derivation d determines a spectral sequence

$$E_r^p = E_r^p(K,W) \Longrightarrow \mathbf{Gr}_W^p H(K,d)$$

defined by

$$E_r^p = \frac{Z_r^p}{B_r^p},$$

where[2]

$$Z_r^p = W^p \cap d^{-1}W^{p+r},$$

$$B_r^p = (dW^{p-r+1}\cap W^p) + (W^{p+1}\cap d^{-1}W^{p+r}) = dZ_{r-1}^{p-(r-1)} + Z_{r-1}^{p+1}.$$

Using the inclusions

$$dB_r^p = dZ_{r-1}^{p+1} \subset B_r^p,\ \ dZ_r^p \subset Z_r^{p+r}$$

we obtain a cochain complex $(E_r(W), d_r)$, where $d_r : E_r^p \to E_r^{p+r}$ is defined by the composition

$$\frac{Z_r^p}{B_r^p} \xrightarrow{d} \frac{dZ_r^p + B_r^p}{B_r^p} \hookrightarrow \frac{Z_r^{p+r}}{B_r^p} \twoheadrightarrow \frac{Z_r^{p+r}}{B_r^{p+r}}$$

We have a natural isomorphism $\alpha_r : E_{r+1} \to H(E_r, d_r)$ defined as the composition

$$\frac{Z_{r+1}^p}{B_{r+1}^p} = \frac{Z_{r+1}^p}{Z_{r+1}^p \cap (dZ_r^{p-r} + Z_{r-1}^{p+1})} \xrightarrow{\psi} \frac{Z_{r+1}^p + Z_{r-1}^{p+1}}{dZ_r^{p-r} + Z_{r-1}^{p+1}}$$

$$\xrightarrow{\sigma^{-1}} \frac{(Z_{r+1}^p + Z_{r-1}^{p+1})/B_r^p}{(dZ_r^{p-r} + Z_{r-1}^{p+1})/B_r^p} = \frac{Z(E_r,d_r)}{B(E_r,d_r)}.$$

The above description identifies E_r with a quotient of submodule of K. We can give a dual description of E_r as the submodule of a quotient of K. More precisely we set

$$\tilde{B}_r^p := dW^{p-(r-1)} + W^{p+1};\ \tilde{Z}_r^p = Z_r^p + \tilde{B}_r^p = W^p \cap d^{-1}W^{p+r} + dW^{p-(r-1)} + W^{p+1}.$$

Observe that

$$Z_r \cap \tilde{B}_r = B_r$$

and thus we have an isomorphism

$$\eta_r^p : E_r^p(W) = \frac{Z_r^p}{Z_r^p \cap \tilde{B}_r^p} \xrightarrow{\psi} \frac{Z_r^p + \tilde{B}_r^p}{\tilde{B}_r^p} = \frac{\tilde{Z}_r^p}{\tilde{B}_r^p} = \mathbf{Im}\big(\, Z_r^p \to K/\tilde{B}_r^p\,\big) =: \tilde{E}_r^p(W).$$

On $\tilde{E}_r$ we have a differential defined by the composition

$$\tilde{d}_r : \frac{\tilde{Z}_r^p}{\tilde{B}_r^p} \xrightarrow{d} \frac{d\tilde{Z}_r^p + \tilde{B}_r^p}{\tilde{B}_r^p} \hookrightarrow \frac{\tilde{Z}_r^{p+r}}{\tilde{B}_r^p} \twoheadrightarrow \frac{\tilde{Z}_r^{p+r}}{\tilde{B}_r^{p+r}}$$

The isomorphism η_r^p is an isomorphism of complexes $(E_r^p, d_r) \to (\tilde{E}_r^p, \tilde{d}_r)$.

We also have a natural identification $\tilde{\alpha}_r : H(\tilde{E}_r, \tilde{d}_r) \to \tilde{E}_{r+1}$ defined as the composition

$$\frac{Z^p(\tilde{E}_r, d_r)}{B^p(\tilde{E}_r, \tilde{d}_r)} = \frac{(W^p \cap d^{-1}W^{p+r+1} + \tilde{B}_r^p)/\tilde{B}_r^p}{(W^p \cap dW^{p-r} + \tilde{B}_r^p)/\tilde{B}_r^p} \xrightarrow{\sigma} \frac{\overbrace{(W^p \cap d^{-1}W^{p+r+1} + \tilde{B}_r^p)}^{:=\hat{Z}_{r+1}^p}}{\underbrace{(W^p \cap dW^{p-r} + \tilde{B}_r^p)}_{:=\hat{B}_{r+1}^p}}$$

[2]Our definition of B_r^p differs from the conventional one $B_r^p = (dW^{p-r+1} \cap W^p)$.

$$= \frac{\hat{Z}^p_{r+1}}{\hat{Z}^p_{r+1} \cap \tilde{B}^p_{r+1}} \xrightarrow{\psi} \frac{\tilde{Z}^p_{r+1}}{\tilde{B}^p_{r+1}} = \tilde{E}^p_{r+1}.$$

We obtain in this fashion a commutative diagram

$$\begin{tikzcd} E_{r+1} \arrow[r, "\eta_{r+1}"] \arrow[d, "\alpha_r"'] & \tilde{E}_{r+1} \\ H(E_r, d_r) \arrow[r, "H(\eta_r)"'] & H(\tilde{E}_r, \tilde{d}_r) \arrow[u, "\tilde{\alpha}_r"'] \end{tikzcd} . \tag{4.3}$$

Suppose F is second decreasing filtration on K compatible with d. It induces a canonical filtration $F = F(r)$ on E^p_r, called the *first direct filtration*, by setting

$$F^k E^p_r = \frac{F^k \cap Z^p_r + B^p_r}{B^p_r}.$$

On the other hand, it induces a filtration $\tilde{F} = \tilde{F}(r)$ on $\tilde{E}^p_r$, called the *second direct filtration* by

$$\tilde{F}^k \tilde{E}^p_r = \frac{F^k \cap \tilde{Z}^p_r + \tilde{B}^p_r}{\tilde{B}^p_r}$$

Observe that

$$\eta^p_r(F(r)) \subset \tilde{F}(r),$$

i.e. η^p_r is compatible with the filtrations, not necessarily strictly.

For $r = 0$ we have

$$Z^p_0 = W^p = \tilde{Z}^p_0, \;\; B^p_0 = W^{p+1} = \tilde{B}^p_0$$

so that we have

$$F(0) = \tilde{F}(0).$$

For $r = 1$ we have

$$Z^p_1 = W^p \cap d^{-1} W^{p+1}, \;\; \tilde{B}^p_1 = dW^p + W^{p+1} = B^p_1$$

and we conclude again that

$$F(1) = \tilde{F}(1).$$

There is a third filtration $\mathbb{F}$ on E^p_r, called the *recurrent filtration* defined inductively by the requirements.

- $\mathbb{F}(r) = F(r), \;\; r = 0, 1.$
- $\mathbb{F}(r)$ induces a canonical filtration on $H(E_r, d_r)$ and using the isomorphism

$$\alpha_r : E_{r+1} \to H(E_r, d_r)$$

we obtain a filtration $\mathbb{F}(r+1)$ on E_{r+1}. and we have

Using the isomorphism η_r and the commutative diagram (4.3) we obtain three filtrations $F, \hat{F}, \tilde{F}$ on E_r satisfying

$$F \subseteq \mathbb{F} \subseteq \tilde{F} \;\; \text{on} \;\; E_r,$$

with equalities for $r = 0, 1$. The differential d_r is compatible with F and $\tilde{F}$ but may not be compatible with $\mathbb{F}$. We have the following result.

Theorem 4.1 (Le lemme de deux filtrations)**.** *Suppose that for every $0 \leq k \leq r$ the differential d_k is* strictly *compatible with the recurrent filtration $\mathbb{F}$. Then d_{r+1} is compatible with $\mathbb{F}$ and for every $0 \leq k \leq r+1$ we have*

$$F = \mathbb{F} = \tilde{F} \text{ on } E_k.$$

For a very elegant proof of this result we refer to [3, §1.3] or [4, §7.2].

5. The degeneration of the spectral sequence $E_r(\mathcal{A}^\bullet(X, \log D), W)$

We now apply the previous abstract considerations to the special case of the complex of $\mathbb{C}$-vector spaces

$$(K^\bullet, d) = \mathcal{A}^\bullet(X, \log D), d)$$

Equipped with the decreasing filtrations

$$W^{-\ell}K = W_\ell \mathcal{A}^\bullet(X, \log D), \quad F^p K = \bigoplus_{k \geq p} \mathcal{A}^{p,q}(X, \log D).$$

As explained in the previous section, on $E_1 = E_1(K, W)$ the three filtrations F, $\mathbb{F}$ and $\tilde{F}$ and $d_1 : E^{-\ell,m} \to E_1^{-\ell+1,m}$ is compatible with F. Since F induces on $E_1^{\bullet,m}$ a pure Hodge structure of weight m we deduce that d_1 is *strict* with respect to F, and in particular $E_2^{-\ell,m}$ is equipped with a canonical pure Hodge structure of weight m. The Hodge filtration is described by the recurrent filtration $\mathbb{F}$.

We now prove by induction over $r \geq 2$ that $d_r = 0$. Since d_0 and d_1 are strictly compatible with $F = \mathbb{F}$ we deduce from Theorem 4.1 that d_2 is compatible with $\hat{F}$. In particular we deduce that

$$d_2 : E_2^{-\ell,m} \to E_2^{-\ell+2,m-1}$$

is a morphism of pure Hodge structures. Since the weight of the codomain $E_2^{-\ell+2,m-1}$ of d_2 is strictly smaller than the weight of the domain $E_2^{-\ell,m}$ we deduce $d_2 = 0$. Indeed we have

$$d_2(\mathbb{F}^p \cap \bar{\mathbb{F}}^q \cap E_2^{\bullet,m}) \subset \mathbb{F}^p \cap \bar{\mathbb{F}}^q \cap E_2^{\bullet,m-1}$$

and we have

$$E_2^{\bullet,m} = \bigoplus_{p+q=m} \mathbb{F}^p \cap \bar{\mathbb{F}}^q \cap E_2^{\bullet,m}, \quad \mathbb{F}^p \cap \bar{\mathbb{F}}^q \cap E_2^{\bullet,m-1} = 0, \quad \forall p+q > m-1.$$

Assume we have proved the vanishing of d_k, $2 \leq k \leq r$ and we prove it for $k = r+1$. The vanishing implies that d_k is strictly compatible with $\mathbb{F}$ for $0 \leq k \leq r$ and that $E_{r+1}^{\bullet,m}$ has a pure Hodge structure of weight m with Hodge filtration $\mathbb{F}$. By Theorem 4.1 the differential d_{r+1} is compatible with $\mathbb{F}$ and thus induces a morphism of pure Hodge structures

$$d_{r+1} : E_{r+1}^{\bullet,m} \to E_{r+1}^{\bullet,m-r}$$

and we conclude as before that $d_{r+1} = 0$ because the weight of $E_{r+1}^{\bullet,m-r}$ is strictly smaller than the weight of $E_{r+1}^{\bullet,m}$.

We deduce that

$$\mathbf{Gr}_\ell^W H^m(X^*, \mathbb{C}) = \mathbf{Gr}_{W_-}^{-\ell} H^m(X^*, \mathbb{C}) \cong E_2^{-\ell,m+\ell}$$

is equipped with a pure Hodge structure of weight $m + \ell$. Observe now that

$$\mathbf{Gr}_\ell^W = \mathbf{Gr}_{m+\ell}^{W[-m]}$$

so that the decreasing filtration $\mathbb{F}$ and the increasing filtration $W[-m]$ define a mixed Hodge structure on $H^m(X^*, \mathbb{C})$. From the equality (3.4) we deduce

$$W_m H^m(X^*) = j^* H^m(X). \tag{5.1}$$

6. Functoriality

It is time to stop and reflect on the things we have done so far. We started with a smooth quasi-projective variety X^*, We chose a smooth compactification X of X^* such that the locus at infinity $X \setminus X^*$ is a normal crossings divisor. Then, using the embedding $j : X^* \hookrightarrow X$ we produced a mixed Hodge structure on $H^\bullet(X^*)$. We have the following result.

Theorem 6.1 (Functoriality of mixed Hodge structures)**.** *(a) The above mixed Hodge structure on $H^\bullet(X^*)$ is independent of the compactification X.*
(b) If $f : X \to Y$ is a holomorphic map between two smooth quasi-projective manifolds then the induced morphism

$$f^* : H^\bullet(Y) \to H^\bullet(X)$$

is a morphism of mixed Hodge structures. □

The proof of this theorem makes heavy use of Hironaka's resolution theorem. For details we refer to [3, §3.2].

7. Examples

We want to discuss a few simple examples to get a feeling of the complexity of the objects involved. We begin by introducing some notations.

If V is a complex vector space equipped with a mixed Hodge structure (F, W) we set

$$h^{p,q}(V) := \dim_{\mathbb{C}} (\mathbf{Gr}^W_{p+q})^{p,q},$$

and we define the Poincaré-Hodge polynomial $P_V(z, \bar{z})$ to be

$$P_V(z, \bar{z}) = \sum_{p,q} h^{p,q}(V) z^p \bar{z}^q.$$

For a smooth quasi-projective manifold X we set

$$\mathcal{P}_X = \mathcal{P}_X(t, z, \bar{z}) = \sum_{k \geq 0} t^k P_{H^k(X)}(z, \bar{z}), \;\; \mathcal{E}^{p,q}(X; z, \bar{z}) = \sum_{k \geq 0} (-1)^k h^{p,q}(H^k(X)\,) \in \mathbb{Z}[z, \bar{z}],$$

$$\mathcal{E}(X; z, \bar{z}) := \mathcal{P}_X(t, z, \bar{z})\,|_{t=-1} = \sum_{p,q} \mathcal{E}^{p,q}(X; z, \bar{z}) \in \mathbb{Z}[z, \bar{z}].$$

We begin with the simplest situation when the divisor D is smooth and irreducible.

Example 7.1. Suppose $Y \overset{i}{\hookrightarrow} X$ is a smooth hypersurface in the projective manifold X. We would like to understand the mixed Hodge structure on $X^* = X \setminus Y$. Denote by j the natural inclusion $X^* \hookrightarrow X$.

Observe that in this case we have $\tilde{D}_\ell = \emptyset$ for $\ell > 1$. In particular $E_1^{-\ell, m}$ is zero for $\ell \neq 0, 1$. We have

$$E_1^{0,m} = H^m(X), \;\; E_1^{-1,m} = H^{m-2}(Y).$$

The E_1-term has the shape

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
 & & & \\
0 & H^2(Y) \xrightarrow{i_!} & H^4(X) \to & 0 \\
 & & & \\
0 & H^1(Y) \xrightarrow{i_!} & H^3(X) \to & 0 \\
 & & & \\
0 & H^0(Y) \xrightarrow{i_!} & H^2(X) \to & 0 \\
 & & & \\
0 & 0 & H^1(X) \to & 0 \\
 & & & \\
0 & 0 & H^0(X) \to & 0
\end{array} \tag{7.1}$$

Denote by $H^\bullet(Y)_{van}$ the kernel of the Gysin map $i_!$. We deduce

$$E_2^{-1,m} = H^{m-2}(Y)_{van}.$$

Using the long exact sequence

$$\cdots \to H^\bullet(Y)[-2] \xrightarrow{i_!} H^\bullet(X) \xrightarrow{j} H^\bullet(X^*) \xrightarrow{\text{Res}} H^\bullet(Y)[-1] \to \cdots$$

we deduce

$$E_2^{0,m} = H^m(X)/\,\mathbf{Im}\, i_! = H^m(X)/\ker j^* \cong j^* H^m(X)$$

Thus we can identify $E_2^{0,m}$ with the subspace of $H^m(X^*)$ consisting of cohomology classes which extend over X. If we denote by W the weight filtration on $H^m(X^*)$ we deduce

$$W_k = 0,\ \ \forall k < m,\ \ W_p = H^m(X^*),\ \ \forall p > m+1$$

$$W_m H^m(X^*) = j^* H^m(X),\ \ \mathbf{Gr}^W_{m+1} H^m(X^*) \cong H^{m-1}(Y)_{van}.$$

In special cases we can say more about the differential d_1. Assume $i : Y \hookrightarrow X$ is a smooth *very ample* divisor. We then have the following result. For a proof we refer to [9, §2.3]

Theorem 7.2 (Hard Lefschetz Theorem)**.** *Set $m = \frac{1}{2}\dim_{\mathbb{R}} Y$ and denote by $[Y]$ the line bundle associated to Y.*

(a) For every $k \neq m$ we have

$$H^k_{van}(Y) = 0.$$

Moreover

$$H^m(Y)_{van} \subset H^m(Y)_{prim} \Longleftrightarrow \forall u \in H^m(Y)_{van},\ \ u \cup c_1([Y]) \mid_Y = 0.$$

(b) We have direct sum decompositions

$$H^m(Y) = H^m(Y)_{van} \oplus i^* H^m(X), \tag{7.2a}$$

$$H^m(Y)_{prim} = H^m(Y)_{van} \oplus i^* H^m(X)_{prim}. \tag{7.2b}$$

The summands in each decomposition are orthogonal with respect to the intersection form on $H^m(Y)$. □

This theorem is useful when for example Y is a smooth hypersurface in some $\mathbb{P}^n$. In this case $m = n-1$, $H^{n-1}(\mathbb{P}^n)_{prim} = 0$ and we deduce from (7.2b)

$$H^{n-1}(Y)_{prim} = H^{n-1}(Y)_{van}.$$

Using (7.2a) we conclude

$$\dim H^{n-1}(Y)_{van} = \dim H^{n-1}(Y) - \begin{cases} 1 & \text{if} \quad \dim_{\mathbb{C}} Y \ \text{ is even} \\ 0 & \text{if} \quad \dim_{\mathbb{C}} Y \ \text{ is odd} \end{cases}.$$

In particular, if Y is a hyperplane in $\mathbb{P}^n$ then $H^k(\mathbb{P}^n \setminus H)$ is equipped with a pure Hodge structure of weight k. Let us look at a few other special examples.

Suppose for example that $Y = Y_d$ is a degree d curve in $\mathbb{P}^2$. Set $X^* = \mathbb{P}^2 \setminus Y_d$. Then the Poincaré-Hodge polynomial of Y is

$$P_Y(z, \bar{z}) = 1 + \frac{(d-1)(d-2)}{2} t(z+\bar{z}) + t^2(z\bar{z}).$$

Then

$$H^0(\mathbb{P}^2 \setminus Y) \cong \mathbb{C}$$
$$W_1 H^1(X^*) = 0, \ \ W_2 H^1(X^*) = H^0(Y)_{van} = 0$$
$$W_2 H^2(X^*) = 0, \ \ W_3 H^2(X^*) = H^1(Y)_{van} = H^1(Y)$$

and we deduce

$$\mathcal{P}_{\mathbb{P}^2\setminus Y_d} = 1 + \frac{(d-1)(d-2)}{2} t^2(z\bar{z})(z+\bar{z}) = 1 + tz\bar{z}(\mathcal{P}_Y - 1 - t^2 z\bar{z}).$$

Using the equality

$$\mathcal{P}_{\mathbb{P}^k} = 1 + t^2(z\bar{z}) + \cdots + t^{2k}(z\bar{z})^k$$

we deduce

$$\mathcal{P}_{\mathbb{P}^2\setminus Y_d} = 1 + \frac{(d-1)(d-2)}{2} t^2(z\bar{z})(z+\bar{z}) = 1 + tz\bar{z}(\mathcal{P}_Y - \mathcal{P}_{\mathbb{P}^1}). \tag{7.3}$$

Suppose now that $Y = Y_d$ is a degree d hypersurface in $\mathbb{P}^2$. Set $X^* = \mathbb{P}^3 \setminus Y$. Then

$$h^{2,0}(Y) = h^{0,2}(Y) = \binom{d-1}{3}, \ \ h^{1,1}(Y) = \frac{4d^2 - 12d^2 + 14d}{6}.$$

We have

$$h^{0,0}(X^*) = 1, \ \ H^1(X^*) = H^2(X^*) = 0$$
$$\mathbf{Gr}_3\, H^3(X^*) = 0, \ \mathbf{Gr}_4\, H^3(\mathbb{P}^3 \setminus Y) = H^2(Y)_{van}$$
$$h^{3,1}(H^3(X^*) = h^{2,0}(Y) = h^{1,3}(\, H^3(X^*)\,), \ \ h^{2,2}(H^3(X^*)) = h^{1,1}(Y) - 1$$

since the nonprimitive class in $H^2(Y)$ has type $(1,1)$ and it is the restriction of the hyperplane class which is a $(1,1)$-class. We deduce

$$\mathcal{P}_{\mathbb{P}^3\setminus Y} = 1 + tz\bar{z}(\mathcal{P}_Y - 1 - tz\bar{z} - t^4 z\bar{z}) = 1 + tz\bar{z}(\mathcal{P}_Y - \mathcal{P}_{\mathbb{P}^2}). \tag{7.4}$$

□

Example 7.3. Often Y could be very far from ample. Here is a simple example. Suppose X is a smooth algebraic surface and set $X^* = X \setminus \{pt\}$. As compactification for X^* we can choose $\bar{X}$ = the blowup of X at p. Denote by $E \hookrightarrow \bar{X}$ the exceptional divisor so that

$$X^* = \bar{X} \setminus E.$$

In this case $E \cong \mathbb{P}^1$ and since the morphism $i_* : H_\bullet(E) \to H_\bullet(\bar{X})$ is one-to-one we deduce $H^\bullet(E)_{van} = 0$ so that

$$E_2^{-1,\bullet} = 0.$$

Hence $H^m(\bar{X} \setminus E)$ is equipped with a *pure* Hodge structure of weight m. Hence

$$\mathcal{P}_{X\setminus p} = \mathcal{P}_X - t^4(z\bar{z})^2.$$

□

Example 7.4. Suppose X^* is the complement of a three distinct lines L_1, L_2, L_3 in $\mathbb{P}^2$. We distinguish two cases (see Fig 1)

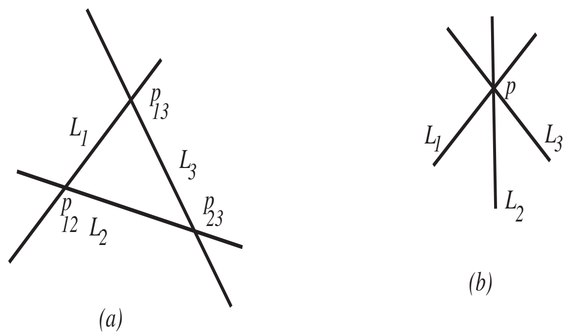

FIGURE 1. *Three lines in the plane.*

(a) *The generic situation.* The three lines are not concurrent. In this case we set $p_{ij} = L_1 \cap L_j$.

(b) *The degenerate situation.* The three lines intersect at a single point p.

To the configuration of lines we associate its *nerve* which is a simplicial complex with one vertex for every line in the configuration, and one edge for pair of intersecting lines, a 2-simplex for every triplet of intersecting lines (see Fig 2) etc.

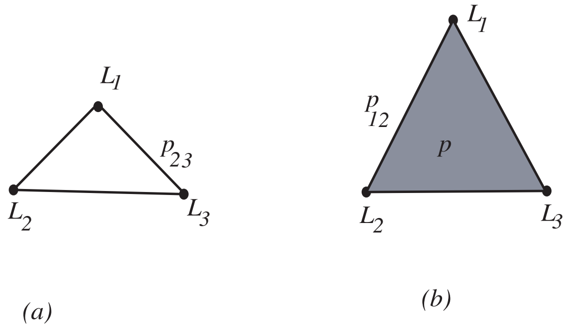

FIGURE 2. *The nerves of the two possible configurations of three lines in the plane.*

In the case (a) the spectral sequence has the form

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
\oplus H^0(p_{ij}) \xrightarrow{\delta} & \oplus H^2(L_i) \xrightarrow{\delta_4} & H^4(\mathbb{P}^2) \to & 0 \\
0 & \bigoplus H^1(L_i) \xrightarrow{\delta_3} & H^3(\mathbb{P}^2) \to & 0 \\
0 & \oplus H^0(L_i) \xrightarrow{\delta_2} & H^2(\mathbb{P}^2) \to & 0 \\
0 & 0 & H^1(\mathbb{P}^2) \to & 0 \\
0 & 0 & H^0(X) \to & 0
\end{array}$$

The row containing $H^4(\mathbb{P}^2)$ is the *augmented* simplicial chain complex corresponding to the simplicial complex $\mathcal{N}$ depicted in Figure 2(a) and thus its homology is the *reduced* homology of the associated space which is a circle. Note that $H^{odd}(L_i) = H^{odd}(\mathbb{P}^2) = 0$ and the differentials $\delta_2, \delta_3, \delta_4$ are onto. We deduce,

$$H^0(X^*) = \mathbb{C}$$

$$\mathbf{Gr}_1\, H^1(X^*) = 0, \;\; \mathbf{Gr}_2\, H^1(X^*) = H^1(X^*) \cong \mathbb{C}^2,$$

$$\mathbf{Gr}_2\, H^2(X^*) = 0, \;\; \mathbf{Gr}_3\, H^2(X^*) = 0, \;\; \mathbf{Gr}_4\, H^2(X^*) = H^2(X^*) \cong H_1(\mathcal{N}) = \mathbb{C}$$

$$H^3(X^*) = H^4(X^*) = 0.$$

As X^* is an affine set, the above computations agree with the Andreotti-Fraenkel theorem which states that an affine set has no homology beyond middle dimension.

For $k = 0, 1, 2$ the space $H^k(X^*)$ has a pure Hodge structure of maximal possible weight $2k$. The Poincaré-Hodge polynomial of X^* is

$$\mathcal{P}_{X^*} = 1 + 2t(z\bar{z}) + t^2(z\bar{z})^2.$$

We deduce that the Euler characteristic is

$$\chi(X^*) = 1 - 2 + 1 = 0.$$

Equivalently we can compute this as

$$\chi(X^*) = \chi(\mathbb{P}^2) - \chi(L_1 \cup L_2 \cup L_3) = 3 - (6 - 3) = 0.$$

(b) In the degenerate case we blow up $\mathbb{P}^2$ at the triple intersection point p. Denote by X the result of this blowup, by E the exceptional divisor and by $\bar{L}_i$ the strict transform of L_i. We get a configuration of four rational curves as depicted in Figure 3.

The second homology of $\bar{X}$ is the direct sum $\mathbb{C}\langle [L] \rangle \oplus \mathbb{C}\langle [E] \rangle$, where $[L]$ denotes the homology class determined by a line in $\mathbb{P}^2 \setminus \{p\} \subset \bar{X}$. Then

$$[\bar{L}_i] = [L] + [E], \;\; \forall i = 1, 2, 3. \tag{7.5}$$

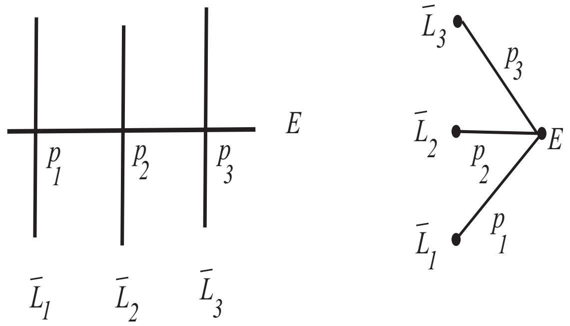

FIGURE 3. *A configuration of rational curves and its associated nerve.*

The Poincaré dual of the E_1-term of the spectral sequence has the form

$$\begin{array}{ccc} \oplus_{i=1}^3 H_0(p_i) \to & \big(\oplus_{i=1}^3 H_0(\bar{L}_i)\big) \oplus H_0(E) \longrightarrow & H_0(\bar{X}) \\ 0 & 0 & \\ 0 & \big(\oplus_{i=1}^3 H_2(\bar{L}_i\big) \oplus H_2(E) \stackrel{\delta_2}{\longrightarrow} & H_2(\bar{X}) \\ 0 & 0 & 0 \\ 0 & 0 & H_4(X) \end{array} .$$

The top row is the augmented simplicial chain complex associated to the nerve of the collection os curves $\bar{L}_i$ and E. Since the nerve is contractible we deduce that the homology of the top row is trivial.We deduce

$$H^k(X^*) = 0, \ ;\forall k > 2$$

From the equation (7.5) we deduce that the map δ_2 is onto and we conclude

$$E_2^{0,2} = 0 \Longrightarrow H^2(X^*) = 0.$$

Finally we deduce that $W_2H^1(X^*) = 0$, and thus

$$H^1(\mathbb{C}^*) \cong \ker \delta_2 \cong \mathbb{C}^2$$

so that $H^1(X^*)$ has a pure Hodge structure of weight 2. $H^0(X^*)$ has a pure Hodge structure of weight 0. The associated Poincaré-Hodge polynomial is

$$P_{X^*} = 1 + 2t(z\bar{z}).$$

□

Remark 7.5. V. Danilov and A. Khovanskii have shown in [2] that for any quasi-projective variety X the cohomology with compact supports $H_c^\bullet(X)$ is equipped with a natural mixed Hodge structure. We define

$$\mathcal{E}_c^{p,q}(X; z, \bar{z}) = \sum_{k\geq 0} (-1)^k h^{p,q}(\, H_c^k(X)\,) z^p \bar{z}^q, \ \ \mathcal{E}_c(X; z, \bar{z}) = \sum_{p,q} \mathcal{E}_c^{p,q}(X; z, \bar{z}).$$

These quantities are *motivic* in the sense that if S is a Zariski closed subvariety of X then

$$\mathcal{E}_c(X; z, \bar{z}) = \mathcal{E}_c(X \setminus S; z, \bar{z}) + \mathcal{E}_c(S; z, \bar{z}).$$

Since every algebraic variety admits a filtration by Zariski closed subsets

$$S_0 \subset S_1 \subset \cdots \subset S_n = X$$

such that $S_k \setminus S_{k-1}$ is smooth we deduce from the equality

$$\mathcal{E}_c(X) = \sum_{k \geq 0} \mathcal{E}_c(S_k \setminus S_{k-1})$$

that the Euler-Hodge characteristic

$$X \mapsto \mathcal{E}_c(X; z, \bar{z}) \in \mathbb{Z}[z, \bar{z}]$$

is uniquely determined by its values on smooth varieties X.

For smooth varieties the Poincaré duality

$$H^\bullet(X) \times H_c^\bullet(X) \to \mathbb{C}$$

defines a canonical pairing between Deligne's mixed Hodge structure on $H^\bullet(X)$ and the above mixed Hodge structure on $H_c^\bullet(X)$ so that each one of these mixed Hodge structures canonically determines the other. If X is smooth and projective then $\mathcal{E}_c(X)$ completely determines all the Hodge-Betti numbers of X. For example

$$\mathcal{E}_c(\mathbb{P}^k) = \sum_{j=0}^{k} (z\bar{z})^j = \frac{1 - (z\bar{z})^{k+1}}{1 - z\bar{z}}.$$

Since

$$\mathbb{P}^k \setminus \mathbb{P}^{k-1} \cong \mathbb{C}^k \Longrightarrow \mathcal{E}_c(\mathbb{C}^k) = (z\bar{z})^k$$

For the complement X^* of three generic lines L_1, L_1, L_3 in $\mathbb{P}^2$ as in Figure 1(a) we have

$$\mathcal{E}_c(X^*) = \mathcal{E}_c(\mathbb{P}^2) - \mathcal{E}_c(L_1 \cup L_2 \cup L_3) = \mathcal{E}_c(\mathbb{P}_2) - \sum_i \mathcal{E}_c(L_i) + \sum_{i \neq j} \mathcal{E}_c(L_i \cap L_j)$$

$$= \mathcal{E}_c(\mathbb{P}^2) - 3\mathcal{E}_c(\mathbb{P}^1) + 3\mathcal{E}_c(\mathbb{C}^0) = (z\bar{z})^2 - 2(z\bar{z}) + 1 = \mathcal{P}_{X^*}(t, z, \bar{z})\,|_{t=-1}\ .$$

□

Appendix A. Gysin maps and the differential d_1

Let us recall the definition of the Gysin map. Suppose X_0, X_1 are two, compact, oriented smooth manifolds without boundary of dimensions n_0, n_1 and $f : X_0 \to X_1$ is a smooth map. For $k = 0, 1$ we denote by $(-,-)_k$ the intersection pairing

$$(-,-)_k : H^\bullet(X_k, \mathbb{C}) \times H^{n_k - \bullet}(X_k, \mathbb{C}) \to \mathbb{C}, \;\; (\alpha, \beta)_k := \int_{X_k} \alpha \wedge \beta.$$

The Poincaré duality theorem states that the induced map

$$PD_{X_k} : H^\bullet(X_k, \mathbb{C}) \to H^{n_k - \bullet}(X_k, \mathbb{C})^*$$

is an isomorphism. The smooth map induces a pullback morphism

$$f^* : H^\bullet(X_1, \mathbb{C}) \to H^\bullet(X_0, \mathbb{C})$$

and a transpose

$$(f^*)^t : H^\bullet(X_0, \mathbb{C})^* \to H^\bullet(X_1, \mathbb{C})^*.$$

The Gysin map is the morphism

$$f_! : H^\bullet(X_0, \mathbb{C}) \to H^{\bullet + (n_1 - n_0)}(X_1, \mathbb{C}), \;\; f_! = PD_{X_1}^{-1} \circ (f^*)^t \circ PD_{X_0}$$

defined by the commutative diagram

$$\begin{array}{ccc} H^{n_0-\bullet}(X_0,\mathbb{C})^* & \xrightarrow{(f^*)^t} & H^{n_0-\bullet}(X_1,\mathbb{C})^* \\ {\scriptstyle PD_{X_0}}\uparrow & & \downarrow{\scriptstyle PD_{X_1}^{-1}} \\ H^\bullet(X_0,\mathbb{C}) & \xrightarrow[f_!]{} & H^{\bullet+(n_1-n_0)}(X_1,\mathbb{C}) \end{array}.$$

Equivalently, the Gysin map is uniquely determined by the equality

$$\int_{X_0}\alpha\wedge f^*\beta=\int_{X_1}(f_!\alpha)\wedge\beta,\ \ \forall\alpha\in H^k(X_0,\mathbb{C}),\ \ \beta\in H^m(X_1,\mathbb{C}),\ \ k+m=n_0. \tag{A.1}$$

We want to give a more explicit description of the Gysin map in the special case when $X_1=Z$ is a compact Kähler manifold, X_0 is a smooth divisor Y on Z and f is the natural inclusion $i:Y\hookrightarrow Z$. For this we need an auxiliary result. Consider the Poincaré residue map

$$\mathrm{Res}:\mathcal{A}_Z^{p,q}(\log Y)\to i_*\mathcal{A}_Y^{p-1,q}$$

and denote by $\mathcal{A}^{p,q}_{Z,Y}$ its kernel. Set

$$\mathcal{A}^m_{Z,Y}=\oplus_{p+q=m}\mathcal{A}^{p,q}_{Z,Y}.$$

Note that we have natural injections

$$\mathcal{A}^{p,q}_Z\hookrightarrow\mathcal{A}^{p,q}_{Z,Y}.$$

However these are not isomorphisms. For example if $Z=\mathbb{C}$, $Y=0$ then the form $\frac{\bar{z}}{z}$ is a section of $\mathcal{A}^{1,0}_{\mathbb{C},0}$ with a singularity at 0. However, its residue is trivial.

The sheaves $\mathcal{A}^{p,q}_{Z,Y}$ are fine and

$$d\mathcal{A}^m_{Z,Y}\hookrightarrow\mathcal{A}^{m+1}_{Z,Y},\ \ \bar{\partial}\mathcal{A}^{p,q}_{Z,Y}\hookrightarrow\mathcal{A}^{p,q+1}_{Z,Y}.$$

Lemma A.1. *The inclusions*

$$(\Omega^\bullet_Z,\partial)\to(\mathcal{A}^\bullet_{Z,Y},d) \tag{A.2}$$

and

$$(\mathcal{A}^\bullet_Z,d)\to(\mathcal{A}^\bullet_{Z,Y},d) \tag{A.3}$$

are quasi-isomorphisms.

Proof Note that we have a commutative diagram of complexes sheaves

$$\begin{array}{ccccc} 0\to(\Omega^\bullet_Z,\partial) & \longrightarrow & (\Omega^\bullet_Z(\log Y),\partial) & \xrightarrow{\mathrm{Res}} & i_*(\Omega^\bullet_Y[-1],\partial)\to 0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ 0\to(\mathcal{A}^\bullet_{Z,Y},d) & \longrightarrow & (\mathcal{A}^\bullet_Z(\log Y),d) & \xrightarrow{\mathrm{Res}} & i_*(\mathcal{A}^\bullet_Y[-1],d)\to 0 \end{array} \tag{†}$$

The rows are exact, the middle vertical arrow is a quasi-isomorphism by (2.1) and the last vertical arrow is a quasi-isomorphism (Dolbeault-to-DeRham). Using the five lemma we deduce that first vertical lemma is a quasi-isomorphism as well. This proves (A.2).

To prove the quasi-isomorphism (A.3) note that we have a commutative diagram of complexes of sheaves

$$\begin{tikzcd}
(\Omega^\bullet_Z, \partial) \arrow[r, hook, "j_1"] \arrow[dr, hook, "j"'] & (\mathcal{A}^\bullet_Z, d) \arrow[d, hook, "j_2"] \\
& (\mathcal{A}^\bullet_{Z,Y}, d)
\end{tikzcd}$$

where j_1 and j are quasi-isomorphisms.

□

The divisor Y determines a pair (L, s), where $L \to Z$ is a holomorphic line bundle and $s : Z \to L$ is a holomorphic section such that $Y = s^{-1}(0)$. Fix a hermitian metric h on L and consider

$$\eta = \eta_{s,h} = \frac{1}{2\pi\mathbf{i}} \partial \log |s|^2_h.$$

Locally, on a trivializing neighborhood $U \subset Z$ for the line bundle L the section s is described by a holomorphic function f vanishing along $Y \cap U$ and we have

$$|s|^2_h = e^{2u}|f|^2$$

for some smooth real valued function u. Using the equality $|f|^2 = f\bar{f}$ we deduce

$$\eta = \frac{1}{2\pi\mathbf{i}} \frac{df}{f} + \frac{1}{\pi\mathbf{i}} \partial u \in \mathcal{A}^{1,0}_Z(U, \log Y). \tag{A.4}$$

$$\omega_{s,h} = d\eta = \bar{\partial}\eta = \frac{1}{\pi\mathbf{i}} \bar{\partial}\partial u \in \mathcal{A}^{1,1}_Z(U). \tag{A.5}$$

In particular, η_h satisfies

$$\operatorname{Res}_Y \eta_h = 1.$$

The closed form $\omega_{s,h}$ represents the first Chern class of L. Observe that if we change the metric h to $h_w = e^{2w}h$ then we have

$$\eta_{s,h_w} = \eta_{s,h} + \frac{1}{\pi\mathbf{i}} \partial w, \;\; \omega_{s,h_w} = \omega_{s,h} + \frac{1}{\pi\mathbf{i}} \bar{\partial}\partial w.$$

Note for later use that

$$\omega_{s,h_w} = \omega_{s,h} + \frac{1}{2\pi\mathbf{i}} d(\partial w - \bar{\partial} w) = \omega_{s,h} + dd^c w, \tag{A.6}$$

where the operator d^c is defined as in [5, p.109]

Suppose $\alpha \in \mathcal{A}^{p,q}_Y(Y)$ is a closed form and denote by $[\alpha]$ the class it determines in $H^{p+q}(Y, \mathbb{C})$. Set $m := p + q$. Fix $\tilde{\alpha} \in \mathcal{A}^{p,q}_Z(Z)$ such that

$$\tilde{\alpha}\,|_Y = \alpha$$

and set

$$\Gamma(\alpha) := -d(\eta \wedge \tilde{\alpha}) \in \mathcal{A}^{p+1,q+1}_Z(Z, \log Y).$$

From the local description (A.4) we deduce

$$G(\Gamma(\alpha)) = -\operatorname{Res}(d(\eta \wedge \tilde{\alpha}) = \operatorname{Res}(\eta \wedge d\tilde{\alpha}) = d\alpha = 0,$$

i.e.

$$G(\alpha) \in \mathcal{A}^{p+1,q+1}_{Z,Y}(Z).$$

Clearly $d\alpha = \bar{\partial}\alpha = 0$ and using Lemma A.1 we deduce that $\Gamma(\alpha)$ determines a cohomology class

$$[\Gamma(\alpha)] \in H^{q+1}(Z, \Omega_Z^{p+1}) = H^{p+1,q+1}(Z) \subset H^{m+2}(Z, \mathbb{C}).$$

Remark A.2. We can avoid the use of Lemma A.1 as follows. Denote by T_ε a tubular neighborhood of radius $\varepsilon > 0$ around Y defined by the inequality

$$|s|_h \leq \varepsilon.$$

Identify T_ε of Y in Z with a tubular neighborhood of Y in the normal bundle of the imbedding $Y \hookrightarrow Z$. Thus we can produce a submersion $\pi_Y : T_\varepsilon \to Y$ and then we construct $\tilde{\alpha} \in \Gamma(Z, \mathcal{A}_Z^m)$ such that $\tilde{\alpha} \,|_{T_{\varepsilon/2}} = \pi_Y^* a$. Then $d\tilde{\alpha} = 0$ on $T_{\varepsilon/2}$ and $d(\eta \wedge \tilde{\alpha}) = d\eta \wedge \tilde{\alpha} - \eta \wedge d\tilde{\alpha}$. Using (A.5) we deduce that $d(\eta \wedge \tilde{\alpha})$ is *smooth* form of degree $(m+2)$ on Z. We will say that $\tilde{\alpha}$ is a *good* extension of α. In fact, as explained in [1, Thm. 5.7], we can define the projection π_Y carefully so the pullback by π_Y preserves the Hodge type, i.e.

$$\alpha \in \Gamma(Y, \mathcal{A}_Y^{p,q}) \Longrightarrow \pi_Y^* \alpha \in \Gamma(T_\varepsilon, \mathcal{A}_Z^{p,q}).$$

In other words, we can choose an extension $\tilde{\alpha}$ of α with the same Hodge type as α such that $G(\tilde{\alpha})$ is smooth on Z. □

Proposition A.3. *For every $\alpha \in \Gamma(Y, \mathcal{A}_Y^{p,q})$ such that $d\alpha = 0$ and for every extension $\tilde{\alpha}$ of α as a m-form on Z ($m = p + q$) we have*

$$[G(\tilde{\alpha})] = i_![\alpha],$$

where $i_!$ denotes the Gysin map induced by the inclusion $i : Y \hookrightarrow Z$. In particular, $i_!$ is a morphism of pure Hodge structures of type $(1,1)$.

Proof We need to prove that

$$\int_Z G(\tilde{\alpha}) \wedge \beta = \int_Y \alpha \wedge \beta \,|_Y, \;\; \forall \beta \in \Gamma\big(Z, \mathcal{A}_Z^{2\dim_{\mathbb{C}} Z - m - 2}\big), \;\; d\beta = 0.$$

We orient ∂T_ε using the outer-normal-first convention. We have

$$G(\alpha) \wedge \beta = -d(\eta \wedge \tilde{\alpha} \wedge \beta)$$

and

$$\int_Z G(\alpha) \wedge \beta = -\lim_{\varepsilon \searrow 0} \int_{Z \setminus T_\varepsilon} d(\eta \wedge \tilde{\alpha} \wedge \beta)$$

Using Stokes formula we deduce

$$-\int_{Z \setminus T_\varepsilon} d(\eta \wedge \tilde{\alpha} \wedge \beta) = -\int_{\partial(Z \setminus T_\varepsilon)} \eta \wedge \tilde{\alpha} \wedge \beta = \int_{\partial T_\varepsilon} \eta \wedge \tilde{\alpha} \wedge \beta.$$

Using partitions of unity we can reduce this to a local computation and we can assume we work in a coordinate neighborhood $(U, (z_i)) = (\mathbb{D}_r^n, (z_i))$ where $f = z_1$. We have a natural projection

$$\pi : \partial T_\varepsilon \to Y, \;\; (z_1, z_2, \cdots, z_n) \mapsto (0, z_2, \cdots, z_n)$$

Integrating along the fibers of π and using the Cauchy residue formula we deduce

$$\int_{\partial T_\varepsilon / Y} \eta \wedge \tilde{\alpha} \wedge \beta = \alpha \wedge \beta$$

so that for $\varepsilon > 0$ sufficiently small we have

$$\int_{\partial T_\varepsilon} \eta \wedge \tilde{\alpha} \wedge \beta = \int_Y \alpha \wedge \beta \,|_Y \,.$$

□

Remark A.4. (a) If $\tilde{\alpha}$ is a good extension of α then we say that $G(\tilde{\alpha})$ is a *good* representative of $i_![\alpha]$.

(b) Proposition A.3 can be rephrased more conceptually as follows. We consider the short exact sequence of complexes of fine sheaves

$$0 \to (\mathcal{A}^\bullet_{Z,Y}, d) \to (\mathcal{A}^\bullet_Z(\log Y), d) \xrightarrow{\text{Res}} (i_*\mathcal{A}^\bullet_Y[-1], d) \to 0.$$

Applying the functor $\Gamma(X, -)$ we obtain a long exact sequence in (hyper) cohomology. We have $\mathbb{H}^\bullet(X, i_*\mathcal{A}^\bullet_Y[-1]) = H^\bullet(Y, \mathbb{C})[-1]$. Given a cohomology class $u \in H^\bullet(Y, \mathbb{C})[-1]$ represented by a closed form α on Y we deduce that $d_1 u$ is represented by $\pm\Gamma\alpha$.

The above short exact sequence is quasi-isomorphic to the sequence

$$0 \to (\Omega^\bullet_Z, \partial) \to (\Omega^\bullet_Z(\log Y), \partial) \xrightarrow{\text{Res}} (i_*\Omega^\bullet_Y[-1], \partial) \to 0$$

and we deduce that the connecting morphism in the hypercohomology long exact sequence

$$d_1 : \mathbb{H}^\bullet\big(Y, (\Omega^\bullet_Y[-1], \partial)\big) \to \mathbb{H}^\bullet\big(Z, (\Omega^\bullet_Z, \partial)\big)[1] \tag{A.7}$$

is, up to a sign, the Gysin morphism. This long exact sequence is essentially the long exact sequence of the pair (Z, Z^*), $Z^* = Z \setminus Y$. To see this note that by excision we have

$$H^\bullet(Z, Z^*) \cong H^\bullet(T_\varepsilon, T^*_\varepsilon) \cong H^\bullet(T_\varepsilon, \partial T_\varepsilon) \cong H^\bullet(Y)[-2],$$

where the last isomorphism is given by the composition

$$H^\bullet(T_\varepsilon, \partial T_\varepsilon) \overset{PD}{\cong} H^{2\dim Z - \bullet}(T_\varepsilon) \cong H^{2\dim Z - \bullet}(Y) \overset{PD}{\cong} H^\bullet(Y)[-2].$$

The long exact sequence is then

$$\cdots \to H^\bullet(Y)[-2] \xrightarrow{i_!} H^\bullet(Z) \to H^\bullet(U) \xrightarrow{\text{Res}} H^\bullet(Y)[-1] \to \cdots.$$

□

To relate the above construction to the differential d_1 in the spectral sequence $E_r(K, W_-)$, $K = (\mathcal{A}^\bullet_X(\log D), d)$ we need to recall an abstract result. More precisely, the differential d_1 is described by the connecting morphism

$$\delta : H^q(\mathbf{Gr}^\ell_{W_-} K) \to H^{q+1}(\mathbf{Gr}^{\ell+1}_{W_-} K)$$

of the long exact sequence corresponding to the short exact sequence

$$0 \to W^{\ell+1}_-/W^{\ell+2}_- \to W^\ell_-/W^{\ell+2}_- \to W^\ell_-/W^{\ell+1}_\to 0. \tag{A.8}$$

In our special case, the connecting morphism is given by the connecting morphism in the hyper-cohomology long exact sequence associated to the short exact sequence of complexes of sheaves we have a commutative diagram of complexes of sheaves

$$0 \to (\mathbf{Gr}^W_{\ell-1}\Omega^\bullet_X(\log D), \partial) \to \Big(\frac{W^\ell\Omega^\bullet_X(\log D)}{W^{\ell-2}\Omega^\bullet_X(\log D)}, \partial\Big) \to (\mathbf{Gr}^W_\ell \Omega^\bullet_X(\log D), \partial) \to 0.$$

As before note that we have a commutative diagram

$$\begin{tikzcd}
0 \to (W_{l-1}\Omega^\bullet_X(\log D),\partial) \arrow[r] \arrow[d, hook, "f"'] & \big(W^\ell\Omega^\bullet_X(\log D),\partial\big) \arrow[r] \arrow[d, hook, "g"'] & (\mathbf{Gr}^W_\ell\,\Omega^\bullet_X(\log D),\partial) \to 0 \arrow[d, "\operatorname{Res}^\ell"'] \\
0 \to (W'_\ell\mathcal{A}^\bullet_X(\log D),d) \arrow[r] & \big(W^\ell\mathcal{A}^\bullet_X(\log D),d\big) \arrow[r, "\operatorname{Res}^\ell"] & (a_*\mathcal{A}^\bullet[-\ell],d) \to 0
\end{tikzcd}$$

in which $W'_\ell = \ker\operatorname{Res}^\ell$, the rows are exact, and the vertical arrows are quasi-isomorphisms of complexes of sheaves. To understand the

The bottom row consists of complexes of acyclic sheaves and the short exact sequence (A.8) is obtained by applying the functor $\Gamma(X,-)$ to the bottom row. The resulting long exact sequence is naturally isomorphic to the hypercohomology long exact sequence obtained by applying the derived functor $R\Gamma(X,-)$ to the first row. We denote by δ the connecting morphism.

To understand the connecting morphism we need to introduce some notations. Fix hermitian holomorphic line bundle (L_i,h_i), $i=1,\cdots,\nu$ and holomorphic sections $s_i\in\Gamma(X,\mathcal{O}_{L_i})$ such that $D_i=s_i^{-1}(0)$. Denote by $\eta_i=\eta_{s_i,h_i}$ the associated $(1,0)$-form. For every ordered multi-index $I=(i_1<\cdots<i_\ell)$, $|I|=\ell$, we set

$$\eta_I=\eta_{i_1}\wedge\cdots\wedge\eta_{i_\ell}\in W_\ell\mathcal{A}^\bullet(X,\log D).$$

Note that

$$d\eta_I\in W_{\ell-1}\mathcal{A}^\bullet(X,\log D).$$

Fix tubular neighborhoods T_I of D_I in X with projections $\pi_I:T_I\to D_I$. For every closed form α_I on D_I we denote by $\pi_I^*\alpha_I$ a good extension and set

$$G_I(\alpha_I)=(-1)^{|I|}d(\eta_I\wedge\pi_I^*\alpha_I)\in W_{\ell-1}\mathcal{A}^\bullet(X,\log D).$$

Its image in $W_{\ell-1}/W_{\ell-2}$ represents $\delta[\alpha_I]$. Using the identification

$$\operatorname{Res}^{\ell-1}:H^\bullet(X,\mathbf{Gr}_{\ell-1}\,\mathcal{A}^\bullet_X(\log D))\xrightarrow{\cong}H^\bullet(\tilde{D}_{\ell-1})[\ell-1]$$

we can identify $\delta[\alpha_I]$ with $\operatorname{Res}^{\ell-1}G_I(\alpha_I)$. Note that

$$\operatorname{Res}^{\ell-1}G_I(\alpha_I)\in\bigoplus_{k=1}^{\ell}H^\bullet(D_{I\setminus i_k}).$$

More precisely, if $[\alpha_I]\in H^m(D_I)$ then we have[3]

$$\operatorname{Res}^{\ell-1}G_I(\alpha_I)=\delta_I[\alpha_I]:=\bigoplus_{k=1}^{\ell}(-1)^{k-1}(\iota_k)_!([\alpha]_I)\in\bigoplus_{k=1}^{\ell}H^{m+2}(D_{I\setminus i_k}),$$

where ι_k denotes the inclusion $D_I\hookrightarrow D_{I\setminus i_k}$. Now define

$$\delta_l=\sum_{|I|=\ell}\delta_I:\bigoplus_{|L|=\ell}H^\bullet(D_L)\to\bigoplus_{|L'|=\ell-1}H^\bullet(D_{L'})[2]\subset H^\bullet(\tilde{D}_{\ell-1})[2].$$

[3]Here we skipped some computational details.

The differential d_1 on $E_1^{-\ell,m} \to E_1^{-\ell+1,m}$ is then given by the commutative diagram

$$\begin{tikzcd} E_1^{-\ell,m} \arrow[r, "d_1"] \arrow[d, "\mathrm{Res}"'] & E_1^{-\ell+1,m} \arrow[d, "\mathrm{Res}"] \\ H^{m-2\ell}(\tilde{D}_\ell) \arrow[r, "\delta_\ell"'] & H^{m-2\ell+2}(\tilde{D}_{\ell-1}) \end{tikzcd} . \tag{A.9}$$

References

[1] C.H. Clemens: *Degeneration of Kähler manifolds*, Duke Math. J., **44**(1977), p. 215-290.
[2] V.I. Danilov, A.G. Khovanskii: *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*, Math. USSR Izvestyia, **29**(1987), 279-298.
[3] P. Deligne: *Théorie de Hodge, II*, Publ. IHES, **40**(1971), 5-57.
[4] P. Deligne: *Théorie de Hodge, III*, Publ. IHES, **44**(1974), 5-77.
[5] P. Griffiths, J. Harris: *Principles of Algebraic Geometry*, John Wiley& Sons, 1978.
[6] P.A. Griffiths, W. Schmid: *Recent developments in Hodge theory: a discussion of techniques and results*, p. 31-127, *Discrete Subgroups of Lie Groups and Applications to Moduli Spaces*, Bombay Colloquium, 1975.
[7] H. K. Nickerson: *On the complex form of the Poincaré lemma*, Proc. A.M.S., **9**(1958), 183-188.
[8] C. Voisin: *Hodge Theory and Complex Algebraic Geometry, I*, Cambridge Studies in Advanced Math, vol. 76, Cambridge University Press, 2002.
[9] C. Voisin: *Hodge Theory and Complex Algebraic Geometry, II*, Cambridge Studies in Advanced Math, vol. 77, Cambridge University Press, 2002.

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