REGULARIZATION OF CERTAIN DIVERGENT SERIES OF POLYNOMIALS

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ABSTRACT. We investigate the generalized convergence and sums of series of the form $\sum_{n\geq 0} a_n \boldsymbol{T}^n P(x)$, where $P \in \mathbb{R}[x]$, $a_n \in \mathbb{R}$, $\forall n \geq 0$, and $\boldsymbol{T} : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear operator that commutes with the differentiation $\frac{d}{dx} : \mathbb{R}[x] \to \mathbb{R}[x]$.

CONTENTS

1. THE MAIN RESULT

We consider series of the form

$$\sum_{n\geq 0} a_n \boldsymbol{T}^n P(x), \tag{$\dagger$}$$

where $P \in \mathbb{R}[x]$, and $\boldsymbol{T} : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear operator such that

$$\boldsymbol{T}D = D\boldsymbol{T}, \tag{$*$}$$

where D is the differentiation operator $D = \frac{d}{dx}$. The condition ($*$) is equivalent with the translation invariance of $\boldsymbol{T}$, i.e.,

$$\boldsymbol{T}\boldsymbol{U}^h = \boldsymbol{U}^h\boldsymbol{T}, \quad \forall h \in \mathbb{R}, \tag{I}$$

where $\boldsymbol{U}^h : \mathbb{R}[x] \to \mathbb{R}[x]$ is the translation operator

$$\mathbb{R}[x] \ni p(x) \mapsto p(x+h) \in \mathbb{R}[x].$$

For simplicity we set $\boldsymbol{U} := \boldsymbol{U}^1$. Clearly $\boldsymbol{U}^h \in \mathcal{O}$ so a special case of the series ($\dagger$) is the series

$$\sum_{n\geq 0} a_n \boldsymbol{U}^{nh} P(x) = \sum_{n\geq 0} a_n P(x+nh), \quad h \in \mathbb{R}, \tag{$\ddagger_h$}$$

which is typically divergent.

We denote by $\mathcal{O}$ the $\mathbb{R}$-algebra of translation invariant operators. We have a natural map

$$\mathcal{Q} : \mathbb{R}[[t]] \to \mathcal{O}, \quad \mathbb{R}[[t] \ni \sum_{n\geq 0} c_n \frac{t^n}{n!} \mapsto \sum_{n\geq 0} \frac{c_n}{n!} D^n.$$

It is known (see [1, Prop. 3.47]) that this map is an isomorphism of rings. We denote by $\boldsymbol{\sigma}$ the inverse of $\mathcal{Q}$

$$\boldsymbol{\sigma} : \mathcal{O} \to \mathbb{R}[[t]], \quad \mathcal{O} \ni \boldsymbol{T} \mapsto \boldsymbol{\sigma}_{\boldsymbol{T}} \in \mathbb{R}[[t]].$$

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For $\boldsymbol{T} \in \mathcal{O}$ we will refer to the formal power series $\boldsymbol{\sigma_T}$ as the *symbol* of the operator $\boldsymbol{T}$. More explicitely

$$\boldsymbol{\sigma_T}(t) = \sum_{n\geq 0} \frac{c_n(\boldsymbol{T})}{n!} t^n, \quad c_n(\boldsymbol{T}) = (\boldsymbol{T}x^n)|_{x=0} \in \mathbb{R}.$$

We denote by $\mathbb{N}$ the set of nonnegative integers, and by $\mathbf{Seq}$ the vector space of real sequences, i.e., maps $a : \mathbb{N} \to \mathbb{R}$. Let $\mathbf{Seq}^c$ the vector subspace of $\mathbf{Seq}$ consisting of all convergent sequences.

A *generalized notion of convergence*[1] or *regularization method* is a pair $\mu = ({}^{\mu}\lim, \mathbf{Seq}_{\mu})$, where

- $\mathbf{Seq}_{\mu}$ is a vector subspace of $\mathbf{Seq}$ containing $\mathbf{Seq}^c$ and,
- ${}^{\mu}\lim$ is a linear map

$$ {}^{\mu}\lim : \mathbf{Seq}_{\mu} \to \mathbb{R}, \quad \mathbf{Seq}_{\mu} \ni a \mapsto {}^{\mu}\lim_{n} a(n) \in \mathbb{R}$$

such that for any $a \in \mathbf{Seq}^c$ we have

$$ {}^{\mu}\lim a = \lim_{n\to\infty} a(n).$$

The sequences in $\mathbf{Seq}_{\mu}$ are called *μ-convergent* and ${}^{\mu}\lim$ is called the *μ-limit*. To any sequence $a \in \mathbf{Seq}$ we associate the sequence $\boldsymbol{S}[a]$ of partial sums

$$\boldsymbol{S}[a](n) = \sigma_{k=0}^{n} a(k). \tag{1.1}$$

A series $\sum_{n\geq 0} a(n)$ is said to by μ-convergent if the sequence $\boldsymbol{S}[a]$ is μ-convergent. We set

$$ {}^{\mu}\sum_{n\geq 0} a(n) := {}^{\mu}\lim_{n} \boldsymbol{S}[a](n).$$

We say that ${}^{\mu}\sum_{n\geq 0} a(n)$ is the *μ-sum of the series*. The regularization method is said to be *shift invariant* if it satisfies the additional condition

$$ {}^{\mu}\sum_{n\geq 0} a(n) = a(0) + {}^{\mu}\sum_{n\geq 1} a(n). \tag{1.2}$$

We refer to the classic [3] for a large collection of regularization methods.

For $x \in \mathbb{R}$ and $k \in \mathbb{N}$ we set

$$[x]_k := \begin{cases} \prod_{i=0}^{k-1}(x-i), & k \geq 1 \\ 1, & k = 0, \end{cases}, \quad \binom{x}{k} := \frac{[x]_k}{k!}.$$

We can now state the main result of this paper.

Theorem 1.1. *Let μ be a regularization method, $\boldsymbol{T} \in \mathcal{O}$ and $f(t) = \sum_{n\geq 0} a_n t^n \in \mathbb{R}[[t]]$. Set $c := c_0(\boldsymbol{T}) = \boldsymbol{T}1$. Suppose that f is μ-*regular *at $t = c$, i.e.,*

$$\text{for every } k \in \mathbb{N} \text{ the series } \sum_{n\geq 0} a_n [n]_k\, c^{n-k} \text{ is } \mu\text{-convergent.} \tag{μ}$$

We denote by $f^{(k)}(c)_{\mu}$ its μ-sum

$$f^{(k)}(c)_{\mu} := {}^{\mu}\sum_{n\geq 0} a_n [n]_k\, c^{n-k}.$$

Then for every $P \in \mathbb{R}[x]$ the series $\sum_{n\geq 0} a_n (\boldsymbol{T}^n P)(x)$ is μ-convergent and its μ-sum is

$$ {}^{\mu}\sum a_n (\boldsymbol{T}^n P)(x) = f(\boldsymbol{T})_{\mu} P\,(x),$$

[1] Hardy refers to such a notion of convergence as convergence in some ‘Pickwickian’ sense.

where $f(\boldsymbol{T})_\mu \in \mathcal{O}$ is the operator

$$f(\boldsymbol{T})_\mu := \sum_{n\geq 0} \frac{f^{(k)}(c)_\mu}{k!}(\boldsymbol{T}-c)^k. \tag{1.3}$$

Proof. Set $\boldsymbol{R} := \boldsymbol{T} - c$ and let $P \in \mathbb{R}[x]$. Then

$$\boldsymbol{R} = \sum_{n\geq 1} \frac{c_n(\boldsymbol{T})}{n!} D^n$$

so that

$$\boldsymbol{R}^n P = 0, \quad \forall n > \deg P. \tag{1.4}$$

In particular this shows that $f(\boldsymbol{T})_\mu$ is well defined. We have

$$a_n \boldsymbol{T}^n P = a_n (c+\boldsymbol{R})^n P = a_n \sum_{k=0}^{n} \binom{n}{k} c^{n-k} \boldsymbol{R}^k P = \sum_{k=0}^{\deg P} \binom{n}{k} c^{n-k} \boldsymbol{R}^k P.$$

At the last step we used (1.4) and the fact that

$$\binom{n}{k} = 0, \text{ if } k > n.$$

This shows that the formal series $\sum_{n\geq 0} a_n (\boldsymbol{T}^n P)(x)$ can be written as a *finite* linear combination of formal series

$$\sum_{n\geq 0} a_n (\boldsymbol{T}^n P)(x) = \sum_{k=0}^{\deg P} \frac{\boldsymbol{R}^k P(x)}{k!} \left(\sum_{n\geq 0} a_n [n]_k \, c^{n-k} \right).$$

From the linearity of the μ-summation operator we deduce

$$ {}^{\mu}\sum_{n\geq 0} a_n (\boldsymbol{T}^n P)(x) = \sum_{k=0}^{\deg P} \frac{\boldsymbol{R}^k P(x)}{k!} \left({}^{\mu}\sum_{n\geq 0} a_n [n]_k \, c^{n-k} \right)$$

$$= \left(\sum_{k=0}^{\deg P} \frac{f^{(k)}(c)_\mu}{k!} \boldsymbol{R}^k \right) P(x) = f(\boldsymbol{T})_\mu P(x)$$

□

2. Some applications

To describe some consequences of Theorem 1.1 we need to first describe some classical facts about regularization methods.

For any sequence $a \in \mathbf{Seq}$ we denote by $\boldsymbol{G}_a(t) \in \mathbb{R}[[t]]$ its generating series. We regard the partial sum construction $\boldsymbol{S}$ in (1.1) as a linear operator $\boldsymbol{S} : \mathbf{Seq} \to \mathbf{Seq}$. Observe that

$$\boldsymbol{G}_{\boldsymbol{S}[a]}(t) = \frac{1}{1-t} \boldsymbol{G}_a(t).$$

We say that a regularization method $\mu_1 = ({}^{\mu_1}\lim, \mathbf{Seq}_{\mu_1})$ is stronger than the regularization method $\mu_0 = ({}^{\mu_1}\lim, \mathbf{Seq}_{\mu_0})$, and we write this $\mu_0 \prec \mu_1$, if

$$\mathbf{Seq}_{\mu_0} \subset \mathbf{Seq}_{\mu_1} \quad \text{and} \quad {}^{\mu_1}\lim_n a(n) = {}^{\mu_0}\lim_n a(n), \quad \forall a \in \mathbf{Seq}_{\mu_0}.$$

The *Abel regularization method*[2] A is defined as follows. We say that a sequence a is A convergent if

- the radius of convergence of the series $\sum_{n\geq 0} a_n t^n$ is at least 1 and
- the function $t \mapsto (1-t)\sum_{n\geq 0} a_n t^n$ has a finite limit as $t \to 1^-$.

Hence

$$ {}^A\lim a(n) = \lim_{t\to 1^-}(1-t)\sum_{n\geq 0} a_n t^n, $$

and $\mathbf{Seq}_A$ consists of sequence for which the above limit exists and it is finite. Using (2) we deduce that a series $\sum_{n\geq 0} a(n)$ is A-convergent if and only if the limit

$$ \lim_{t\to 1^-}\sum_{n\geq 0} a_n t^n $$

exists and it is finite. We have the following immediate result.

Proposition 2.1. *Suppose that $f(z)$ is a holomorphic function defined in an open neighborhood of the set $\{1\}\cup\{|z|\}\subset\mathbb{C}$. If $\sum_{n\geq 0} a_n z^n$ is the Taylor series expansion of f at $z=0$ then the corresponding formal power series $[f]=\sum_{n\geq 0} a_n t^n$ is A-regular at $t=1$,*

$$ [f]^{(k)}(1)_A = f^k(1), $$

and the series

$$ [f](r)_A = \sum_k \frac{[f]^{(k)}(1)_A}{k!} r^k $$

coincides with the Taylor expansion of f at $z=1$, and it converges to $f(1+r)$.

Corollary 2.2. *Suppose that $f(z)$ is a holomorphic function defined in an open neighborhood of the set $\{1\}\cup\{|z|\}\subset\mathbb{C}$ and $\sum_{n\geq 0} a_n z^n$ is the Taylor series expansion of f at $z=0$. Then for every $\boldsymbol{T}$ in $\mathcal{O}$ such that $c_0(\boldsymbol{T})=1$, any $P\in\mathbb{R}[x]$, and any $x\in\mathbb{R}$ we have*

$$ {}^A\sum_n a_n \boldsymbol{T}^n P(x) = \sum_{k\geq 0}\frac{f^k(1)}{k!}(\boldsymbol{T}-1)^k P(x). \qquad \square $$

Let $k\in\mathbb{N}$. A sequence $a\in\mathbf{Seq}$ is said to be C_k-*convergent* (or *Cesàro convergent of order* k) if the limit

$$ \lim_{n\to\infty}\frac{\boldsymbol{S}^k[a](n)}{\binom{n+k}{k}} $$

exists and it is finite. We denote this limit by ${}^{C_k}\lim a(n)$. A series $\sum_{n\geq 0} a(n)$ is said to be C_k-convergent if the sequence of partial sums $\boldsymbol{S}[a]$ is C_k convergent. Thus the C_k-sum of this series is

$$ {}^{C_k}\sum_{n\geq 0} a(n) = \lim_{n\to\infty}\frac{\boldsymbol{S}^{k+1}[a](n)}{\binom{n+k}{k}}. $$

More explicitly, we have (see [3, Eq.(5.4.5)])

$$ {}^{C_k}\sum_{n\geq 0} a(n) = \lim_{n\to\infty}\frac{1}{\binom{n+k}{k}}\left(\sum_{\nu=0}^{n}\binom{\nu+k}{k}a(n-\nu)\right) $$

[2] This was apparently known and used by Euler.

Hence

$$ {}^{C_k}\sum_{n\geq 0} a(n) \Longleftrightarrow \boldsymbol{S}^{k+1}[a](n) \sim A\binom{n+k}{k} \sim A\frac{n^k}{k!}, $$

where

$$ a\sim b \Longleftrightarrow \lim_{n\to\infty}\frac{a(n)}{b(n)} = 1, $$

if $a(n), b(n)\neq 0$, for $n \gg 0$.

The C_0 convergence is equivalent with the classical convergence and it is known (see [3, Thm. 43, 55]) that

$$ C_k \prec C_{k'} \prec A,\ \ \forall k < k'. $$

Given this fact, we define a sequence to be C-convergent (Cesàro convergent) if it is C_k-convergent for some $k\in\mathbb{N}$. Note that $C\prec A$. Both the C and A methods are shift invariant, i.e., they satisfy the condition (1.2).

We want to comment a bit about possible methods of establishing C-convergence. To formulate a general strategy we need to introduce a classical notation. More precisely, if $f(t)=\sum_{n\geq 0} a_nt^n$ is a formal power series we let $[t^n]f(t)$ denote the coefficient of t^n in this power series, i.e. $[t^n]f(t) = a_n$.

Let $f(t)=\sum_{n\geq 0} a_nt^n$. Then the series $\sum_{n\geq 0} a_nt^n$ C-converges to A if and only if there exists a nonnegative *real* number α such that

$$ [t^n]\Big((1-t)^{-(\alpha+1)}f(t)\Big) \sim A\frac{n^\alpha}{\Gamma(\alpha+1)}, $$

where Γ is Euler's Gamma function. For a proof we refer to [3, Thm. 43].

Definition 2.3. We say that a power series $f(t)=\sum_{n\geq 0} a_nt^n$ is *Cesàro convenient (or C-convenient) at* 1 *if the following hold.*

(i) The radius of convergences of the series is ≥ 1

(ii) The function f is regular at $z=1$ and has finitely many singularities $\zeta_1,\dots,\zeta_\nu\neq 1$ on the unit circle $\{|z|=1\}$.

(iii) There exist $\varepsilon>0$ and $\theta\in(0,\frac{\pi}{2})$ such that f admits a continuation to the dimpled disk

$$ \Delta_{\varepsilon,\theta} := \left\{ z\in\mathbb{C};\ \ |z|<1+\varepsilon,\ \ \arg\Big(\frac{z}{\zeta_j}-1\Big) > \theta,\ \ \forall j=1,\dots,\nu \right\}. $$

(iv) For every singular point ζ_j there exists a positive integer m_j such that

$$ f(z) = O\big((z-\zeta_j)^{-m_j}\big)\ \ \text{as } z\to\zeta_j,\, z\in\Delta. $$

□

The results in [2, Chap. VI] implies that the collection $\mathcal{R}_C$ of C-convenient power series is a ring satisfying

$$ f\in\mathcal{R}_C \Longleftrightarrow \frac{df}{dt}\in\mathcal{R}_C. $$

Invoking [2, Thm VI.5] we deduce the following useful consequence.

Corollary 2.4. *Let $f\in\mathbb{R}[[t]]$ be a power series C-convenient at* 1*. Then f is C-regular at* 1 *and*

$$ f^k(1)_C = f^{(k)}(1)_A = f^{(k)}(1). $$ □

Using [2, VII.7] we obtain the following useful result.

Corollary 2.5. *(a) The power series*

$$(1+t)^{-m} = \sum_{n\geq 0}\binom{n+m-1}{n}(-t)^n,\ \ m\geq 1,\ \ \log(1+t) = \sum_{n\geq 1}(-1)^{n+1}\frac{t^n}{n}$$

are C-regular at 1.
(b) If $f(z)$ is an algebraic function defined on the unit disk $|z|<1$ and regular at $z=1$ then the Taylor series of f at $z=0$ is C-regular at 1.

Recall that the Cauchy product of two sequences $a,b\in\mathbf{Seq}$ is the sequence $a*b$,

$$a*b(n) = \sum_{i=0}^{n} a(n-i)b(i),\ \ \forall n\in\mathbb{N}.$$

A regularization method is said to be *multiplicative* if

$$^{\mu}\sum_n a*b(n) = \left(^{\mu}\sum_n a(n)\right)\left(^{\mu}\sum_n b(n)\right),$$

for any μ-convergent series $\sum_{n\geq 0}a(n)$ and $\sum_{n\geq 0}b(n)$. The results of [3, Chap.X] show that the C and A methods are multiplicative.

For any regularization method μ and $c\in\mathbb{R}$ we denote by $\mathbb{R}[[t]]_\mu$ the set of series that are μ-regular at $t=1$.

Proposition 2.6. *Let μ be a multiplicative regularization method. Then $\mathbb{R}[[t]]_\mu$ is a commutative ring with one and we have the product rule*

$$(f\cdot g)^{(n)}(1)_\mu = \sum_{k=0}^{n}\binom{n}{k}f^{(k)}(1)_\mu\cdot g^{(n-k)}(1)_\mu.$$

Moreover, if $\boldsymbol{T}\in\mathcal{O}$ is such that $c_0(T)=1$ then the map

$$\mathbb{R}[[t]]_\mu \ni f\mapsto f(\boldsymbol{T})_\mu\in\mathcal{O}$$

is a ring morphism.

Proof. The product formula follows from the iterated application of the equalities

$$D_t(fg) = (D_tf)g + f(D_tg),\ \ (fg)(1)_\mu = f(1)_\mu\cdot g(1)_\mu,\ \ f'(1)_\mu = (D_tf)(1)_\mu,$$

where $D_t:\mathbb{R}[[t]]\to\mathbb{R}[[t]]$ is the formal differentiation operator $\frac{d}{dt}$. The last statement is an immediate application of the above product rule. □

Remark 2.7. The inclusion $\mathbb{R}[[t]]_C\subset\mathbb{R}[[t]]_A$ is strict. For example, the power series

$$f(z) = e^{1/(1+z)}$$

satisfies the assumption of Proposition 2.1 so that the associated formal power series $[f]$ is A-regular at 1. On the other hand, the arguments in [3, §5.12] show that $[f]$ is not C-regular at 1. □

Consider the translation operator $\boldsymbol{U}^h\in\mathcal{O}$. From Taylor's formula

$$p(x+h) = \sum_{n\geq 0}\frac{h^n}{n!}D^np(x)$$

we deduce that

$$\boldsymbol{\sigma}_{\boldsymbol{U}^h}(t) = e^{th}.$$

Set $\Delta_h := \boldsymbol{U}^h - 1$. Using Corollary 2.5 and Theorem 1.1 we deduce the following result.

Corollary 2.8. *For any $P \in \mathbb{R}[x]$ we have*

$$ {}^{C}\sum_{n\geq 0}(-1)^n P(x+nh) = \frac{1}{2}\left(\sum_{n\geq 0}\frac{(-1)^n}{2^n}\Delta_h^n\right)P(x). \tag{2.1}$$

Observe that

$$\left(1+\frac{1}{2}\Delta_h\right)\left(\sum_{n\geq 0}\frac{(-1)^n}{2^n}\Delta_h^n\right) = 1$$

so that $\frac{1}{2}\sum_{n\geq 0}\frac{(-1)^n}{2^n}\Delta_h^n$ is the inverse of the operator $2+\Delta_h$. We thus have

$$ {}^{C}\sum_{n\geq 0}(-1)^n P(x+nh) = (2+\Delta_h)^{-1}P(x) = (1+\boldsymbol{U}^h)^{-1}P(x). \tag{2.2}$$

Remark 2.9. Here is a heuristic explanation of the equality (2.2) assuming the Cesàro convergence of the series $\sum_{n\geq 0}(-1)^n P(x+nh)$. Denote by $S(x)$ the Cesàro sum of this series. Then

$$S(x+h) = {}^{C}\sum_{n\geq 0}(-1)^n P\big(\,x+(n+1)h\,\big)$$

$$\stackrel{(1.2)}{=} -{}^{C}\sum_{n\geq 0}(-1)^n P(x+h) + P(x) = -S(x) + P(x).$$

Hence

$$S(x+h)+S(x) = P(x),\quad \forall x\in\mathbb{R}.$$

If we knew that $S(x)$ is a polynomial we would then deduce

$$S(x) = (1+\boldsymbol{U}^h)^{-1}P(x). \qquad \square$$

The inverse of $1+\boldsymbol{U}^h$ can be explicitly expressed using Euler numbers and polynomials, [4, Eq. (14), p.134]. The Euler numbers E_k are defined by the Taylor expansion

$$\frac{1}{\cosh t} = \frac{2}{e^t+e^{-t}} = \sum_{k\geq 0}\frac{E_k}{k!}t^k.$$

Since $\cosh t$ is an even function we deduce that $E_k = 0$ for odd k. They satisfy the recurrence relation

$$E_n + \binom{n}{2}E_{n-2} + \binom{n}{4}E_{n-4} + \cdots = 0,\quad n\geq 2. \tag{2.3}$$

Here are the first few Euler numbers.

n	0	2	4	6	8	10	12	14	16
E_n	1	-1	5	-61	$1,385$	$-50,521$	$2,702,765$	$-199,360,981$	$19,391,512,145$

Then

$$\frac{1}{1+\boldsymbol{U}^h} = \frac{\boldsymbol{U}^{-\frac{h}{2}}}{\boldsymbol{U}^{\frac{h}{2}}+\boldsymbol{U}^{-\frac{h}{2}}} = \frac{\boldsymbol{U}^{-\frac{1}{2}}}{e^{\frac{D}{2}}+e^{-\frac{D}{2}}} = \frac{1}{2}\boldsymbol{U}^{-\frac{h}{2}}\frac{1}{\cosh\frac{hD}{2}} = \frac{1}{2}\boldsymbol{U}^{-\frac{h}{2}}\sum_{k\geq 0}\frac{E_k h^k}{2^k k!}D^k.$$

Hence

$$ {}^{C}\sum_{n\geq 0}(-1)^n P(x+nh) = \frac{1}{2}\sum_{k\geq 0}\frac{E_k h^k}{2^k k!}P^{(k)}\left(x-\frac{h}{2}\right). \tag{2.4}$$

When $P(x)=x^m$, $h=1$, we have
$$\overset{C}{\sum_{n\geq 0}}(-1)^n(x+n)^m=\frac{1}{2}\sum_{k\geq 0}\binom{m}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{m-k}. \tag{2.5}$$
Setting $x=0$ and using the equality $E_{2j+1}=0$, $\forall j$ we conclude that
$$\overset{C}{\sum_{n\geq 0}}(-1)^n n^m=\frac{1}{2^{m+1}}\sum_{k\geq 0}(-1)^{m-k}E_k\binom{m}{k}=\frac{(-1)^m}{2^{m+1}}\sum_{k\geq 0}E_{2k}\binom{m}{2k}. \tag{2.6}$$
Using (2.3) we deduce that when m is even, $m=2m'$, $m'>0$ we have
$$\overset{C}{\sum_{n\geq 0}}(-1)^n n^{2m'}=0. \tag{2.7}$$
For example
$$1-1+1-1+\cdots\overset{C}{=}\frac{1}{2}, \tag{$\dagger_0$}$$
$$-1+2-3+4-\cdots\overset{C}{=}-\frac{1}{4}, \tag{$\dagger_1$}$$
$$-1+2^3-3^3+4^3-\cdots\overset{C}{=}\frac{1}{8}, \tag{$\dagger_3$}$$
$$-1^5+2^5-3^5+4^5-\cdots\overset{C}{=}-\frac{1}{4}. \tag{$\dagger_5$}$$
When $P(x)=\binom{x}{m}$, $x=0$, $h=1$ then it is more convenient to use (2.1) because
$$\Delta\binom{x}{k}=\binom{x}{k-1},\;\;\forall k,x.$$
We deduce
$$\overset{C}{\sum_{n\geq 0}}(-1)^n\binom{n}{m}=\frac{1}{2}\sum_{k=1}^{m}\frac{(-1)^k}{2^k}\binom{0}{m-k}=\frac{(-1)^m}{2^{m+1}}. \tag{2.8}$$

Example 2.10. Consider the translation invariant operator
$$\boldsymbol{T}:\mathbb{R}[x]\to\mathbb{R}[x],\;\;P(x)\mapsto\int_0^\infty e^{-s}P(x+s)dx.$$
Set $\boldsymbol{R}=\boldsymbol{T}-1$. As explained in [1, II.3.B], the operators $\boldsymbol{T}$ and $\boldsymbol{R}$ are intimately related to the Laguerre polynomials. We have $\boldsymbol{R}=D\boldsymbol{T}=\boldsymbol{T}D$ and[3]
$$\sigma_{\boldsymbol{T}}(t)=\frac{1}{1-t}\sigma_{\boldsymbol{R}}(t)=\frac{1}{1-t}-1=\frac{t}{1-t}.$$
If $P\in\mathbb{R}[x]$ is a polynomial of degree m then
$$\boldsymbol{T}^kP(x)_{x=0}=(1+D+\cdots+D^m)P(x)_{x=0}$$
$$=\int_{\mathbb{R}^k_{\geq 0}}e^{-(s_1+s_2+\cdots+s_k)}P(s_1+\cdots+s_k)ds_1\cdots ds_k.$$
For $t\geq 0$ we denote by $\Delta_k(t)$ the $(k-1)$ simplex
$$\Delta_{k-1}(t):=\left\{(s_1,\dots,s_k)\in\mathbb{R}^k_{\geq 0};\;\;s_1+\cdots+s_k=t\right\}.,$$

[3]We can write formally $\boldsymbol{T}=\int_0^\infty e^{-s}\boldsymbol{U}^s ds=\int_0^\infty e^{-s(1-D)}ds=(1+D)^{-1}$, so that $\sigma_{\boldsymbol{T}}(t)=\frac{1}{1-t}$.

and by $dV_{k-1}(t)$ the Euclidean volume element on $\Delta_{k-1}(t)$. Integrating along the fibers of the function $f:\mathbb{R}^k_{\geq 0}\to[0,\infty)$, $f(s_1,\dots,s_k)=s_1+\cdots+s_k$ we deduce

$$\int_{\mathbb{R}^k_{\geq 0}} e^{-(s_1+s_2+\cdots+s_k)}P(s_1+\cdots+s_k)ds_1\cdots ds_k=\int_0^\infty\left(\int_{\Delta_{k-1}(t)}\frac{1}{|\nabla f|}dV_{k-1}(t)\right)e^{-t}P(t)dt$$
$$=\frac{v_{k-1}}{\sqrt{k}}\int_0^\infty e^{-s}s^{k-1}P(s)ds,$$

where v_{k-1} is the $(k-1)$-dimensional volume of the $(k-1)$-simplex $\Delta_{k-1}=\Delta_{k-1}(t)_{t=1}$.

To compute the volume v_{k-1} we view Δ_k is a regular k-simplex with distinguished base Δ_k, and distinguished vertex $(0,\dots,0,1)\in\mathbb{R}^{k+1}$. The distance d_k from the vertex to the base is the distance from the vertex to the center of the base. We have

$$d_k^2=1+\frac{1}{k},\ \ d_k=\sqrt{\frac{k+1}{k}},\ \ v_k=\frac{1}{k}d_kv_{k-1}=\left(\frac{k+1}{k^3}\right)^{1/2}v_{k-1}.$$

Since $v_0=1$ we deduce

$$v_k=\frac{(k+1)^{1/2}}{k!},\ \ \boldsymbol{T}^kP(x)_{x=0}=\frac{1}{(k-1)!}\int_0^\infty e^{-s}s^{k-1}P(s)ds,$$

and

$$\boldsymbol{R}^kP(x)_{x=0}=\frac{1}{(k-1)!}\int_0^\infty e^{-s}s^{k-1}P^{(k)}(s)ds.$$

Using Theorem 1.1 and Corollary 2.4 with the C-convenient series $f(t)=(1+t)^{-1}$ we deduce

$$ {}^C\sum_{n\geq 0}(-1)^n\boldsymbol{T}^nP(x)_{x=0}={}^C\sum_{n\geq 0}(-1)^n\frac{1}{(n-1)!}\int_0^\infty e^{-s}s^{n-1}P(s)ds$$
$$=\int_0^\infty\left(\sum_{k=0}^{\deg P}\frac{(-1)^k}{2^{k+1}(k-1)!}s^{k-1}P^{(k)}(s)\right)ds.$$

If we let $P(s)=s^m$ we deduce

$$\int_0^\infty e^{-s}s^{n-1}P(s)ds=(m+n-1)!,\ \ \int_0^\infty e^{-s}s^{k-1}P^{(k)}(s)ds=[m]_k(m-1)!=[m-1]_{k-1}m!,$$

and

$$ {}^C\sum_{n\geq 0}(-1)^n\binom{m+n-1}{m}=\sum_{k=0}^m\frac{(-1)^k}{2^{k+1}}\binom{m-1}{k-1}. \tag{2.9}$$

Let us point out that (2.9) can be obtained from (2.8) using the shift-invariance of the Cesàro regularization method. □

REFERENCES

[1] M. Aigner: *Combinatorial Theory*, Springer Verlag, 1997.
[2] P. Flajolet, R. Sedgewick: *Analytic Combinatorics*, Cambridge University Press, 2009.
[3] G.H. Hardy: *Divergent Series*, Chelsea Publishing Co. 1991.
[4] N.E. Nörlund: *Mémoire sur les polynomes de Bernoulli*, Acta Math. **43**(1922), 121-194.

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