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Tame Flows

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Contents

Abstract

The tame flows are "nice" flows on "nice" spaces. The nice (tame) sets are the pfaffian sets introduced by Khovanski, and a flow $\Phi : \mathbb{R} \times X \to X$ on pfaffian set X is tame if the graph of Φ is a pfaffian subset of $\mathbb{R} \times X \times X$. Any compact tame set admits plenty tame flows. We prove that the flow determined by the gradient of a generic real analytic function with respect to a generic real analytic metric is tame. The typical tame gradient flow satisfies the Morse-Smale condition, and we prove that in the tame context, under certain spectral constraints, the Morse-Smale condition implies the fact that the stratification by unstable manifolds is Verdier and Whitney regular. We explain how to compute the Conley indices of isolated stationary points of tame flows in terms of their unstable varieties, and then give a complete classification of gradient like tame flows with finitely many stationary points. We use this technology to produce a Morse theory on posets generalizing R. Forman's discrete Morse theory. Finally, we use the Harvey-Lawson finite volume flow technique to produce a homotopy between the DeRham complex of a smooth manifold and the simplicial chain complex associated to a triangulation.

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Introduction

Loosely speaking, the tame sets (respectively tame flows) are sets (respectively continuous flows) which display very few pathologies. Technically speaking, they are sets or flows definable within a tame structure.

The subject of *o*-minimal or tame geometry is not as popular as it ought to be in geometric circles, although this situation is beginning to change. The tame geometry is a vast generalization of the more classical subject of real algebraic geometry. One such extension of real algebraic geometry was conceived and investigated by A. Khovanski in [**28**], and our tame sets are generalizations of Khovanski's pfaffian sets. In particular, all the tame sets will be subsets of Euclidean spaces.

If we think of a flow as generated by a system of ordinary differential equations then, roughly speaking, the tame flows correspond to first order ordinary differential equations which we can solve explicitly by quadratures, with one important caveat: the resulting final description of the solutions should not involve trigonometric functions because tame flows do not have periodic orbits. For example, an autonomous linear system of ordinary differential equations determines a tame flow if and only if the defining matrix has only real eigenvalues.

Given that the tame sets display very few pathologies, they form a much more restrictive class of subsets of Euclidean spaces, and in particular, one might expect that the tame flows are not as plentiful. In the present paper we set up to convince the reader that there is a rather large supply of such flows, and that they are worth investigating due to their rich structure.

The paper is structured around three major themes: examples of tame flows, properties of tame flows, and applications of tame flows.

To produce examples of tame flows we describe several general classes of tame flows, and several general surgery like operations on tame flows which generate new flows out of old ones. These operations have a simplicial flavor: we can *cone* and *suspend* a flow, we can *join* two flows, or we can *glue* two flows along a common, closed invariant subset.

The simplest example of tame flow is the trivial flow on a set consisting of single point. An iterated application of the cone operations produces canonical tame flows on any affine m-simplex, and then by gluing, on any triangulated tame set. Since any tame set can be triangulated, we conclude that there exist many tame flows on any tame set.

Another class of tame flows, which cannot be obtained by the cone operation, consists of the gradient flows of "most" real analytic functions on a real analytic manifold equipped with a real analytic metric.

More precisely, we prove that, for any real analytic f function on a real analytic manifold M, there exists a dense set of real analytic metrics g with the property that the flow generated by $\nabla^g f$ is tame. This is a rather nontrivial result, ultimately based on the Poincaré-Siegel theorem concerning the canonical form of a vector field in a neighborhood of a stationary point. The Poincaré-Siegel theorem plays the role of the more elementary Morse lemma.

The usual techniques pioneered by Smale show that a tame gradient flow can be slightly modified to a gradient like tame flow satisfying the Morse-Smale regularity conditions.

We investigate the stratification of a manifold given by the unstable manifolds of the downward gradient flow of some real analytic function. We prove that this stratification satisfies the Whitney regularity condition (a) if and only if the flow satisfies the Morse-Smale transversality conditions.

The method of proof is essentially a "microlocalization" of the Morse flow and allows us to draw even stronger conclusions. More precisely, we show that if the tame gradient flow associated to a real analytic function f and metric g satisfies the Morse-Smale condition, and if for every unstable critical point x of f, the spectrum Σ_x of the Hessian of f at x satisfies the *clustering condition*

$$\max \Sigma_x^+ < \operatorname{dist}(\Sigma_x^+, 0) + \operatorname{dist}(\Sigma_x^-, 0), \text{ where } \Sigma_x^{\pm} := \big\{ \lambda \in \Sigma_x;\ \ \pm\lambda > 0 \big\},$$

then the stratification by unstable manifolds satisfies the Verdier regularity condition. Again, the Poincaré-Siegel theorem shows that set of tame gradient flows satisfying the spectral clustering condition above is nonempty and "open".

In the tame world, the Verdier condition implies the Whitney regularity conditions. We deduce that the unstable manifolds of a tame Morse-Smale flows satisfying the spectral clustering condition form a Whitney stratification. The results of F. Laundebach [**30**] on the local conical structure of the stratification by the unstable manifolds follow from the general results on the local structure of a Whitney stratified space.

The clustering condition is in a sense necessary because we produced examples of Morse-Smale flows violating this condition, and such that the stratification by unstable manifold is not Whitney regular, and thus, not Verdier regular; see Remark 8.8 (b),(c).

As far as (stratified) Morse theory goes, the Verdier regularity condition is a more appropriate condition than Whitney's regularity condition since, according to Kashiwara-Schapira [**26**, Cor. 8.3.24], a Verdier stratification has no exceptional points in the sense defined by Goresky-MacPherson in [**17**, Part I, Sect. 1.8].

Let us observe that if the stratification by the unstable manifolds of the downward gradient flow of a Morse function f on a compact real analytic manifold M satisfies the Verdier condition, then the Morse function can be viewed as a *stratified* Morse function with respect to two different stratifications. The first one, is the trivial stratification with a single stratum, the manifold M itself. The second stratification is the stratification given by the unstable manifolds.

We also investigate Morse like tame flows on singular spaces, i.e., tame flows which admit a Lyapunov function. We explain how to compute the (homotopic) Conley index of an isolated stationary point in terms of the unstable variety of that point. We achieve this by proving a singular counterpart of the classical result in Morse theory: crossing a critical level of a Morse function corresponds homotopically to attaching a cell of a certain dimension. Since we are working on singular spaces the change in the homotopy type is a bit more complicated, but again, crossing a critical level has a similar homotopic flavor. The sublevel sets of the Lyapunov function change by a cone attachment. The cone has a very precise dynamical description, namely it is the cone spanned by the trajectories of the flow "exiting" the stationary point.

The arguments used in the computation of these Conley indices lead to an almost complete classification of gradient like tame flows on compact tame spaces. This classification resembles the classical result of Smale: the gradient flow of a Morse function produces a handle decomposition of the underlying manifold, and conversely, any handle decomposition can be obtained in this fashion. When working on singular spaces, the operation of handle addition is replaced by a so called flip-flop. This mimics the classical operation in algebraic geometry, a blowdown followed by a blowup.

We use the Conley index computation in the study of certain Morse like flows on simplicial complexes. The nerves of finite posets[1] are special examples of simplicial complexes. To any poset $(P,<)$, and any isotone map $\pi : (P,<) \to (Q,\prec)$ such that every nonempty fiber $\pi^{-1}(q) \subset P$ has a unique $<$-minimal element, we associate a tame flow on the nerve of P whose stationary points are the vertices of the nerve, i.e., the elements of P. These are gradient like flows in the sense that they admit piecewise linear functions decreasing strictly along the nonconstant trajectories. The Conley indices are determined from the combinatorics of the map $\pi : P \to Q$.

When we specialize the general theory to the case of poset of faces $\mathcal{F}(X)$ of a regular CW decomposition of a space X we obtain, as a very special case, R. Forman's discrete Morse theory, [**14**]. The combinatorial Morse functions of Forman correspond to isotone maps $(\mathcal{F}(X),<) \to (Q,\prec)$ such that the fiber over each point consists of an order interval of length ≤ 1.

In fact, even in this case the general theory suggests a more flexible definition of what should constitute a combinatorial Morse function which addresses one limitation of combinatorial Morse theory, namely, the scarcity of combinatorial Morse functions. We describe an increasing sequence $\mathcal{M}_1(K) \subset \mathcal{M}_2(K) \subset \cdots$ of sets of Morse like functions defined on the faces of a simplicial complex K. Their union is denoted by $\mathcal{M}(K)$.

The smallest of these sets, $\mathcal{M}_1(K)$, consists of the functions introduced by R. Forman himself. As we go higher in this sequence, we obtain larger supplies of Morse like functions, but we have to pay a price for this, since the local structure of their critical points becomes more complicated. However, we still have a simple way of eliminating the homotopically irrelevant faces.

A function $f \in \mathcal{M}(K)$ defines a piecewise linear function $\tilde{f}$ on the geometric realization of K. The function $\tilde{f}$ is a genuine stratified Morse function with respect to the stratification given by the open faces of the first barycentric subdivision.

A function $f \in \mathcal{M}(K)$ also defines a canonical tame flow on K such that the faces of K are invariant subsets. The stationary points of this flow are the barycenters of the faces of K. These stationary points also coincide with the critical points of the corresponding stratified Morse function $\tilde{f}$, and the Goresky-MacPherson local Morse datum of a stratified critical point is homotopic with the Conley index of that point viewed as a stationary point of the associated tame flow.

We also blend the tameness with the finite volume techniques of Harvey-Lawson to prove that the DeRham complex of a compact, orientable smooth manifold is naturally homotopic to the simplicial chain complex (with real coefficients) of a

[1]The nerve of a poset is the (combinatorial) simplicial complex whose simplices are the linearly ordered subsets of P.

triangulation of the manifold. This implies, among other things, that for a compact oriented, real analytic manifolds, the DeRham complex, Morse-Floer complex associated to a a Morse-Smale flow, and simplicial complex associated to a triangulation are *naturally* homotopic, so they define isomorphic objects in the derived category of bounded complexes of real vector spaces.

Here is briefly the organization of the paper. Chapter 1 is a crash course in tame geometry where we define precisely the meaning of the attribute "tame" and list without proofs a few geometric consequences of tameness used throughout the paper.

In Chapters 2 and 3 we describe a large list of examples of tame flows, and prove several elementary properties of an arbitrary tame flow. In particular, in these Chapters we describe in detail some canonical tame flows on affine simplices (Example 2.10), and on Grassmannians (Example 2.13) which will play an important role in the paper.

Chapters 4-8 are devoted to gradient flows determined by real analytic functions on real analytic manifolds equipped with real analytic metrics. We prove that "most" of these flows are tame (Theorem 4.5), they satisfy the Morse-Smale condition (Theorem 5.1), and moreover, that the Morse-Smale condition is (essentially) equivalent with the fact that the stratification by unstable manifolds is Verdier and Whitney regular (Theorem 8.1, 8.2).

In Chapter 9 we describe how to compute the Conley index of an isolated stationary point of a tame flow admitting Lyapunov functions in terms of the unstable variety of that point (Theorem 9.10).

In Chapter 10 (Theorem 10.4) we produce a complete topological classification of gradient like tame flows with finitely many stationary points on compact tame spaces.

In Chapter 11 we use the Conley index computations to investigate the homotopy type of posets by using certain tame flows associated to certain discrete Morse like functions on posets (Theorem 11.3, 11.18). We also prove (Proposition 11.12, Corollary 11.14) a generalization of a theorem of M. Chary [**5**] and D. Kozlov [**29**, Thm. 11.2, 11.4].

In the last Chapter we explain how to use the Harvey-Lawson techniques to produce results about the homotopy type of the DeRham complex (Theorem 12.11).

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CHAPTER 1

Tame spaces

Since the subject of tame geometry is not very familiar to many geometers we devote this section to a brief introduction to this topic. Unavoidably, we will have to omit many interesting details and contributions, but we refer to [**8, 10, 12**] for more systematic presentations. For every set X we will denote by $\mathcal{P}(X)$ the collection of all subsets of X.

An $\mathbb{R}$-*structure*[1] is a collection $\mathcal{S} = \{ \mathcal{S}^n \}_{n \geq 1}$, $\mathcal{S}^n \subset \mathcal{P}(\mathbb{R}^n)$, with the following properties.

- **E**$_1$**:** $\mathcal{S}^n$ contains all the real algebraic subvarieties of $\mathbb{R}^n$, i.e., the zero sets of finite collections of polynomial in n real variables.
- **E**$_2$**:** For every linear map $L : \mathbb{R}^n \to \mathbb{R}$, the half-plane $\{\vec{x} \in \mathbb{R}^n; \;\; L(x) \geq 0\}$ belongs to $\mathcal{S}^n$.
- **P**$_1$**:** For every $n \geq 1$, the family $\mathcal{S}^n$ is closed under boolean operations, $\cup$, $\cap$ and complement.
- **P**$_2$**:** If $A \in \mathcal{S}^m$, and $B \in \mathcal{S}^n$, then $A \times B \in \mathcal{S}^{m+n}$.
- **P**$_3$**:** If $A \in \mathcal{S}^m$, and $T : \mathbb{R}^m \to \mathbb{R}^n$ is an affine map, then $T(A) \in \mathcal{S}^n$.

Example 1.1 (Semialgebraic sets). Denote by $\mathcal{S}_{alg}$ the collection of real semialgebraic sets. Thus, $A \in \mathcal{S}^n_{alg}$ if and only if A is a finite union of sets, each of which is described by finitely many polynomial equalities and inequalities. The celebrated Tarski-Seidenberg theorem states that $\mathcal{S}_{alg}$ is a structure.

For example, the set

$$A = \big\{ (x, a_0, \dots, a_{n-1}) \in \mathbb{R}^{n+1}; a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n = 0 \big\}$$

is real algebraic, and Tarski-Seidenberg theorem implies that its projection on the plane with coordinates a_i, $0 \leq i \leq n-1$,

$$\big\{ (a_0, \dots, a_{n-1}) \in \mathbb{R}^n; \;\; \exists x \in \mathbb{R} : \;\; a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n = 0 \big\}$$

is semialgebraic. □

Given a structure $\mathcal{S}$, then an $\mathcal{S}$-*definable* set is a set that belongs to one of the $\mathcal{S}^n$-s. If A, B are $\mathcal{S}$-definable, then a function $f : A \to B$ is called $\mathcal{S}$-*definable* if its graph

$$\Gamma_f := \big\{ (a, b) \in A \times B; \;\; b = f(a) \big\}$$

is $\mathcal{S}$-definable. The reason these sets are called definable has to do with mathematical logic.

[1]This is a highly condensed and special version of the traditional definition of structure. The model theoretic definition allows for ordered fields, other than $\mathbb{R}$, such as extensions of $\mathbb{R}$ by "infinitesimals". This can come in handy even if one is interested only in the field $\mathbb{R}$.

A *formula*[2] is a property defining a certain set. For example, the two different looking formulas

$$\{x \in \mathbb{R};\ \ x \geq 0\},\ \ \{x \in \mathbb{R};\ \ \exists y \in \mathbb{R}:\ \ x = y^2\},$$

describe the same set, $[0, \infty)$.

Given a collection of formulas, we can obtain new formulas, using the logical operations $\wedge, \vee, \neg$, and quantifiers $\exists$, $\forall$. If we start with a collection of formulas, each describing an $\mathcal{S}$-definable set, then any formula obtained from them by applying the above logical transformations will describe a definable set.

To see this, observe that the operators $\wedge, \vee, \neg$ correspond to the boolean operations, $\cap, \cup$, and taking the complement. The existential quantifier corresponds to taking a projection. For example, suppose we are given a formula $\phi(a, b)$, $(a, b) \in A \times B$, A, B definable, describing a definable set $C \subset A \times B$. Then the formula

$$\{\, a \in A;\ \exists b \in B:\ \ \phi(a, b)\,\}$$

describes the image of the subset $C \subset A \times B$ via the canonical projection $A \times B \to A$. If $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$, then the projection $A \times B \to A$ is the restriction to $A \times B$ of the linear projection $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{P}_3$ implies that the image of C is also definable. Observe that the universal quantifier can be replaced with the operator $\neg\exists\neg$.

EXAMPLE 1.2. (a) The composition of two definable functions $A \stackrel{f}{\to} B \stackrel{g}{\to} C$ is a definable function because

$$\Gamma_{g \circ f} = \{\, (a, c) \in A \times C; \exists b \in B : (a, b) \in \Gamma_f,\ \ (b, c) \in \Gamma_g \,\}.$$

Note that any polynomial with real coefficients is a definable function.
(b) The image and the preimage of a definable set via a definable function is a definable set. Using $\mathbf{E}_2$ we deduce that any semialgebraic set $\mathcal{S}$ is definable. In particular, the Euclidean norm

$$|\bullet| : \mathbb{R}^n \to \mathbb{R},\ \ |(x_1, \dots, x_n)| = \Big(\sum_{i=1}^n x_i^2\Big)^{1/2}$$

is $\mathcal{S}$-definable.
(c) Suppose $A \subset \mathbb{R}^n$ is definable. Then its closure $\boldsymbol{cl}(A)$ is described by the formula

$$\{x \in \mathbb{R}^n;\ \ \forall \varepsilon > 0,\ \ \exists a \in A:\ \ |x - a| < \varepsilon\,\},$$

and we deduce that $\boldsymbol{cl}(A)$ is also definable. Let us examine the correspondence between the operations on formulas and operations on sets on this example.

We rewrite this formula as

$$\forall \varepsilon \Big(\, (\varepsilon > 0) \Rightarrow \exists a (a \in A) \wedge (x \in \mathbb{R}^n) \wedge (|x - a| < \varepsilon)\,\Big).$$

In the above formula we see one free variable x, and the set described by this formula consists of those x for which that formula is a true statement.

The above formula is made of the "atomic" formulæ,

$$(a \in A),\ \ (x \in \mathbb{R}^n),\ \ (|x - a| < \varepsilon),\ \ (\varepsilon > 0),$$

[2]We are deliberately vague on the meaning of formula.

which all describe definable sets. The logical connector $\Rightarrow$ can be replaced by $\vee\neg$. Finally, we can replace the universal quantifier to rewrite the formula as a transform of atomic formulas via the basic logical operations.

$$\neg\Big\{\exists\varepsilon\neg\Big((\varepsilon>0)\Rightarrow\exists a(a\in A)\wedge(x\in\mathbb{R}^n)\wedge(|x-a|<\varepsilon)\Big)\Big\}. \qquad \square$$

Given an $\mathbb{R}$-structure $\mathcal{S}$, and a collection $\mathcal{A}=(\mathcal{A}_n)_{n\geq 1}$, $\mathcal{A}_n\subset\mathcal{P}(\mathbb{R}^n)$, we can form a new structure $\mathcal{S}(\mathcal{A})$, which is the smallest structure containing $\mathcal{S}$ and the sets in $\mathcal{A}_n$. We say that $\mathcal{S}(\mathcal{A})$ is obtained from $\mathcal{S}$ by *adjoining the collection* $\mathcal{A}$.

DEFINITION 1.3. An $\mathbb{R}$-structure is called *o-minimal* (order minimal) or *tame* if it satisfies the property

O: Any set $A\in\mathcal{S}^1$ is a *finite* union of open intervals (a,b), $-\infty\leq a<b\leq\infty$, and singletons $\{r\}$. $\square$

EXAMPLE 1.4. (a) The collection $\mathcal{S}_{alg}$ of real semialgebraic sets is a tame structure.
(b)(Gabrielov, Hironaka, Hardt, [**15, 24, 22**]) A *restricted* real analytic function is a function $f:\mathbb{R}^n\to\mathbb{R}$ with the property that there exists a real analytic function $\tilde{f}$ defined in an open neighborhood U of the cube $C_n:=[-1,1]^n$ such that

$$f(x)=\begin{cases}\tilde{f}(x) & x\in C_n\\ 0 & x\in\mathbb{R}^n\setminus C_n.\end{cases}$$

we denote by $\mathcal{S}_{\mathrm{an}}$ the structure obtained from $\mathcal{S}_{alg}$ by adjoining the graphs of all the restricted real analytic functions. Then $\mathcal{S}_{\mathrm{an}}$ is a tame structure, and the $\mathcal{S}_{\mathrm{an}}$-definable sets are called *globally subanalytic sets.*
(c)(Wilkie, van den Dries, Macintyre, Marker, [**11, 50**]) The structure obtained by adjoining to $\mathcal{S}_{an}$ the graph of the exponential function $\mathbb{R}\to\mathbb{R}$, $t\mapsto e^t$, is a tame structure.
(d)(Khovanski, Speissegger, Wilkie, [**28, 43, 50**]) There exists a tame structure $\mathcal{S}'$ with the following properties

(d_1) $\mathcal{S}_{an}\subset\mathcal{S}'$
(d_2) If $U\subset\mathbb{R}^n$ is open, connected and $\mathcal{S}'$-definable, $F_1,\dots,F_n:U\times\mathbb{R}\to\mathbb{R}$ are $\mathcal{S}'$-definable and C^1, and $f:U\to\mathbb{R}$ is a C^1 function satisfying

$$\frac{\partial f}{\partial x_i}=F_i(x,f(x)),\ \ \forall x\in\mathbb{R},,\ \ i=1,\dots,n, \tag{1.1}$$

then f is $\mathcal{S}'$-definable.

The smallest structure satisfying the above two properties, is called the *pfaffian closure*[3] of $\mathcal{S}_{\mathrm{an}}$, and we will denote it by $\widehat{\mathcal{S}}_{\mathrm{an}}$.

Observe that if $f:(a,b)\to\mathbb{R}$ is C^1, $\widehat{\mathcal{S}}_{\mathrm{an}}$-definable, and $x_0\in(a,b)$ then the antiderivative $F:(a,b)\to\mathbb{R}$

$$F(x)=\int_{x_0}^x f(t)dt,\ \ x\in(a,b),$$

is also $\widehat{\mathcal{S}}_{\mathrm{an}}$-definable. $\square$

[3]Our definition of pfaffian closure is more restrictive than the original one in [**28, 43**], but it suffices for the geometrical applications we have in mind.

The definable sets and function of a tame structure have rather remarkable *tame* behavior which prohibits many pathologies. It is perhaps instructive to give an example of function which is not definable in any tame structure. For example, the function $x \mapsto \sin x$ is not definable in a tame structure because the intersection of its graph with the horizontal axis is the countable set $\pi\mathbb{Z}$ which violates the o-minimality condition **O**.

We will list below some of the nice properties of the sets and function definable in a tame structure $\mathcal{S}$. Their proofs can be found in [**10**].

• (*Piecewise smoothness of one variable tame functions.*) If $f : (0,1) \to \mathbb{R}$ is an $\mathcal{S}$-definable function, and p is a positive integer, then there exists

$$0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$$

such that the restriction of f to each subinterval (a_{i-1}, a_i) is C^p and monotone. Moreover f admits right and left limits at any $t \in [0,1]$.

• (*Closed graph theorem.*) Suppose X is a tame set and $f : X \to \mathbb{R}^n$ is a tame bounded function. Then f is continuous if and only if its graph is closed in $X \times \mathbb{R}^n$.

• (*Curve selection.*) If A is an $\mathcal{S}$-definable set, and $x \in \boldsymbol{cl}(A) \setminus A$, then there exists an $\mathcal{S}$ definable continuous map

$$\gamma : (0,1) \to A$$

such that $x = \lim_{t\to 0} \gamma(t)$.

• Any definable set has finitely many connected components, and each of them is definable.

• Suppose A is an $\mathcal{S}$-definable set, p is a positive integer, and $f : A \to \mathbb{R}$ is a definable function. Then A can be partitioned into finitely many $\mathcal{S}$ definable sets $S_1, \dots, S_k$, such that each S_i is a C^p-manifold, and each of the restrictions $f|_{S_i}$ is a C^p-function.

• (*Triangulability.*) For every compact definable set A, and any finite collection of definable subsets $\{S_1, \dots, S_k\}$, there exists a compact simplicial complex K, and a definable homeomorphism

$$\Phi : K \to A$$

such that all the sets $\Phi^{-1}(S_i)$ are unions of relative interiors of faces of K.

• (*Definable selection.*) Suppose A, Λ are $\mathcal{S}$-definable. Then a *definable* family of subsets of A parameterized by Λ is a definable subset

$$S \subset A \times \Lambda.$$

We set

$$S_\lambda := \big\{ a \in A;\ \ (a,\lambda) \in S \big\},$$

and we denote by Λ_S the projection of S on Λ. Then there exists a definable function $s : \Lambda_S \to A$ such that

$$s(\lambda) \in S_\lambda,\ \ \forall \lambda \in \Lambda_S.$$

• (*Dimension.*) The dimension of an $\mathcal{S}$-definable set $A \subset \mathbb{R}^n$ is the supremum over all the nonnegative integers d such that there exists a C^1 submanifold of $\mathbb{R}^n$ of dimension d contained in A. Then $\dim A < \infty$, and

$$\dim(\boldsymbol{cl}(A) \setminus A) < \dim A.$$

Moreover, if $(S_\lambda)_{\lambda \in \Lambda}$ is a definable family of definable sets then the function

$$\Lambda \ni \lambda \mapsto \dim S_\lambda$$

is definable.

• (*Definable triviality of tame maps.*) We say that a tame map $\Phi : X \to S$ is *definably trivial* if there exists a definable set F, and a definable homeomorphism $\tau : X \to F \times S$ such that the diagram below is commutative

$$\begin{array}{ccc} X & \xrightarrow{\ \tau\ } & S \times F \\ & {\scriptstyle\Phi}\searrow \quad \swarrow{\scriptstyle\pi_S} & \\ & S & \end{array}$$

If $\Psi : X \to Y$ is a definable map, and p is a positive integer, then there exists a partition of Y into definable C^p-manifolds $Y_1, \dots, Y_k$ such that each the restrictions

$$\Psi : \Psi^{-1}(Y_k) \to Y_k$$

is definably trivial.

• (*Definability of Euler characteristic.*) Suppose $(S_\lambda)_{\lambda \in \Lambda}$ is a definable family of compact tame sets. Then the map

$$\Lambda \ni \lambda \mapsto \chi(S_\lambda) = \text{the Euler characteristic of } S_\lambda \in \mathbb{Z}$$

is definable. In particular, the set

$$\big\{\, \chi(S_\lambda);\ \ \lambda \in \Lambda \,\big\} \subset \mathbb{Z}$$

is finite.

• (*Scissor equivalence principle.*) Suppose S_0, S_1 are two tame sets. We say that they are *scissor equivalent* if there exist a tame bijection $F : S_0 \to S_1$. (The bijection F *need not be continuous.*) Then S_0 and S_1 are scissor equivalent if and only if they have the same dimension and the same Euler characteristic.

• (*Crofton formula.*, [**4**], [**13**, Thm. 2.10.15, 3.2.26]) Suppose E is an Euclidean space, and denote by $\mathbf{Graff}^k(E)$ the Grassmannian of affine subspaces of codimension k in E. Fix an invariant measure μ on $\mathbf{Graff}^k(E)$. μ is unique up to a multiplicative constant. Denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. Then there exists a constant $C > 0$, depending only on μ, such that for every compact, k-dimensional tame subset $S \subset E$ we have

$$\mathcal{H}^k(S) = C \int_{\mathbf{Graff}^k(E)} \chi(L \cap S)\, d\mu(L).$$

• (*Finite volume.*) Any compact k-dimensional tame set has finite k-dimensional Hausdorff measure.

• (*Tame quotients.*) Suppose X is a tame set, and $E \subset X \times X$ is a tame subset defining an equivalence relation on X. We assume that the natural projection $\pi : E \to X$ is *definable proper*, i.e., for any compact tame subset $K \subset X$ the preimage $\pi^{-1}(K) \subset E$ is compact. Then the quotient space X/E can be realized as a tame set, i.e., there exists a tame set Y, and a tame continuous surjective map $p : X \to Y$ satisfying the following properties:

(Q_1) $p(x) = p(y) \Longleftrightarrow (x, y) \in E$.
(Q_2) p is definable proper.

The pair (Y, p) is called the *definable quotient* of X mod E. It is a quotient in the category of tame sets and tame continuous map in the sense that, for any tame continuous function $f : X \to Z$ such that $(x, y) \in E \Longrightarrow f(x) = f(y)$, there exists a unique tame continuous map $\bar{f} : Y \to Z$ such that the diagram below is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Z \\ {\scriptstyle p}\downarrow & \nearrow_{\bar{f}} & \\ Y & & \end{array}$$

□

In the sequel we will work *exclusively* with the tame structure $\widehat{\mathcal{S}}_{\mathrm{an}}$. We will refer to the $\widehat{\mathcal{S}}_{\mathrm{an}}$-definable sets (functions) as *tame* sets (or functions), or *definable* sets (functions).

CHAPTER 2

Basic properties and examples of tame flows

We can now introduce the subject of our investigation.

DEFINITION 2.1. A *tame flow* on a tame set X is a continuous flow

$$\Phi : \mathbb{R} \times X \to X, \ \ \mathbb{R} \times X \ni (t, x) \to \Phi_t(x),$$

such that Φ is a tame map. □

If Φ is a tame flow on a tame set X, we denote by $\mathbf{Cr}_\Phi$ the set of stationary points of the flow. Observe that $\mathbf{Cr}_\Phi$ is a tame subset of X.

DEFINITION 2.2. Suppose Φ is a tame flow on the tame set X. Then a *tame Lyapunov* function for Φ is a tame continuous function $f : X \to \mathbb{R}$ which decreases strictly along the nonconstant trajectories of Φ, and it is constant on the path components of $\mathbf{Cr}_\Phi$. We say that a tame flow is *gradient like* if it admits a Lyapunov function. □

PROPOSITION 2.3. *(a) If Φ is a tame flow on the tame set X, and $F : X \to Y$ is a tame homeomorphism then the conjugate*

$$\Psi_t = F \circ \Phi_t \circ F^{-1} : Y \to Y$$

is also a tame flow.
(b) If Φ is a tame flow on the tame set X, and Ψ is a tame flow on the tame set Y, then the product flow on $X \times Y$,

$$\Phi \times \Psi : \mathbb{R} \times X \times Y \to X \times Y, \ \ (t, x, y) \mapsto (\Phi_t(x), \Psi_t(y))$$

is tame. Moreover, if f is a tame Lyapunov function for Φ, and g is a tame Lyapunov function for Ψ, then

$$f \boxplus g : X \times Y \to \mathbb{R}, \ \ f \boxplus g(x, y) = f(x) + g(y),$$

is a tame Lyapunov function for $\Phi \times \Psi$.
(c) If Φ is a tame flow, then its opposite $\tilde{\Phi}_t := \Phi_{-t}$ is also a tame flow.
(d) If Φ is a tame flow on the tame space X and Y is a Φ-invariant tame subspace then the restriction of Φ to Y is also a tame flow.
(e) Suppose X is a tame set, and Y_1, Y_2 are compact tame subsets. Suppose Φ^k is a tame flow on Y_k, $k = 1, 2$, such that $Y_1 \cap Y_2$ is Φ^k invariant, $\forall k = 1, 2$, and

$$\Phi^1|_{Y_1 \cap Y_2} = \Phi^2|_{Y_1 \cap Y_2}.$$

Then there exists a unique tame flow Φ on X such that

$$\Phi|_{Y_k} = \Phi^k, \ \ k = 1, 2.$$

Moreover, if $f_k : Y_k \to \mathbb{R}$, $k = 1, 2$ is a tame Lyapunov function for Φ^k and

$$f_1|_{Y_1 \cap Y_2} = f_2|_{Y_1 \cap Y_2}$$

then the function

$$f_1 \# f_2 : X \to \mathbb{R}, \quad (f_1 \# f_2)(x) = \begin{cases} f_1(x) & x \in Y_1 \\ f_2(x) & x \in Y_2 \end{cases}$$

is a tame Lyapunov function for Φ.

PROOF. We prove only (a). The map $\Psi : \mathbb{R} \to X \to X$, $(t, x) \mapsto F \circ \Phi_t(F^{-1}(x))$ can be written as the composition of tame maps

$$\mathbb{R} \times Y \stackrel{\mathbb{1} \times F^{-1}}{\longrightarrow} \mathbb{R} \times X \stackrel{\Phi}{\longrightarrow} X \stackrel{F}{\to} Y.$$

□

EXAMPLE 2.4. The translation flow on $\mathbb{R}$ given by

$$T_t(x) = x + t, \ \ \forall t, x \in \mathbb{R}$$

is tame since its graph is the graph of $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The identity $I_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is a tame Lyapunov function for the opposite flow. □

EXAMPLE 2.5. Let $X = [0, 1]$, and consider the flow Φ on X generated by the vector field

$$\xi = x(x - 1)\partial_x.$$

The function $t \mapsto x(t) = \Phi_t(x_0)$ satisfies the initial value problem

$$\dot{x} = x(x - 1), \ \ x(0) = x_0.$$

If $x_0 \in \{0, 1\}$ then $x(t) \equiv x_0$. If $x_0 \in (0, 1)$ then we deduce

$$\frac{dx}{x(x - 1)} = dt \Longleftrightarrow \frac{dx}{x} - \frac{d(1 - x)}{1 - x} = -dt$$

so that

$$\log \frac{x}{1 - x} - \log \frac{x_0}{(1 - x_0)} = -t.$$

Hence

$$\frac{x}{1 - x} = r(x_0, t) := e^{-t} \frac{x_0}{1 - x_0} \Longleftrightarrow x(t) = \frac{r(x_0, t)}{1 + r(x_0, t)} = \frac{e^{-t} x_0}{1 - x_0 + e^{-t} x_0}. \tag{2.1}$$

This shows that Φ is tame and its restriction to $(0, 1)$ is tamely conjugate to the translation flow. The identity function $[0, 1] \to [0, 1]$ is a Lyapunov function for this flow. We will refer to Φ as the *canonical downward flow* on $[0, 1]$. □

EXAMPLE 2.6. Consider the unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2; \ \ x^2 + y^2 = 1\}.$$

The height function $h_0 : S^1 \to \mathbb{R}$, $h_0(x, y) = y$, is a real analytic Morse function on S^1. Define

$$U^+ := S^1 \cap \{x > 0\}, \ \ U^- := S^1 \cap \{x < 0\}.$$

Along U^+ we can use y as coordinate, and we have $d(x^2 + y^2) = 0$, so that

$$dx = -\frac{y}{x} dy \Longrightarrow dx^2 + dy^2 = \frac{1}{1 - y^2} dy^2.$$

The gradient of h_0 with respect to the round metric $\frac{1}{1-y^2}dy^2$ is then $\xi_0 := (1-y^2)\partial_y$ so that the descending gradient flow of h (with respect to this metric) is given in the coordinate y by

$$\dot{y} = -(1-y^2).$$

This flow is tamely conjugate to the flow in Example 2.5 via the linear increasing homeomorphism $[-1,1] \to [0,1]$. Thus the gradient flow of the height function on the round circle is tame. Note that this flow is obtained by gluing two copies of the standard decreasing flow on $[0,1]$. □

EXAMPLE 2.7 (*A simple non tame flow*). Consider the rotational flow on the unit circle

$$R : \mathbb{R} \times S^1 \to S^1, \;\; R_t(e^{\boldsymbol{i}\theta}) = e^{\boldsymbol{i}(t+\theta)}.$$

This flow is not tame because the set

$$A = \big\{ t \in \mathbb{R}; \;\; R_t(1) = e^{\boldsymbol{i}t} = 1 \,\big\} = 2\pi\mathbb{Z}$$

is not tame.

We deduce from this simple example that a tame flow cannot have nontrivial periodic orbits because the restriction of the flow to such an orbit is tamely equivalent to the rotation flow which is not tame. This contradicts Proposition 2.3(d). □

EXAMPLE 2.8 (*A tame flow with no Lyapunov functions*). Consider the vector field V in the plane given by

$$V = (x^2 + |y|)\partial_y.$$

Observe that V has a unique zero located at the origin. The flow lines are the solutions of

$$\dot{x} = 0, \;\; \dot{y} = (x^2 + |y|), \;\; x(0) = x_0, \;\; y(0) = y_0.$$

Note that $y(t)$ increases along the flow lines. Thus, if $y_0 \geq 0$, we deduce

$$\dot{y} = x_0^2 + y \Longrightarrow \frac{d}{dt}(e^{-t}y) = e^{-t}x_0^2 \Longrightarrow e^{-t}y(t) - y_0 = x_0^2(1-e^{-t})$$

so that

$$y(t) = e^t y_0 + x_0^2(e^t - 1).$$

If $y_0 < 0$ then while $y < 0$ we have

$$\dot{y} + y = x_0^2 \Longrightarrow e^t y + |y_0| = x_0^2(e^t - 1).$$

Thus

$$y(t) = 0 \Longleftrightarrow e^t x_0^2 = x_0^2 + |y_0| \Longrightarrow t = T(x_0, y_0) := \log(x_0^2 + |y_0|) - \log x_0^2$$

We deduce that if $y_0 < 0$ we have

$$y(t) = \begin{cases} x_0^2(1-e^{-t}) + y_0 e^{-t} & \text{if} \quad t \leq T(x_0,y_0) \\ x_0^2(e^{t-T(x_0,y_0)} - 1) & \text{if} \quad t > T(x_0,y_0) \end{cases}.$$

The trajectories of this flow are depicted in the top half of Figure 1

From the above description it follows immediately that this flow is tame and extends to a tame flow on S^2, the one-point compactification of the plane. The flow on the eastern hemisphere ($X \geq 0$) is depicted at the bottom of Figure 1. Observe

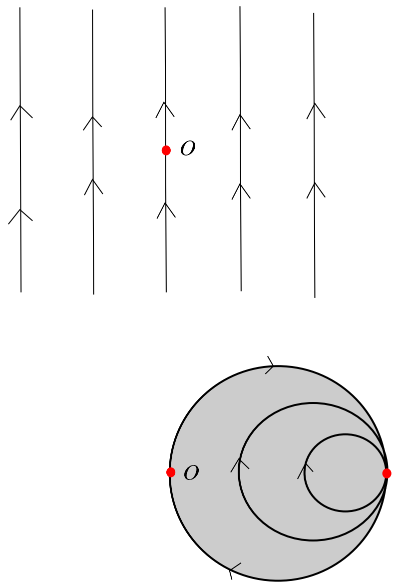


FIGURE 1. *A tame flow with lots of homoclinic orbits.*

that all but two of the orbits of this flow are homoclinic so that this flow does not admit Lyapunov functions. □

EXAMPLE 2.9 (*The cone construction*). Suppose Φ is a tame flow over a compact tame set $X \subset \mathbb{R}^N$. We form the *cone over* Φ as follows.

First, define the cone over X to be the tame space $C(X) \subset [0,1] \times \mathbb{R}^N \subset \mathbb{R}^{N+1}$ defined as the definable quotient

$$[0,1] \times X/\{1\} \times X.$$

The time 1-slice $\{1\} \times X$ is mapped to the *vertex* of the cone, denoted by $*$. Denote by $\pi : [0,1] \times X \to C(X)$ the natural projection. Observe that π is a bijection

$$[0,1) \times X \to C(X)^* = C(X) \setminus \{*\}.$$

We thus have two maps

$$\sigma : C(X)^* \to X, \ \ \alpha : C(X) \to [0,1].$$

called the *shadow*, and respectively *altitude*. Any point on the cone, other than the vertex, is uniquely determined by its shadow and altitude.

The product of the standard decreasing flow Ψ on $[0,1]$ with the flow Φ on X produces a flow on $[0,1] \times X$ which descends to a flow on the cone $C(X)$ called the *downward cone* of Φ which we denote by C^Φ. The vertex is a stationary point of this flow. If $p \in C(X)^*$ then, to understand the flow line $t \mapsto C_t^\Phi(p)$, it suffices to keep track of the evolution of its shadow and its height. The shadow of $C_t^\Phi(p)$ is $\Phi_t\sigma(p)$, while the height of $C_t^\Phi(p)$ is $\Psi_t\alpha(p)$.

Observe that if f is a Lyapunov function for Φ on x, then for every positive constant λ the function

$$f_\lambda : C(X) \to \mathbb{R}, \quad f_\lambda(x) = \begin{cases} \lambda & x = * \\ \lambda\alpha(x) + \big(1 - \alpha(x)\big) f(\sigma(x)) & x \neq * \end{cases}$$

is a Lyapunov function for C^Φ. □

EXAMPLE 2.10 (*The canonical tame flow on an affine simplex*). We want to investigate the cone construction in a very special case. Suppose E is a finite dimensional affine space. For every subset $V \subset E$ we denote by $\mathbf{Aff}(V)$ the affine subspace spanned by V. The set V is called *affinely independent* if $\dim \mathbf{Aff}(V) = \#V - 1$.

If $V = \{v_0, \dots, v_k\}$, and $\dim \mathbf{Aff}(V) = k$ we define

$$[V] = [v_0, \dots, v_k] := \operatorname{conv}(\{v_0, \dots, v_k\}),$$

where "conv" denotes the convex hull operation. We will refer to $[v_0, \dots, v_k]$ as the affine k-simplex with vertices $v_0, \cdots, v_k$. Its *relative interior*, denoted by $\boldsymbol{Int}[v_0, \dots, v_k]$ is defined by

$$\boldsymbol{Int}[v_0, \dots, v_k] := \Big\{ \sum_{i=0}^k t_i v_i; \;\; t_i > 0, \;\; \sum_{i=0}^k t_i = 1 \Big\}.$$

Given a linearly ordered, affinely independent finite subset of E we can associate in a canonical fashion a tame flow on the affine simplex spanned by this set. For another description of this flow we refer to [**38**, p.166-167].

Fix an affine k-simplex S in the affine space E with vertex set V. A linear ordering on V is equivalent to a bijection

$$\{0, 1, \cdots, k\} \to V, \;\; i \mapsto v_i \;\text{ so that }\; v_i < v_j \iff i < j.$$

Recall the *affine cone construction.*

Let Y be a subset in an affine space E, and $v \in E \setminus \mathbf{Aff}(Y)$. The *cone on Y with vertex v* is the set

$$C_v(Y) := \big\{ x = (1-t)v + ty = v + t(y - v); \;\; t \in [0,1], \;\; y \in Y \big\}.$$

In other words, $C_v(Y)$ is the union of all segments joining v to a point $y \in Y$. Note that since $v \in E \setminus \mathbf{Aff}(Y)$ two such segments have only the vertex v in common. This means that any point $p = C_v(Y) \setminus \{v\}$ can be written uniquely as an affine combination

$$p = v + t(y - v), \;\; t \in (0,1], \;\; y \in Y.$$

If $S = [v_0, \dots, v_k]$ is an affine k-simplex, then

$$[v_0, \dots, v_i, v_{i+1}] = C_{v_{i+1}}([v_0, \cdots, v_i])$$

so that

$$S_k = C_{v_k} \circ \cdots \circ C_{v_1}(\{v_0\}) := C_{v_k}\big(\cdots C_{v_1}(\{v_0\}) \cdots\big).$$

The cone construction extends to sets equipped with vector fields.

Suppose $Y \subset E$, $v \in E \setminus \mathbf{Aff}(Y)$, and $Z : Y \to TE$ is a vector field on Y. Temporarily, we impose no regularity constraints on Z. A priori, it could even be discontinuous. Define

$$\hat{Z} = C_v(Z) : C_v(Y) \to TE,$$

by setting for $t \in [0,1]$, and $y \in Y$,

$$\hat{Z}\big(\, v + t(y-v)\,\big) = (1-t)\cdot t(y-v) + tZ(y).$$

Note that $\hat{Z}(v) = 0$ and $\hat{Z}(y) = Z(y)$, $\forall y \in Y$. We let

$$S_i := [v_0, \dots, v_i],$$

and define

$$Z_i := C_{v_i} \circ \cdots C_{v_1}(\vec{0}),$$

where $\vec{0}$ denotes the trivial vector field on the set $\{v_0\}$. By construction we have

$$Z_{i+1}|_{S_i} = Z_i, \;\; Z_i(v_j) = 0, \;\; \forall j \leq i.$$

Observe that along the segment $[v_0, v_1]$ we have

$$Z_1(v_1 + t(v_1 - v_0)) = -t(1-t)\overrightarrow{v_0 v_1}.$$

Its flow is the canonical downward flow on a segment and it is depicted in Figure 2.

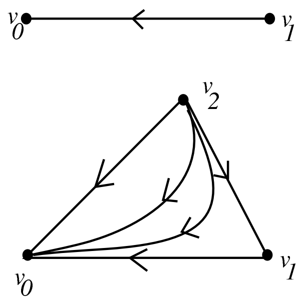


FIGURE 2. *Gradient like tame flows on low dimensional simplices.*

To understand the nature of the vector fields Z_i we argue inductively. Let $p \in [v_0, \cdots, v_k, v_{k+1}]$, $p \neq v_{k+1}$, and denote by q the intersection of the line $v_{k+1}p$ with $[v_0, \cdots, v_k]$ (see Figure 3). If $(t_0, \dots, t_{k+1})$ denote the barycentric coordinates of p in S_{k+1}, and $(s_0, \dots, s_k)$ denote the barycentric coordinates of q in S_k, then

$$s_i = \frac{t_i}{t_0 + \cdots + t_k} = \frac{t_i}{1 - t_{k+1}}, \;\; 0 \leq i \leq k$$

and

$$(p - v_{k+1}) = (1 - t_{k+1})(q - v_{k+1}).$$

Then

$$Z_{k+1}(p) = t_{k+1}(1 - t_{k+1})(q - v_{k+1}) + (1 - t_{k+1})Z_k(q)$$

Since S_k is described in S_{k+1} by $t_{k+1} = 0$ and $Z_{k+1}|_{S_k} = Z_k$ we can rewrite the last equality as

$$Z_{k+1}(t_0, \dots, t_k, t_{k+1}) = t_{k+1}(1 - t_{k+1})\left\{ \Big(\sum_{i=0}^{k} \frac{t_i}{1 - t_{k+1}} v_i\Big) - v_{k+1} \right\}$$

$$+ (1 - t_{k+1})Z_k\Big(\frac{t_0}{1 - t_{k+1}}, \cdots, \frac{t_k}{1 - t_{k+1}}\Big).$$

This shows inductively that Z_k is Lipschitz continuous, and even smooth on the relative interiors of the faces of S_k.

Denote by Φ_t^k the (local) flow defined by Z_k. For any vector $\vec{\lambda} = (\lambda_0, \dots, \lambda_k) \in \mathbb{R}^{k+1}$ such that

$$\lambda_0 < \lambda_1 < \cdots < \lambda_k,$$

we define $f_{\vec{\lambda}} : S_k \to \mathbb{R}$ to be the unique affine function on S_k satisfying

$$f_{\vec{\lambda}}(v_i) = \lambda_i, \ \ \forall i = 0, 1, \dots, k.$$

We want to show that for every $k \geq 0$ the following hold.

Fact 1. The flow Φ_t^k exists for all t on S_k, it is tame, and it is of the triangular type (1.1).

Fact 2. The linearization of Z_k at a vertex v_ℓ, $\ell = 0, 1, \dots, k$ is diagonalizable, its spectrum is $\{-1, 1\}$ and the eigenvalue 1 has multiplicity ℓ.

Fact 3. The function $f_{\vec{\lambda}}$ is a Lyapunov function for Φ^k.

Fact 1. To show that the flow Φ_t^k is tame we argue by induction over k. The case $k = 1$ follows from Example 2.5. For the inductive step we fix a vertex u of S_k, and relabel the other $u_1, \dots, u_k$.

We think of u as the origin of the affine space $\mathbf{Aff}(S_{k+1})$, and we introduce the vectors

$$\vec{e}_i := \overrightarrow{uu_i} = u_i - u, \ \ \vec{e}_{k+1} := \overrightarrow{uv_{k+1}} = v_{k+1} - u.$$

These define linear coordinates $(x_1, \dots, x_k, x_{k+1})$ on $\mathbf{Aff}(S_{k+1})$ so that

$$\mathbf{Aff}(S_k) = \{x_{k+1} = 0\}.$$

We say that these are the *linear coordinates determined by the vertex* u.

Consider the point $p \in S_{k+1} \setminus v_{k+1}$ with linear coordinates

$$p \longleftrightarrow (x_1, \dots, x_k, x_{k+1}).$$

Denote by $p' \in S_k$ the projection of p on S_k parallel to e_{k+1}, and by q the intersection of the line $v_{k+1}p$ with S_k (see Figure 3). We say that q is the *shadow* of p. Then p' has coordinates

$$p' \longleftrightarrow (x_1, \dots, x_k, 0),$$

while the shadow q has coordinates

$$q \longleftrightarrow \Big(\frac{x_1}{1 - x_{k+1}}, \cdots, \frac{x_k}{1 - x_{k+1}}, 0 \Big).$$

Since $\overrightarrow{v_{k+1}p} = (1 - x_{k+1})\overrightarrow{v_{k+1}q}$ we deduce

$$Z_{k+1}(x_1, \cdots, x_k, x_{k+1}) = x_{k+1}(1 - x_{k+1})\overrightarrow{v_{k+1}q} + (1 - x_{k+1})Z_k(q)$$

$$= x_{k+1}(1 - x_{k+1})\Big\{ \Big(\sum_{i=1}^{k} \frac{x_i}{1 - x_{k+1}} \vec{e}_i \Big) - \vec{e}_{k+1} \Big\}$$

$$+ (1 - x_{k+1}) Z_k \Big(\frac{x_1}{1 - x_{k+1}}, \cdots, \frac{x_k}{1 - x_{k+1}} \Big)$$

$$= -x_{k+1}(1 - x_{k+1})\vec{e}_{k+1} + x_{k+1} \sum_{i=1}^{k} x_i \vec{e}_i + (1 - x_{k+1}) Z_k \Big(\frac{x_1}{1 - x_{k+1}}, \cdots, \frac{x_k}{1 - x_{k+1}} \Big).$$

If we write

$$Z_k = \sum_{i=0}^{k} Z_k^i \vec{e}_i$$

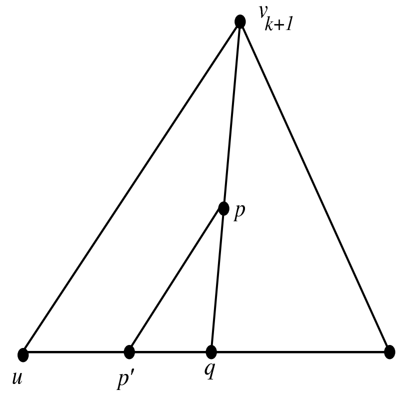


FIGURE 3. *Dissecting the cone construction.*

then we deduce

$$Z^{k+1}_{k+1} = x_{k+1}(x_{k+1}-1),\ \ Z^i_{k+1} = x_{k+1}x_i+(1-x_{k+1})Z^i_k\Big(\frac{x_1}{1-x_{k+1}},\cdots,\frac{x_k}{1-x_{k+1}}\Big).$$

This shows inductively that, away from the vertex v_{k+1} of the simplex S_{k+1}, the vector field Z_{k+1} has upper triangular form in the coordinates $x_1,\cdots,x_{k+1}$, i.e., the i-th component $Z^i_{k+1}(x_1,\cdots,x_{k+1})$ depends only on the variables $x_i,\cdots,x_{k+1}$. This defines a system of differential equations of the pfaffian type (1.1).

We want to prove that the vector field Z_{k+1} determines a *globally defined* flow on the simplex S_k. From the inductive assumption we deduce that for any k simplex with linearly oriented vertex set the corresponding vector field determines a globally defined tame flow. Consider a point

$$p \in S_{k+1} \setminus \{v_{k+1}\}.$$

We will use the linear coordinates $(x_1,\dots,x_{k+1})$ determined by the vertex v_0. Assume that the linear coordinates of p are

$$p = (a_1,\dots,a_{k+1})$$

The flow line of Z_{k+1} through p is a path

$$t \stackrel{\gamma}{\longmapsto} (x_1(t),\dots,x_{k+1}(t))$$

satisfying the initial value problem

$$\dot{x}_{k+1} = -x_{k+1}(1-x_{k+1}),\ \ x_{k+1}(0)=a_{k+1} \tag{2.2}$$

$$\dot{x}_i = x_{k+1}x_i + (1-x_{k+1})Z^i_k\big(\frac{x_i}{(1-x_{k+1})},\cdots,\frac{x_k}{(1-x_{k+1})}\big),\ \ x_i(0)=a_i. \tag{2.3}$$

For simplicity we write $x := x_{k+1}$. We introduce the *shadow* coordinates

$$s_i = \frac{x_i}{(1-x_{k+1})} \Longleftrightarrow x_i = s_i(1-x_{k+1}),\ \ i=1,2,\cdots,k.$$

The projection of the path $\gamma(t)$ from the vertex v_{k+1} onto the face $[v_0,\dots,v_k]$ is given in linear coordinates by the shadow path $t \mapsto (\,s_1(t),\dots,s_k(t)\,)$.

Since $\dot{x} = -x(1-x)$ we deduce

$$\frac{d}{dt}x_i = \frac{d}{dt}(s_i(1-x)) = \dot{s}_i(1-x) - s_i\dot{x} = \dot{s}_i(1-x) + s_i x(1-x).$$

Using this in (2.2) and (2.3) we deduce

$$\dot{x} = -x(1-x) \;\; \dot{s}_i = Z_k^i(s_i, \cdots, s_k), \;\; j = 1, \ldots, k. \tag{2.4}$$

This computation, coupled with the inductive assumption show that $\Phi_t^{k+1}(p) \in S_{k+1}$, $\forall t \in \mathbb{R}$.

The flow Φ_t^{k+1} can be given the following simple interpretation. Denote by Φ_t^k the flow on $[v_0, \cdots, v_k]$, and by s the shadow map

$$s : \mathrm{Int}\,[v_0, \ldots, v_{k+1}] \to \mathrm{Int}\,[v_0, \ldots, v_k], \quad s(p) := v_{k+1}p \cap [v_0, \ldots, v_k],$$

where $v_k p$ denotes the line passing through v_{k+1} and p. We assume v_0 is the origin of our affine space so we can describe a point in the simplex $[v_0, v_1, \ldots, v_i]$ by its linear coordinates $(x_1, \ldots, x_i)$. Given

$$p_0 = (a_1, \ldots, a_{k+1}) \in \mathrm{Int}\,[v_0, \ldots, v_{k+1}],$$

we set $q_0 = s(p_0)$ and then we have

$$\Phi_t^{k+1}(p_0) = x(t)v_{k+1} + \big(1 - x(t)\big)\Phi_t^k(s(p_0)), \quad x(t) = \frac{e^{-t}a_{k+1}}{1 - a_{k+1} + e^{-t}a_{k+1}}.$$

The path $\Phi_t^{k+1}(p_0)$ can be visualized using a natural moving frame.

Denote by s_0 the shadow of p_0. Now let s_0 go with the flow Φ^k, $s(t) = \Phi_t^k(s_0)$. The point $p(t) = \Phi^{k+1}(p_0)$ lies on the segment $[s(t), v_{k+1}]$. If we affinely identify this segment with the segment $[0,1]$ so that $s(t) \longleftrightarrow 0$ and $v_{k+1} \longleftrightarrow 1$, then the motion of the point $p(t)$ along the (moving) segment $[s(t), v_{k+1}]$ is mapped to the motion on the unit segment $[0,1]$ governed by the canonical downward flow on $[0,1]$. In other words, the flow Φ^{k+1} is the negative cone on the flow Φ^k. This proves that Φ^{k+1} is tame.

To see how this works in concrete examples, suppose S_2 is the 2-simplex

$$\{(x,y) \in \mathbb{R}^2;\; 0 \le x, y, \;\; x + y \le 1\}$$

with vertices $v_0 = (0,0)$, $v_1 = (1,0)$, $v_2 = (0,1)$. Consider the point $p_0 = (x_0, y_0)$ in the interior of this simplex. If Φ is the flow defined by the vector field Z_2 then

$$\Phi_t(x_0, y_0) = \Big(\big(1 - y(t)\big)\frac{e^{-t}x_0}{1 - x_0 - y_0 + e^{-t}x_0},\; y(t) \Big), \;\; y(t) = \frac{e^{-t}y_0}{1 - y_0 + e^{-t}y_0}.$$

Fact 2. The statement about the linearization of Z_k at the vertices of S_k is again proved by induction. The statement is obvious for $k = 1$. For the inductive step, denote by u one of the vertices of S_k, and label the remaining ones by $u_1, \ldots, u_k$. We again think of u as the origin of $\mathbf{Aff}(S_{k+1})$ and as such we obtain a basis

$$\vec{e}_i = u_i - u, \;\; \vec{e}_{k+1} = v_{k+1} - u,$$

and linear coordinates $(x_1, \ldots, x_{k+1})$. The point u has linear coordinates $x_i = 0$, $0 \le i \le k+1$ in S_{k+1}. Denote by ∇ the trivial connection on the tangent bundle of $\mathbf{Aff}(S_{k+1})$. For $i = 0, \ldots, k$ we have

$$\nabla_{e_i} Z_{k+1}(x_1, \ldots, x_k) = x_{k+1}\vec{e}_i + \nabla_{e_i} Z_k\Big(\frac{x_1}{1 - x_{k+1}}, \cdots, \frac{x_k}{1 - x_{k+1}}\Big)$$

and

$$\nabla_{e_{k+1}} Z_{k+1} = \sum_{i=1}^{k}\big(x_i\vec{e}_i + (2x_{k+1} - 1)\vec{e}_{k+1}\big) - Z_k\Big(\frac{x_1}{1 - x_{k+1}}, \cdots, \frac{x_k}{1 - x_{k+1}}\Big)$$

$$+\frac{1}{1-x_{k+1}}\sum_{i=1}^{k}x_i\nabla_{e_i}Z_k\Big(\frac{x_1}{1-x_{k+1}},\cdots,\frac{x_k}{1-x_{k+1}}\Big)$$

Observe that at $(x_1,\cdots,x_{k+1})=0$ we have

$$\nabla_{e_i}Z_{k+1}=\nabla_{e_i}Z_k,\quad 1\leq i\leq k,$$

and

$$\nabla_{e_{k+1}}Z_{k+1}=-e_{k+1}$$

This proves the statement about the linearization of Z_{k+1} at $u\in\{v_0,\dots,v_k\}$.

Finally, we want to prove that the linearization of Z_{k+1} at v_{k+1} is the identity. Since $Z_k(v_i)=\vec{0}$, $\forall i=0,1,\dots,k$ we deduce that at a point p on the line segment $[v_{k+1},v_i]$ given by

$$p=v_{k+1}+(1-t)(v_i-v_{k+1}),$$

the vector field Z_{k+1} is described by

$$Z_{k+1}(p)=t(1-t)(v_i-v_{k+1}).$$

If we fix the origin of $\mathbf{Aff}(S_{k+1})$ at v_{k+1}, and we set $\vec{f_i}:=\overrightarrow{v_{k+1}v_i}$, then

$$Z_{k+1}(v_{k+1}+s\vec{f_i})=s(1-s)\vec{f_i},\ \ \nabla_{f_i}Z_{k+1}(v_k)=\vec{f_i},$$

so that the linearization of Z_{k+1} at v_{k+1} is the identity operator

$$\boldsymbol{I}:T_{v_{k+1}}\mathbf{Aff}(S_{k+1})\to T_{v_{k+1}}\mathbf{Aff}(S_{k+1}).$$

Fact 3. We again argue by induction. The statement is true for $k=1$. For the inductive step, denote by $(x_1,\dots,x_{k+1})$ the linear coordinates on S_{k+1} determined by the vertex v_0. In these coordinates we have

$$f_{\vec{\lambda}}=\lambda_0+\sum_{j=1}^{k+1}(\lambda_j-\lambda_0)x_j.$$

If we write $x=x_{k+1}$, and again we introduce the shadow coordinates $s_j=\frac{x_j}{1-x}$, we deduce

$$f_{\vec{\lambda}}(s_1,\dots,s_k,x)=\lambda_0+(\lambda_{k+1}-\lambda_0)x+(1-x)\sum_{j=1}^{k}(\lambda_j-\lambda_0)s_j.$$

If we differentiate $f_{\vec{\lambda}}(s_1,\dots,s_k,x)$ along a flow line we deduce

$$\frac{d}{dt}f_{\vec{\lambda}}(s_1,\dots,s_k,x)=(\lambda_{k+1}-\lambda_0)\dot{x}-\dot{x}\sum_{j=1}^{k}(\lambda_j-\lambda_0)s_j+(1-x)\sum_{j=1}^{k}(\lambda_j-\lambda_0)\dot{s}_j.$$

Using (2.4) we deduce

$$\frac{d}{dt}f_{\vec{\lambda}}(s_1,\dots,s_k,x)$$

$$=-(\lambda_{k+1}-\lambda_0)x(1-x)+x(1-x)\sum_{j=1}^{k}(\lambda_j-\lambda_0)s_j+(1-x)\sum_{j=1}^{k}(\lambda_j-\lambda_0)\dot{s}_j$$

$$=x(1-x)\left(\sum_{j=1}^{k}(\lambda_j-\lambda_0)s_j-(\lambda_{k+1}-\lambda_0)\right)+(1-x)\sum_{j=1}^{k}(\lambda_j-\lambda_0)\dot{s}_j$$

The first term is strictly negative because $\lambda_j < \lambda_{k+1}$, and on S_k we have

$$\sum_j s_j \leq 1, \quad s_j \geq 0,$$

where at least one of these inequalities is strict. The second term is negative since the restriction of $f_{\vec{\lambda}}$ to the face S_k is a Lyapunov function for Φ^k. □

The above example has an important consequence.

PROPOSITION 2.11. *On any compact tame space there exist gradient like tame flows with finitely many stationary points.*

PROOF. Suppose X is a compact tame space. Choose an affine simplicial complex Y and a tame homeomorphism $F : Y \to X$. Denote by $V(Y)$ the vertex set of Y and choose a map $\ell : V(Y) \to \mathbb{R}$ which is injective when restricted to the vertex set of any simplex of Y. We can now use the map ℓ to linearly order the vertex set of any simplex σ of Y by declaring

$$u < v \Longleftrightarrow \ell(u) < \ell(v).$$

This ordering induces as in Example 2.10 a tame flow $\Phi^\sigma = \Phi_t^{\sigma,\ell}$ on any face σ of Y such that

$$\Phi^\tau|_\sigma = \Phi^\sigma, \quad \forall \sigma \prec \tau.$$

Thus the tame flows on the faces of Y are compatible on overlaps and thus define a tame flow on Y. Note that the function ℓ defines a piecewise linear function $\ell : Y \to \mathbb{R}$ which decreases strictly along the trajectories of Φ. Using the homeomorphism F we obtain a tame flow $F \circ \Phi \circ F^{-1}$ on X. Its stationary points correspond via F with the vertices of Y, and $F \circ \ell \circ F^{-1}$ is a tame Lyapunov function. □

EXAMPLE 2.12. Suppose E is a finite dimensional real Euclidean space, and $A \in \operatorname{End}(E)$ is a symmetric endomorphism. Then the linear flow

$$\Phi^A : \mathbb{R} \times E \to E, \quad \Phi_t^A(x) = e^{At}x, \quad x \in E,$$

is a tame flow. Similarly, the flow

$$\Psi^A : \mathbb{R} \times \operatorname{End} E \to \operatorname{End} E, \quad \Psi_t^A(S) = e^{At}Se^{-At}, \quad S \in \operatorname{End} E$$

is a tame flow. □

EXAMPLE 2.13. Suppose E is a finite dimensional real Euclidean space, and $A \in \operatorname{End}(E)$ is a symmetric endomorphism. Denote by $\mathbf{Gr}_k(E)$ the Grassmannian of k-dimensional subspaces of E. For every $L \in \mathbf{Gr}_k(E)$ we denote by P_L the orthogonal projection onto L. The map

$$\mathbf{Gr}_k(E) \ni L \mapsto P_L \in \operatorname{End} E$$

embeds $\mathbf{Gr}_k(E)$ as a real algebraic submanifold of $\operatorname{End} E$.

On $\operatorname{End} E$ we have and inner product given by

$$\langle S, T \rangle = \operatorname{tr}(ST^*),$$

and we denote by $|\bullet|$ the corresponding Euclidean norm on $\operatorname{End} E$. This inner product induces a smooth Riemann metric on $\mathbf{Gr}_k(E)$.

The flow

$$\mathbf{Gr}_k(E) \ni L \mapsto e^{At}L \in \mathbf{Gr}_k(E)$$

is tame. To see this, consider an orthonormal basis of eigenvectors of A, $e_1, \dots, e_n$, $n = \dim E$ such that

$$Ae_i = \lambda_i e_i, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

For every subset $I \subset \{1, \dots, n\}$ we write

$$E_I := \operatorname{span}\{e_i, \quad i \in I\}, \quad I^\perp := \{1, \dots, n\} \setminus I.$$

For $\#I = k$ we set

$$\mathbf{Gr}_k(E)_I = \bigl\{L \in \mathbf{Gr}_k; \quad L \cap E_I^\perp = 0, \bigr\}.$$

$\mathbf{Gr}_k(E)_I$ is a semialgebraic open subset of $\mathbf{Gr}_k(E)$ and

$$\mathbf{Gr}_k(E) = \bigcup_{\#I=k} \mathbf{Gr}_k(E)_I.$$

A subspace $L \in \mathbf{Gr}_k(E)_I$ can be represented as the graph of a linear map $S = S_L : E_I \to E_I^\perp$, i.e.,

$$L = \bigl\{x + Sx; \quad x \in E_I\bigr\}.$$

Using the basis $(e_i)_{i \in I}$ and $(e_\alpha)_{\alpha \in I^\perp}$ we can represent S as a $(n-k) \times k$ matrix

$$S = [s_{\alpha i}]_{i \in I, \ \alpha \in I^\perp}.$$

The subspaces E_I and $E_I^\perp$ are A invariant. Then $e^{At}L \in \mathbf{Gr}_k(E)_I$, and it is represented as the graph of the operator $S_t = e^{At}Se^{-At}$ described by the matrix

$$\operatorname{Diag}(e^{\lambda_\alpha t}, \quad \alpha \in I^\perp) \cdot S \cdot \operatorname{Diag}(e^{-\lambda_i t}, \quad i \in I) = [e^{(\lambda_\alpha - \lambda_i)t} s_{\alpha i}]_{i \in I, \ \alpha \in I^\perp}.$$

This proves that the flow is tame.

Let us point out that this flow is the gradient flow of the function

$$f_A : \mathbf{Gr}_k(E) \to \mathbb{R}, \quad f_A(L) = \operatorname{tr} AP_L = \langle A, P_L \rangle.$$

This is a Morse-Bott function. We want to describe a simple consequence of this fact which we will need later on.

Suppose U is a subspace of E, $\dim U \leq k$, and define

$$A := P_{U^\perp} = \mathbb{1}_E - P_U.$$

Then

$$f_A(L) = \operatorname{tr}(P_L - P_L P_U) = \dim L - \operatorname{tr}(P_L P_U).$$

On the other hand, we have

$$|P_U - P_U P_L|^2 = \operatorname{tr}(P_U - P_U P_L)(P_U - P_L P_U) = \operatorname{tr}(P_U - P_U P_L P_U)$$
$$= \operatorname{tr} P_U - \operatorname{tr} P_U P_L P_U = \dim U - \operatorname{tr} P_U P_L.$$

Hence

$$f_A(L) = |P_U - P_U P_L|^2 + \dim L - \dim U,$$

so that

$$f_A(L) \geq \dim L - \dim U,$$

with equality if and only if $L \supset U$. Thus, the set of minima of f_A consists of all k-dimensional subspaces containing U. We denote this set with $\mathbf{Gr}_k(E)_U$. Since f_A is a Morse-Bott function we deduce that

$$\forall j \leq k, \quad \forall U \in \mathbf{Gr}_j(E) \ \ \exists C = C(U) > 1, \quad \forall L \in \mathbf{Gr}_k(E):$$

(2.5)
$$\frac{1}{C}\operatorname{dist}(L, \mathbf{Gr}_k(E)_U)^2 \leq |P_U - P_U P_L|^2 \leq C \operatorname{dist}(L, \mathbf{Gr}_k(E)_U)^2.$$

In a later section we will prove more precise results concerning the asymptotics of this Grassmannian flow. □

CHAPTER 3

Some global properties of tame flows

We would like to present a few general results concerning the long time behavior of a tame flow.

Definition 3.1. Suppose $\Phi : \mathbb{R} \times X \to X$ is a continuous flow on a topological space X. Then for every set $A \subset X$ we define

$$\Phi_+(A) = \bigcup_{t\geq 0} \Phi_t(A) = \Phi([0,\infty) \times A), \quad \Phi_-(A) = \bigcup_{t\leq 0} \Phi_t(A) = \Phi((-\infty, 0] \times A)$$

$$\Phi(A) = \Phi(\mathbb{R} \times A) = \Phi_+(A) \cup \Phi_-(A).$$

We will say that $\Phi_\pm(A)$ is the forward/backward *drift* of A along Φ, and that $\Phi(A)$ is the *complete drift*. □

The next result follows immediately from the definitions.

Proposition 3.2. *If Φ is a tame flow on X then for every tame subset $A \subset X$ the sets $\Phi_\pm(A)$ and $\Phi(A)$ are tame.* □

Theorem 3.3. *Suppose Φ is a continuous flow on the tame set X. Consider the flow $G_\Phi := T \times \tilde{\Phi} \times \Phi$ on $\mathbb{R} \times X \times X$, where T denotes the translation flow on $\mathbb{R}$ and $\tilde{\Phi}_t = \Phi_{-t}$. Denote by Δ_0 the initial diagonal*

$$\Delta_0 = \big\{ (0, x, x) \in \mathbb{R} \times X \times X \big\}.$$

The following conditions are equivalent.

(a) Φ is a tame flow.

(b) The complete drift of Δ_0 along G_Φ is a tame subspace of $\mathbb{R} \times X \times X$.

Proof. (a) $\Longrightarrow$ (b). Since Φ is tame we deduce that G_Φ is tame and we conclude using Proposition 3.2.

(b) $\Longrightarrow$ (a). Observe that

$$G_\Phi(\Delta_0) = \big\{ (t, x_0, x_1) \in \mathbb{R} \times X \times X; \ \exists x \in X : \ \ x_0 = \Phi_{-t}(x), \ \ x_1 = \Phi_t(x) \big\}.$$

Consider the tame homeomorphism

$$F : \mathbb{R} \times X \times X \to \mathbb{R} \times X \times X, \ \ (t, x_0, x_1) \longmapsto (s, y_0, y_1) := (2t, x_0, x_1)$$

and observe that F maps $G_\Phi(\Delta_0)$ onto the graph of the flow Φ. □

Corollary 3.4. *Suppose that the flow Ψ on the tame space S is tamely conjugate to the translation flow on $\mathbb{R}$. Then a flow Φ on the tame space X is tame if and only if there exists $s_0 \in S$ such that the total drift of the diagonal*

$$\Delta_{s_0} = \{ (s_0, x, x) \in S \times X \times X \}$$

with respect to the flow $\Psi \times \tilde{\Phi} \times \Phi$ *is tame.* □

In applications S will be the open semi-circle

$$S = \{(x, y) \in \mathbb{R}^2;\ \ x^2 + y^2 = 1,\ \ x > 0\}$$

equipped with the negative gradient flow of the height function $h(x, y) = y$. As origin of s we take $s_0 = (1, 0)$. As explained in Example 2.6 this flow is tamely conjugate to the translation flow on $\mathbb{R}$. The following result is an immediate consequence of tameness.

PROPOSITION 3.5. *Suppose X is a tame compact set of dimension d, S is the open semi-circle equipped with the flow Ψ described above, and Φ is a tame flow on X. We set $s_t := \Psi_t(s_0)$, $s_0 = (1, 0) \in S$. Then Φ has finite volume, i.e. the image of the graph of Φ via the tame diffeomorphism*

$$\mathbb{R} \times X \times X \to S \times X \times X,\ \ (t, x_0, x_1) \mapsto (s_t, x_0, x_1)$$

has finite $(d + 1)$-dimensional Hausdorff measure.

PROPOSITION 3.6. *Suppose Φ is a tame flow on the compact space X. Then there exists a positive constant $L = L(X, \Phi)$ such that every orbit of Φ has length $\leq L$.*

PROOF. Consider the roof

$$\begin{array}{ccc} & \Gamma_\Phi \subset \mathbb{R} \times X \times X & \\ {}^{l}\swarrow & & \searrow^{r} \\ X & & X \end{array}$$

where

$$\ell(t, x_0, x_1) = x_0, \quad r(t, x_0, x_1) = x_1.$$

This roof describes the family of subspaces of X, $(\mathcal{O}_x)_{x \in X}$, where

$$\mathcal{O}_x = r\big(\ell^{-1}(x)\big) = \big\{\Phi_t(x);\ \ t \in \mathbb{R}\,\big\}.$$

We see that $\mathcal{O}_x$ is the orbit of the flow through x, and thus the family of orbits is a definable family of tame subsets with diameters bounded from above. The claim in the proposition now follows from the Crofton formula and the definability of Euler characteristic. □

PROPOSITION 3.7. *Suppose that Φ is a tame flow on the compact tame space X. Then for every $x \in X$ the limits $\lim_{t \to \pm\infty} \Phi(x)$ exist and are stationary points of the flow denoted by $\Phi_{\pm\infty}(x)$. Moreover, the resulting maps*

$$\Phi_{\pm\infty} : X \to X$$

are tame.

PROOF. Clearly the limits exist if x is a stationary point. Assume x is not a stationary point. Then the orbit $\Phi(x)$ is a one-dimensional tame subset and its frontier

$$\mathbf{Fr}\,\Phi(x) := (\boldsymbol{cl}\,\Phi(x)) \setminus \Phi(x)$$

is a tame, zero dimensional, Φ-invariant subset. In particular it must be finite collection of stationary points, $\{x_1, \dots, x_\nu\}$. Choose small, disjoint, tame, open neighborhoods $U_1, \dots, U_\nu$ of $x_1, \dots, x_\nu$, and set

$$U = \bigcup_{k=1}^{\nu} U_k.$$

Then the set $S = \{t \in \mathbb{R};\ \Phi_t(x) \in U\}$ is a tame open subset of $\mathbb{R}$, and thus it consists of finitely many, disjoint open intervals, $I_1, \cdots, I_N$. Since the set $\{x_1, \dots, x_\nu\}$ consists of limit points of the orbit, one (and only one) of these intervals, call it I_+, is unbounded from above, and one and only one of these intervals, call it I_-, is unbounded from below. Then there exist $x_\pm \in \{x_1, \dots, x_\nu\}$ such that $\Phi_t(x)$ is near $x_\pm$ when $t \in I_\pm$. We deduce that

$$\mathbf{Fr}\,\Phi(x) = \{x_\pm\} \ \text{ and } \ \lim_{t\to\pm\infty} \Phi_t(x) = x_\pm.$$

Denote by $\Gamma^\pm$ the graph of $\Phi_{\pm\infty}$. We deduce that

$$(x, y) \in \Gamma^+ \iff \forall \varepsilon > 0,\ \ \exists T > 0:\ \ \mathrm{dist}\,(\Phi_t(x), y) < \varepsilon,\ \ \forall t > T.$$

This shows Γ^+ is definable. A similar argument shows that Γ^- is tame. □

DEFINITION 3.8. Suppose Φ is a tame flow on the compact tame space X.
(a) We denote by $\mathbf{Cr}_\Phi$ the set of stationary points of Φ and for every $x \in \mathbf{Cr}_\Phi$ we set

$$W^+(x, \Phi) := \Phi_\infty^{-1}(x),\ \ W^-(x, \Phi) := \Phi_{-\infty}^{-1}(x)$$

and we say that $W^\pm(x, \Phi)$ is the *stable* (resp. *unstable*) variety of x.
(b) For $x_0, x_1 \in \mathbf{Cr}_\Phi$ we set

$$C_\Phi(x_0, x_1) := W^-(x_0, \Phi) \cap W^+(x_1, \Phi) = \{z \in X;\ \ x_0 = \Phi_{-\infty}(z),\ \ x_1 = \Phi_\infty(z)\}.$$

We say that $C_\Phi(x_0, x_1)$ is the Φ-*tunnel* from x_0 to x_1. Observe that all the spaces $\mathbf{Cr}_\Phi$, C_Φ and $W^\pm(-, \Phi)$ are tame subspaces. □

REMARK 3.9. Example 2.8 shows that there exists tame flows Φ admitting stationary points x such that the self-tunnel $C_\Phi(x, x)$ is nonempty. □

Suppose Φ is a tame flow on the compact tame space X. Assume that $\mathbf{Cr}_\Phi$ is finite. Observe that we have a natural action of $\mathbb{R}^2$ on $\mathbb{R} \times X \times X$ given by

$$(s_0, s_1) \cdot (t, x_0, x_1) := \big(t + s_1 - s_0, \Phi_{s_0}(x_0), \Phi_{s_1}(x_1)\big).$$

We denote by $\Gamma \subset \mathbb{R} \times X \times X$ the graph of Φ, and we observe that Γ is invariant with respect to the above action of $\mathbb{R}^2$. We denote by $\Gamma^t \subset X \times X$ the graph of Φ_t, by $\bar{\Gamma}$ the closure of Γ in $[-\infty, \infty] \times X \times X$, and by $\Gamma^{\pm\infty}$ the part of $\bar{\Gamma}$ over $\pm\infty$. Extend the above $\mathbb{R}^2$-action to $[-\infty, \infty] \times X \times X$ by setting

$$(s_0, s_1) \cdot (\pm\infty, x_0, x_1) := \big(\pm\infty, \Phi_{s_0}(x_0), \Phi_{s_1}(x_1)\big).$$

For every subset $S \subset X \times X$ we denote by ${}^*S$ the reflection of S in the diagonal, i.e.

$${}^*S = \{(x_0, x_1) \in X \times X;\ \ (x_1, x_0) \in S\}$$

PROPOSITION 3.10. *$\bar{\Gamma}$ and $\Gamma^{\pm\infty}$ are tame, $\mathbb{R}^2$-invariant subsets of $[-\infty, \infty] \times X \times X$. Moreover,*

$$\Gamma^{-\infty} =^* \Gamma^{\infty},$$
$$\{x\} \times W^-(x, \Phi), \;\; W^+(x) \times \{x\} \subset \Gamma^{\infty},$$
$$\dim \Gamma^{\pm\infty} = \dim X = \dim \bar{\Gamma} - 1.$$

PROOF. The first part follows from the tameness of Γ and the continuity of the $\mathbb{R}^2$-action. Suppose $(x_0, x_1) \in \Gamma_\infty$. Then there exist sequences $(x_n) \subset X$, $t_n \to \infty$ such that

$$(x_n, \Phi_{t_n} x_n)) \to (x_0, x_1).$$

Let $y_n := \Phi_{t_n}(x_n)$. Then $x_n = \Phi_{-t_n}(y_n)$, and we deduce

$$(x_1, x_0) = \lim_{t_n \to \infty} (y_n, \Phi_{-t_n}(y_n)) \in \Gamma^{-\infty}.$$

Let $y \in W^-(x, \Phi)$. Then

$$(y, x) = \lim_{t \to \infty} (y, \Phi_t(y)) \in \Gamma^{-\infty} \Longrightarrow (x, y) \in^* \Gamma^{-\infty} = \Gamma^{\infty}.$$

Hence $\{x\} \times W^-(x, \Phi) \subset \Gamma^\infty$. The inclusion $W^+(x, \Phi) \times \{x\} \subset \Gamma^\infty$ is proved in a similar fashion.

From the equality $\Gamma^\infty \cup \Gamma^{-\infty} = \bar{\Gamma} \setminus \Gamma$ we deduce

$$\dim \Gamma^{\pm\infty} \leq \dim \bar{\Gamma} - 1 = \dim X.$$

On the other hand,

$$\dim \Gamma^\infty \geq \max_{x \in \mathbf{Cr}_\Phi} W^{\pm}(x, \Phi).$$

If we observe that

$$X \setminus \mathbf{Cr}_\Phi = \bigsqcup_{x \in \mathbf{Cr}_\Phi} W^+(x, \Phi) \setminus \{x\} = \bigsqcup_{x \in \mathbf{Cr}_\Phi} W^-(x, \Phi) \setminus \{x\},$$

we deduce from the scissor equivalence principle that

$$\dim X = \max_{x \in \mathbf{Cr}_\Phi} W^+(x, \Phi) = \max_{x \in \mathbf{Cr}_\Phi} W^-(x, \Phi),$$

which proves that $\dim \Gamma^\infty = \dim X$.

□

CHAPTER 4

Tame Morse flows

For any smooth manifold M, $m = \dim M$, and any differentiable function $f : M \to \mathbb{R}$, we denote by $\mathbf{Cr}_f \subset M$ the set of critical points of f, and by $\Delta_f \subset \mathbb{R}$, the *discriminant set* of f, i.e., the set of critical values of f. For every positive integer λ and every positive real number r we denote by $\mathbb{D}^\lambda(r)$ the open Euclidean ball in $\mathbb{R}^\lambda$ of radius r centered at the origin. When $r = 1$ we write simply $\mathbb{D}^\lambda$.

If ξ is a C^2 vector field on M, and $p_0 \in M$ is a stationary point of p_0, then the linearization of ξ at p_0, is the linear map $L_{\xi,p_0} : T_{p_0}M \to T_{p_0}M$ defined by

$$L_{\xi,p_0}X_0 = (\nabla_X \xi)_{p_0}, \;\; \forall X_0 \in T_{p_0}M,$$

where ∇ is any linear connection on TM, and X is any vector field on M such that $X(p_0) = X_0$. The linearization is independent of the choice of ∇.

If $(x^i)_{1\leq i\leq m}$ are local coordinates on M such that

$$\xi = \sum_i \Xi^i \partial_{x^i},$$

then with respect to the basis (∂_{x^i}) of $T_{p_0}M$, the linearization of ξ at p_0 is described by the matrix $\bigl(\partial_{x^j}\Xi^i(p_0)\bigr)_{1\leq i,j\leq m}$.

DEFINITION 4.1. Suppose M is a compact, real analytic manifold of dimension m.

(a) A *Morse pair* on M is a pair (ξ, f), where ξ is a C^2 vector field on M and $f : M \to \mathbb{R}$ is a C^3, Morse function on M satisfying the following conditions.

- (a1) $\xi \cdot f < 0$ on $M \setminus \mathbf{Cr}_f$.
- (a2) For any $p \in \mathbf{Cr}_f$ the Hessian $H : T_pM \times T_pM \to \mathbb{R}$ of f at p satisfies
$$H_p\bigl(L_{\xi,p}X, X\bigr) < 0, \;\; \forall X \in T_pM \setminus 0.$$
- (a3) For every critical point p of f of index λ there exist an open neighborhood U_p of $p \in M$, a C^3-diffeomorphism,
$$\Psi : U_p \to \mathbb{D}^m,$$
and real numbers $\mu_1, \ldots, \mu_m > 0$ such that $\Psi(p) = 0$, and
$$\Psi_*(\xi) = \sum_{i\leq\lambda} \mu_i u^i \partial_{u^i} - \sum_{j>\lambda} \mu_j u^j \partial_{u^j},$$
where (u^i) denote the Euclidean coordinates on $\mathbb{D}^m$.

(b) The Morse pair (ξ, f) is called *tame* if the function f is tame, and the changes of coordinates Ψ are tame.

(c) The coordinate chart (U_p, Ψ) in (a) is said to be *adapted to* (ξ, f) *at* p. Using the coordinates determined by Ψ we define

$$|u_-|^2 := \sum_{j\leq\lambda} |u^j|^2, \quad |u_+|^2 = \sum_{j>\lambda} |u^j|^2, \quad \mu := 2\max(\mu_i) + 1,$$

$$E(u) := \begin{cases} 0 & \text{if} \quad p \text{ is a local max or local min} \\ \underbrace{\Big(\sum_{i\leq\lambda}(u^i)^{\mu/\mu_i}\Big)}_{E^-(u)} \cdot \underbrace{\Big(\sum_{j>\lambda}(u^j)^{\mu/\mu_j}\Big)}_{E^+(u)} & \text{if} \quad p \text{ is a saddle point, } 0<\lambda<m. \end{cases}$$

(d) For every triplet of real numbers $\varepsilon, \delta, r > 0$ we define the *block*

$$\mathcal{B}_p(\varepsilon, \delta, r) := \big\{\, u \in U_p;\;\; |f(u) - f(p)| < \varepsilon, \;\; E(u) < \delta, \;\; |u_-|^2 + |u_+|^2 < r^2 \big\}.$$

(e) A *Morse flow* on a compact real analytic manifold M is the flow generated by a C^3-vector field ξ, where for some C^3-function $f : M \to \mathbb{R}$ the pair (ξ, f) is a Morse pair on M. □

REMARK 4.2. Definition 4.1(a) is a mouthful. Condition (a1) states that f decreases strictly along the flow lines of ξ.

The strong condition is (a3). It says that we can find local coordinates near p so that, in these coordinates, the vector field ξ looks like a linear vector field on $\mathbb{R}^m$. More precisely, this linear vector field can be identified with a linear vector field on T_pM, namely the linearization of ξ at p. In particular, (a3) implies that, with respect to the adapted coordinates at p, the linearization of ξ at p is described by the diagonal matrix

$$L_{\xi,p} = \mathrm{Diag}(\mu_1, \dots, \mu_\lambda, -\mu_{\lambda+1}, \dots, -\mu_m\,).$$

The Hessian at p defines a quadratic function on T_pM and condition (a2) states that this function decreases along the flow lines of the above linear vector field on T_pM.

Note also that $\mu/\mu_i > 2$, for any i, and thus, if the pair (ξ, f) is *tame*, the function $E(u)$ is a *tame* C^2-function. Observe also that in U_p we have

$$\xi \cdot E^{\pm} = \pm\mu E^{\pm},$$

so that $\xi \cdot E = 0$, i.e., the quantity E is conserved along the trajectories of ξ. □

In the sequel we will need the following technical result.

PROPOSITION 4.3. *Suppose* (ξ, f) *is a Morse pair, and* $p \in \mathbf{Cr}_f$. *Fix* C^3-*coordinates* (u_-, u_+) *on open neighborhood* U_p *of* p *which are adapted to* (ξ, f). *Then there exists* $r_0 = r_0(f) > 0$ *such that for every* $r > 0$ *there exist* $\varepsilon_r, \delta_r > 0$ *such that*

$$\mathcal{B}_p(\varepsilon, \delta, r_0) \subset \mathbb{D}^m(r), \;\; \forall 0 < \varepsilon < \varepsilon_r, \;\; 0 < \delta < \delta_r.$$

In other words, no mater how small r *is we can choose* $\varepsilon, \delta > 0$ *sufficiently small so that the isolating block* $\mathcal{B}_p(\varepsilon, \delta, r_0)$*, a priori contained in* $\mathbb{D}^m(r_0)$*, is in fact contained in a much smaller ball* $\mathbb{D}^m(r)$.

Proof. Assume $f(p) = 0$. The statement is obviously true if p is a local min or a local max. We assume p is a saddle point and we denote by H the Hessian of f at p.

There exist $C = C(f) > 0$, and $\alpha = \alpha(\mu_1, \dots, \mu_m) > 1$ such that,

$$|f(u) - \frac{1}{2}H(u)| \le C(|u_-|^3 + |u_+|^3), \;\; E_\pm(u) \ge |u_\pm|^{2\alpha}, \;\; \forall |u_+| + |u_-| \le 1.$$

We deduce that if $u \in \overline{\mathcal{B}_p(\varepsilon, \delta, r)}$ and $r < 1$ then

$$-\varepsilon + C(|u_-|^3 + |u_+|^3) \le \frac{1}{2}H(u) \le \varepsilon + C(|u_-|^3 + |u_+|^3),$$

$$|u_\pm| \le r, \;\; |u_-| \cdot |u_+| \le \delta^{1/\alpha}.$$

The condition[1] (a2) in Definition 4.1 implies that the Hessian H of f is negative definite on the subspace $u_+ = 0$, and positive definite on the subspace $u_- = 0$. With respect to the decomposition $u = u_- + u_+$, the Hessian H is represented by a symmetric matrix with the block decomposition

$$H = \begin{bmatrix} Q_+ & B^* \\ B & Q_+ \end{bmatrix},$$

where Q_- is negative definite, and Q_+ is positive definite. Then

$$H(u_-, u_+) = (Q_- u_-, u_-) + (Q_+ u_+, u_+) + 2(Bu_-, u_+).$$

There exists a constant $\beta > 0$, depending only on B, such that for any $\hbar > 0$ we have

$$\beta\Big(-\frac{1}{\hbar}|u_-|^2 - \hbar|u_+|^2\Big) \le 2(Bu_-, u_+) \le \beta\Big(\hbar|u_-|^2 + \frac{1}{\hbar}|u_+|^2\Big).$$

Hence

$$(Q_- u_-, u_-) - \frac{\beta|u_-|^2}{\hbar} + (Q_+ u_+, u_+) - \beta\hbar|u_+|^2 \le H(u),$$

and

$$H(u) \le (Q_- u_-, u_-) + \beta\hbar|u_-|^2 + (Q_+ u_+, u_+) + \frac{\beta|u_+|^2}{\hbar}.$$

By choosing $\hbar$ sufficiently small we deduce that there exist constants $0 < a < 1 < b$ such that

$$-\frac{1}{a}|u_-|^2 + \frac{1}{b}|u_+|^2 \le \frac{1}{2}H(u) \le -a|u_-|^2 + b|u_+|^2.$$

Putting all of the above together we deduce that there exists $C_1 = C_1(f) > 1$ such that if

$$u \in \mathcal{B}_p(\varepsilon, \delta, r) \;\text{ and }\; r < 1$$

then,

$$\frac{1}{C_1}(-\varepsilon + |u_-|^3 + |u_+|^3) \le -|u_-|^2 + |u_+|^2 \le C_1(\varepsilon + |u_-|^3 + |u_+|^3),$$

and

$$|u_-| \cdot |u_+| \le \delta^{1/\alpha}.$$

Now fix $r_0 = \frac{1}{2C_1} < 1$. We want to show that for every $r < r_0$ there exist $\varepsilon, \delta > 0$ such that $\mathcal{B}_p(\varepsilon, \delta, r_0) \subset \mathbb{D}^m(r)$.

We argue by contradiction, and we assume there exists $0 < \bar{r} < r_0$ such that, for any $\varepsilon, \delta > 0$, we have

$$\mathcal{B}_p(\varepsilon, \delta, r_0) \not\subset \mathbb{D}^m(\bar{r}).$$

[1]The adapted coordinates need not diagonalize the Hessian, so that (a3) $\not\Rightarrow$ (a2).

We deduce that we can find a sequence $u_n \in \mathcal{B}_p(1/n, 1/n, r_0)$ such that $|u_n| \geq \bar{r}$. Set

$$s_n := |(u_n)_-|, \;\; t_n := |(u_n)_+|.$$

We deduce that $s_n^2 + r_n^2 \geq \bar{r}^2$, and

$$\frac{1}{C_1}\Big(-\frac{1}{n} + s_n^3 + t_n^3\Big) < -s_n^2 + t_n^2 \leq C_1\Big(\frac{1}{n} + s_n^3 + t_n^3\Big), \;\; s_n t_n \leq n^{-1/\alpha}, \;\; s_n, \, t_n < r_0.$$

The condition

$$s_n t_n \leq n^{-1/\alpha}, \;\; 0 \leq s_n, \, t_n < r_0 = \frac{1}{2C_1}$$

implies that a subsequence of s_n converges to $s_\infty \in [0, r_0]$, a subsequence of t_n converges to $t_\infty \in [0, r_0]$ and

$$s_\infty t_\infty = 0, \;\; s_\infty^2 + t_\infty^2 \geq \bar{r}^2.$$

We observe that $t_\infty \neq 0$ because, if that were the case, we would have

$$0 < \frac{1}{C_1} s_\infty^3 \leq -s_\infty^2 < 0.$$

Hence we must have $s_\infty = 0$ and $t_\infty \neq 0$. We deduce

$$t_\infty^2 \leq C_1 t_\infty^3, \;\; t_\infty > \bar{r} \Longrightarrow \frac{1}{C_1} \leq t_\infty \leq r_0 = \frac{1}{2C_1}.$$

We have reached a contradiction. This concludes the proof of Proposition 4.3. □

The above proposition implies that for $r < r_0$, and any ε, δ sufficiently small, we have

$$\partial \mathcal{B}_p(\varepsilon, \delta, r_0) \cap \partial \mathbb{D}^m(r) = \emptyset.$$

When this happens we say that $\mathcal{B}_p(\varepsilon, \delta, r_0)$ is an *isolating block* of p. The boundary of such an isolating block has a decomposition

$$\partial \mathcal{B}_p(\varepsilon, \delta, r_0) = \partial_+ \mathcal{B}_p(\varepsilon, \delta, r_0) \cup \partial_- \mathcal{B}_p(\varepsilon, \delta, r_0) \cup \partial_0 \mathcal{B}_p(\varepsilon, \delta, r_0),$$

where

$$\partial_\pm \mathcal{B}_p(\varepsilon, \delta, r_0) = \overline{\mathcal{B}_p(\varepsilon, \delta, r_0)} \cap \{f = f(p) \pm \varepsilon\},$$

and

$$\partial_0 \mathcal{B}_p(\varepsilon, \delta, r_0) = \overline{\mathcal{B}_p(\varepsilon, \delta, r_0)} \cap \{E(u) = \delta\}.$$

The function $E(u)$ is twice differentiable (since $\mu/\mu_i > 2$), and it is constant along the trajectories of ξ while f decreases along these trajectories. This implies that no trajectory of ξ which starts at a point

$$q \in \{f(p) - \varepsilon < f < f(p) + \varepsilon\} \setminus \mathcal{B}_p(\varepsilon, \delta, r_0)$$

can intersect the block $\mathcal{B}_p(\varepsilon, \delta, r_0)$.

THEOREM 4.4. *Suppose (ξ, f) is a tame Morse pair on M such that ξ is real analytic. Then the flow generated by ξ is tame.*

PROOF. First, let us introduce some terminology. Suppose (ξ, f) is a tame Morse pair on the real analytic manifold and $\Phi = \Phi^\xi : \mathbb{R} \times M \to M$ is the flow generated by ξ. For any subset $A \subset M$ we set

$$A^\xi := \Big\{ y \in M; \;\; \exists t \geq 0, \;\; x \in A : \;\; y = \Phi_t(x) \Big\}.$$

In other words, A^{ξ} is the forward drift of A, i.e., the region of M covered by the forward trajectories of ξ which start at a point in A. We define similarly

$$A^{-\xi} := \Big\{ y \in M;\ \ \exists t \leq 0,\ \ x \in A :\ \ y = \Phi_t(x) \Big\}.$$

Step 1. Let $c \in \Delta_f$. We will show that there exists $\sigma = \sigma(c) > 0$ such that, for any $\varepsilon \in (0, \sigma)$, and any tame set $A \subset \{c - \varepsilon < f < c + \varepsilon\}$, the intersection

$$A^{\xi}(\varepsilon) := A^{\xi} \cap \{c - \varepsilon < f < c + \varepsilon\}$$

is a tame set.

Let $\gamma > 0$ such that the only critical value of f in the interval $(c - \gamma, c + \gamma)$ is c, and define

$$\mathbf{Cr}_f^c := \mathbf{Cr}_f \cap \{f = c\}.$$

The set $\mathbf{Cr}_f^c$ is finite. We can find $\varepsilon_0, r_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$, and any $p \in \mathbf{Cr}_f$, the blocks $\mathcal{B}_p := \mathcal{B}_p(\varepsilon, \varepsilon, r_0)$ are isolating, and their closures are pairwise disjoint. Set

$$\sigma := \min(\gamma, \varepsilon_0).$$

For $0 < \varepsilon < \sigma$, $p \in \mathbf{Cr}_f^c$, and any tame subset

$$A \subset \{c - \varepsilon < f < c + \varepsilon\}$$

we set,

$$A_p := A \cap \mathcal{B}_p(\varepsilon, \varepsilon, r_0),\ \ \mathcal{B}_\varepsilon := \bigcup_{p \in \mathbf{Cr}_f^c} \mathcal{B}_p(\varepsilon, \varepsilon, r_0),$$

$$Z_\varepsilon := \{c - \varepsilon < f < c + \varepsilon\} \setminus \mathcal{B}_\varepsilon,\ \ A_* = A \cap Z_\varepsilon.$$

Since

$$A = A_* \cup \Big(\bigcup_{p \in \mathbf{Cr}_f^c} A_p \Big)$$

it suffices to show that each of the subsets $A_*(\varepsilon)^{\xi}$ and $A_p(\varepsilon)^{\xi}$ is definable.

Note first that, since the isolating blocks $\mathcal{B}_p$ are definable sets, each A_p is definable.

For $p \in \mathbf{Cr}_f^c$ we denote by λ_p its index, and we choose a coordinate chart (U_p, Ψ_p) adapted to (ξ, f) near p such that

$$\Phi_t(u) = (e^{\mu_1 t} u^1, \cdots, e^{\mu_\lambda t} u^\lambda, e^{-\mu_{\lambda+1} t} u^{\lambda+1}, \cdots, e^{-\mu_m t} u^m).$$

We deduce that

$$A_p^{\xi} \cap \{c - \varepsilon < f < c + \varepsilon\}$$

$$= \Big\{\, u \in A_p;\ \ \exists t \geq 0 :\ \ (e^{\mu_1 t} u^1, \cdots, e^{\mu_\lambda t} u^\lambda, e^{-\mu_{\lambda+1} t} u^{\lambda+1}, \cdots, e^{-\mu_m t} u^m) \in \mathcal{B}_p,\ \Big\}$$

This shows A_p^{ξ} is definable.

Note that no trajectory of ξ starting on Z_ε will intersect the neighborhood $\mathcal{B}_\varepsilon$ of $\mathbf{Cr}_f^c$. Let

$$m := \inf \Big\{\, |\xi \cdot f(x)|;\ \ x \in Z_\varepsilon \,\Big\}.$$

Observe that $m > 0$. Fix $T > \frac{2\varepsilon}{m}$. We deduce that

$$\forall x \in Z_\varepsilon,\ \ \Phi_T(x) \in \{f < c - \varepsilon\}.$$

Since the vector field ξ is real analytic we deduce from the Cauchy-Kowalewski theorem (in the general form proved in [**9**, I.§7]) that the flow map

$$\Phi : [0, T] \times M \to M$$

is real analytic.

Observe that

$$A_*(\varepsilon)^\xi = \Big\{ y \in M;\ \ |f(y) - c| < \varepsilon,\ \ \exists t \in [0,T],\ \ \exists x \in A_* : \Phi_t(x) = y \Big\}.$$

This shows $A_*^\xi(\varepsilon)$ is definable. In particular, we deduce that for $\varepsilon < \sigma(c)$, and every definable

$$A \subset \{c - \varepsilon \leq f \leq c + \varepsilon\}$$

the set $A^\xi \cap \{c - \varepsilon \leq f \leq c + \varepsilon\}$ is also definable.

Step 2. Suppose the interval $[a,b]$ contains no critical values of f. Then for every tame set $A \subset \{a \leq f \leq b\}$ the set

$$A^\xi \cap \{a \leq f \leq b\},\ \ \text{and}\ \ A^{-\xi} \cap \{a \leq f \leq b\}$$

are also tame. Indeed, let

$$m := \inf\big\{ |\xi \cdot f(x)|;\ \ a \leq f(x) \leq b,\ \big\}.$$

Since the interval $[a,b]$ contains no critical values we deduce that $m > 0$. Fix $T > \frac{b-a}{m}$. Then

$$\forall x \in \{f = b\},\ \ \Phi_T(x) \in \{f < a\}.$$

Observe that

$$A^\xi \cap f^{-1}([a,b]) = \big\{ y \in M;\ \ a \leq f(y) \leq b,\ \ \exists t \in [0,T],\ \ x \in A:\ \ y = \Phi_T(x) \big\}.$$

We deduce from the above description that $A^\xi \cap f^{-1}([a,b])$ is definable since A is so and the map $\Phi : [0,T] \times M \to M$ is real analytic.

Step 3. Suppose A is a tame subset of N. Then A^ξ and $A^{-\xi}$ are also tame. To prove this we must first consider an *f-slicing*. This is a finite collection of real numbers

$$a_0 < a_1 < \cdots < a_n$$

with the following properties.

- $f(M) \subset [a_0, a_n]$.
- a_i is a regular value of f, $\forall i = 0, \cdots, n$.
- Every interval $[a_{i-1}, a_i]$, $1 \leq i \leq n$ contains at most one critical value of f.
- If the interval $[a_{i-1}, a_i]$ contains one critical value of f, then this critical value must be the midpoint

$$c_i = \frac{a_i + a_{i-1}}{2}$$

  Moreover, the interval $[a_{i-1}, a_i]$ is very short, i.e., $(a_i - a_{i-1}) < \sigma(c_i)$.

Fix an f-slicing $a_0 < \cdots < a_n$, and a tame set $A \subset M$. Now define

$$M_i := f^{-1}([a_{i-1}, a_i]),\ \ \partial_- M_i = \{f = a_{i-1}\}.$$

Then

$$A^\xi = \bigcup_i (A \cap M_i)^\xi.$$

We will prove by induction over i that for every tame set $B \subset M_i$ the set B^ξ is also tame. For $i = 1$, the interval $[a_0, a_1]$ must contain a critical value, the absolute minimum and we conclude using Step 1 since

$$B \subset M_1 \Longrightarrow B^\xi \subset M_1 \Longrightarrow B^\xi = B^\xi \cap M_1.$$

Consider now a tame set $B \subset M_{i+1}$. Then

$$B^{\xi} = B^{\xi} \cap M_{i+1} \cup (B^{\xi} \cap \partial_- M_{i+1})^{\xi}.$$

$B^{\xi} \cap M_{i+1}$ is tame by Step 1, if the interval $[a_i, a_{i+1}]$ contains a critical value, or by Step 2, if the interval $[a_i, a_{i+1}]$ contains no critical value.

Now observe that

$$B^{\xi} \cap \partial_- M_{i+1} = (B^{\xi} \cap M_{i+1}) \cap \partial_- M_{i+1}$$

so that $B^{\xi} \cap \partial_- M_{i+1}$ is a tame subset of $\partial_- M_{i+1} \subset M_i$. The induction hypothesis now implies that $(B^{\xi} \cap \partial_- M_{i+1})^{\xi}$ is tame.

Step 4. *Conclusion.* Suppose (ξ, f) is a tame Morse pair on M. We construct a new tame Morse pair $(\hat{\xi}, \hat{f})$ on $S^1 \times M \times M$ defined by

$$\hat{f}(\theta, x, y) = h_0(\theta) - f(x) + f(y), \ \ \forall (p, x, y) \in S^1 \times M \times M,$$

where $h_0 : S^1 \to \mathbb{R}$ is the height function we considered in Example 2.6. Similarly

$$\hat{\xi}(\theta, x, y) = \xi_0(\theta) \oplus -\xi(x) \oplus \xi(y),$$

where ξ_0 is the gradient of $-h_0$. Denote by θ_0 the point $(1, 0)$ on the unit circle and let

$$\Delta = \{(\theta_0, x, x) \in S^1 \times M \times M\}.$$

By Step 3 the set

$$G = \Delta^{\hat{\xi}} \cup \Delta^{-\hat{\xi}} = \Big\{ (\theta, u, v) \in S^1 \times M \times M; \ \ \exists t \in \mathbb{R}, \ \ x \in M : \ \ (\theta, u, v) = \Phi_t^{\hat{\xi}}(\theta_0, x, x) \Big\}$$

is tame. Since the negative gradient flow of h_0 in the open half-circle $S = \{x^2 + y^2 = 1; \ \ x > 0\}$ is tamely conjugate to the translation flow on $\mathbb{R}$ we deduce from Corollary 3.4 that Φ is a tame flow. □

THEOREM 4.5. *Suppose X is a compact, real analytic manifold and $f : X \to \mathbb{R}$ is a real analytic Morse function. Then for every real analytic metric g_0 on X and every $\varepsilon > 0$ there exist a real analytic metric g on X with the following properties.*

- $\|g_0 - g\|_{C^2} \leq \varepsilon$.
- $(f, -\nabla^g f)$ *is a tame Morse pair.*

In particular, the flow generated by $-\nabla^g f$ is a tame Morse flow.

PROOF. The proof is based on a simple strategy. We show that we can find real analytic metrics g arbitrarily C^2-close to g_0 such that the gradient vector field $\nabla^g f_0$ can be linearized by a real analytic change of coordinates localized in a neighborhood of the critical set. The linearizing change of coordinates is obtained by invoking the Poincaré-Siegel theorem [**1**, Chap. 5] on the normal forms of real analytic vector fields in a neighborhood of an isolated stationary point.

We digress to recall the Poincaré-Siegel theorem. Suppose $\vec{Z}$ is a real analytic vector field defined in a neighborhood $\mathcal{N}$ of the origin 0 of the Euclidean vector space $\mathbb{R}^n$. Assume that 0 is an isolated stationary point of the vector field $\vec{Z}$. If we regard $\vec{Z}$ as a real analytic map

$$\vec{Z} : \mathcal{N} \to \mathbb{R}^N$$

then we obtain a Taylor expansion near 0

$$\vec{Z}(x) = L \cdot x + \text{higher order terms}, \ \ x \in \mathcal{N},$$

where $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator. We regard it as a linear operator $T_0\mathbb{R}^n \to T_0\mathbb{R}^n$. As such, it can be identified with the linearization of $\vec{Z}$ at the origin.

The Poincaré-Siegel theorem describes conditions on L which imply the existence of real analytic coordinates $y = (y^1, \cdots, y^n)$ near $0 \in \mathbb{R}^n$ such that in these new coordinates the vector field $\vec{Z}$ is linear,

$$\vec{Z}(y) = Ly = \sum_j y^j L\partial_{y^j}.$$

We describe these conditions only in the case when L is semisimple (diagonalizable) and all its eigenvalues are real since this is the only case of interest to us.

Denote the eigenvalues of L by

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n.$$

We write

$$\vec{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n.$$

We say that L satisfies the *Siegel* (C, ν)-condition if, for any $k = 1, \ldots, n$, and any $\vec{m} = (m_1, \cdots, m_n) \in (\mathbb{Z}_{\geq 0})^n$ such that

$$|\vec{m}| := m_1 + \cdots + m_n \geq 2,$$

we have

$$|\mu_k - (\vec{m}, \vec{\mu})| \geq \frac{C}{|\vec{m}|^\nu}.$$

We denote by $\mathcal{S}_{C,\nu} \subset \mathbb{R}^n$ the set of vectors $\vec{\mu}$ satisfying the Siegel (C, ν)-condition, and we set

$$\mathcal{S}_\nu := \bigcup_{C>0} \mathcal{S}_{C,\nu}.$$

Then the set $\mathbb{R}^n \setminus \mathcal{S}_\nu$ has zero Lebesgue measure if $\nu > \frac{n-2}{2}$, [**1**, §24.C]. In other words, if we fix $\nu > \frac{n-2}{2}$ then almost every vector $\vec{\mu} \in \mathbb{R}^n$ satisfies the Siegel (C, ν)-condition for some $C > 0$. We can now state the Poincaré-Siegel theorem whose very delicate proof can be found in [**1**, Chap.5] or [**37**].

THEOREM 4.6 (Poincaré-Siegel). *Suppose that the eigenvalues $(\mu_1, \ldots, \mu_n)$ satisfy the Siegel (C, ν) condition for some $C > 0$ and $\nu > 0$. Then there exist local, real analytic coordinates $y = (y^1, \ldots, y^n)$ defined in a neighborhood of $0 \in \mathbb{R}^n$ such that, in these coordinates, the vector field $\vec{Z}$ is linear,*

$$\vec{Z}(y) = L(y). \qquad \square$$

After this digression we return to our original problem.

According to [**19**], one can find a real analytic isometric embedding of (X, g_0) in some Euclidean space $\mathbb{R}^N$. For every real analytic metric g on X we set $\vec{Z}_g := \nabla^g f_0 \in \mathrm{Vect}(X)$. For every $p_0 \in \mathbf{Cr}_{f_0}$ we denote by $L_{g,p_0} : T_{p_0}X \to T_{p_0}X$ the linearization of Z_g at p.

The operator L_{g,p_0} is symmetric (with respect to the metric g) and thus diagonalizable. More precisely, if we choose local analytic coordinates on X near $p_0 \in \mathbf{Cr}_{f_0}$ such that $x^i(p_0) = 0$, and the vectors ∂_{x_i} form a g-orthonormal basis of $T_{p_0}X$, then

$$Z_g = \sum_{i,j} g^{ij}(\partial_{x^j} f_0)\partial_{x^i}$$

$$L_{g,p_0}\partial_{x^k} = \sum_{i,j}\Big(\big(\partial_{x^k} g^{ij}\big)(p_0)\cdot(\partial_{x^j} f_0)(p_0) + g^{ij}(p_0)\big(\partial^2_{x^k x^j} f_0\big)(p_0)\Big)\partial_{x^i}.$$

$$= \sum_{i,j}\delta^{ij}\big(\partial^2_{x^k x^j} f_0\big)(p_0)\partial_{x^i} = (\partial^2_{x^k x^k} f_0)(p_0)\partial_{x^k}$$

Thus, for every orthonormal basis of $T_{p_0}M$, the matrix describing the linearization of $\nabla^g f$ at p_0 coincides with the matrix describing the Hessian of f at p_0. This shows that the pair $(-\nabla^g f, f)$ satisfies the conditions (a1) and (a2) in the definition of a Morse pair, Definition 4.1.

We want to prove that arbitrarily close to any real analytic metric g we can find real analytic metrics h such that for every $p_0 \in \mathbf{Cr}_{f_0}$ there exist real analytic coordinates y so that in these coordinates the vector field Z_{h,p_0} has the linear form

$$Z_{h,p_0}(y) = L_{h,p_0}(y).$$

Here is the strategy. Near each p_0 choose local analytic coordinates $(x^i = x^i_{p_0})$ such that (∂_{x^i}) is a g-orthonormal basis of $T_{p_0}X$ which diagonalizes the Hessian matrix, i.e.

$$\partial^2_{x^i x^j} f_0(p_0) = 0,\ \ \forall i \neq j.$$

If h is another metric on X then the above computation shows that

$$L_{h,p_0}\partial_{x^k} = \sum_{i,j} h^{ij}\big(\partial^2_{x^k x^j} f_0\big)(p_0)\partial_{x^i} = \sum_i h^{ik}\big(\partial^2_{x^k x^k} f_0\big)(p_0)\partial_{x^i}.$$

Denote by $\mathrm{Sym}^+(n)$ the space of positive definite, symmetric $n\times n$ matrices. We will show that for any map

$$A : \mathbf{Cr}_{f_0} \to \mathrm{Sym}^+(n),\ \ p \mapsto A_p$$

close to the identity map

$$\boldsymbol{I} : \mathbf{Cr}_{f_0} \to \mathrm{Sym}^+(n),\ \ p \mapsto \boldsymbol{I}_n,$$

there exists a real analytic metric h, close to g, such that, for every $p \in \mathbf{Cr}_{f_0}$, the matrix describing h at p in the coordinates (x^i_p) chosen above is equal to A_p^{-1}. In other words, we want to show that as h runs through a small neighborhood of g, the collection of matrices

$$\mathbf{Cr}_{f_0} \ni p \mapsto \big(h^{ij}(p)\big)_{1\leq i,j\leq n} \in \mathrm{Sym}^+(n)$$

spans a small neighborhood of the identity map. This is achieved via a genericity result. We can then prescribe h so that at every $p \in \mathbf{Cr}_{f_0}$ the linearization $L_{h,p}$ satisfies the conditions of the Poincaré-Siegel theorem.

To prove that we can prescribe the metric h any way we please at the points in $\mathbf{Cr}_{f_0}$ we will prove an elementary genericity result, which can be viewed as a multivariable generalization of the classical Lagrange interpolation formula. To state it we need a bit of terminology.

Fix a finite dimensional Euclidean space E and denote by $\mathcal{P}_d(\mathbb{R}^N, E)$ the vector space of polynomial maps $\mathbb{R}^N \to E$ of degree $\leq d$. For every E-valued, real analytic function f defined in the neighborhood of a point $x \in \mathbb{R}^N$, and every nonnegative integer k we denote by $j_k(f, x) \in \mathcal{P}_k(\mathbb{R}^n, E)$ the k-th jet of f at x.

LEMMA 4.7. *Let $B \subset \mathbb{R}^N$ be an open ball and $S \subset B$ a finite subset. For every integers $d > k > 0$ define the* linear map

$$\mathbf{ev}_{d,k} : \mathcal{P}_d(\mathbb{R}^N, \mathbb{R}^\ell) \to \prod_{s\in S} \mathcal{P}_k(\mathbb{R}^N, \mathbb{R}^\ell), \;\; f \longmapsto (j_k(f,s))_{s\in S}.$$

Then $\mathbf{ev}_{d,k}$ is onto if $d \geq 2(k+1)|S| - 2$.

PROOF. [2] It suffices to show that for every $s_0 \in S$, and every $P_0 \in \mathcal{P}_k(\mathbb{R}^N, \mathbb{R}^\ell)$ there exists $f \in \mathcal{P}_d(\mathbb{R}^n, \mathbb{R}^\ell)$ such that

$$j_k(f, s_0) = j_k(P_0, s_0) \;\; j_k(f, s) = 0, \;\; \forall s \neq s_0.$$

Clearly it suffices to prove this only in the case $\ell = 1$. For every $s \in S$ define

$$\rho_s(x) = |x - s|^2 \in \mathcal{P}_2(\mathbb{R}^N, \mathbb{R}).$$

Observe that

$$j_k(\rho_s^{k+1}, s) = 0, \quad \forall s \in S, \;\; \forall k \geq 0.$$

Now define

$$Q_{s_0} = \prod_{s\neq s_0} \rho_s^{k+1}, \;\; \deg Q_{s_0} = 2(k+1)(|S| - 1).$$

Observe that for every polynomial function p we have

$$j_k(pQ_{s_0}, s) = 0, \;\; \forall s \neq s_0.$$

The function $1/Q_{s_0}$ is real analytic in a neighborhood of s_0, and we denote by $R_{s_0} \in \mathcal{P}_k(\mathbb{R}^N, \mathbb{R})$ the k-th jet of $1/Q_{s_0}$ at s_0. Then

$$j_k(R_{s_0} Q_{s_0}, s_0) = 1.$$

Now define

$$f = P_0 R_{s_0} Q_{s_0}, \;\; \deg f = k + k + (2k+2)(|S| - 1) = 2(k+1)|S| - 2.$$

Then

$$j_k(f, s_0) = j_k\big(\, j_k(P_0, s_0) \cdot j_k(R_{s_0} Q_{s_0}, s_0), s_0 \,\big) = j_k(P_0, s_0),$$

and

$$j_k(f, s) = 0, \;\; \forall s \neq s_0. \qquad \square$$

Suppose now that the set S lies on the compact real analytic submanifold $X \subset \mathbb{R}^N$. By choosing real analytic coordinates on X near each point $s \in S$ we obtain locally defined real analytic embeddings

$$i_s : U_s \subset \mathbb{R}^n \to \mathbb{R}^N, \;\; i_s(0) = s, \;\; i_s(U_s) \subset X.$$

Here, for every $s \in S$, we denoted by U_s a small, open ball centered at $0 \in \mathbb{R}^n$, $n = \dim X$. In particular, for every Euclidean vector space E, and every positive integer k we obtain *surjective* linear maps

$$\pi_s : \mathcal{P}_k(\mathbb{R}^N, E) \to \mathcal{P}_k(\mathbb{R}^n, E), \;\; \mathcal{P}_k(\mathbb{R}^N, E) \ni f \mapsto j_k(\, f \circ i_s, 0).$$

For $x \in X$ we denote by $J_k(X, x, E)$ the space of k-jets at x of E-valued real analytic maps defined in a neighborhood of x. If f is such a map, then we denote by $j_k(f, x) \in J_k(X, x, E)$ its k-th jet. We topologize $\mathcal{P}_d(\mathbb{R}^N, E)$ by setting

$$|f| = \|f\|_{C^2(X)}$$

Lemma 4.7 implies the following result.

[2]The idea of this proof arose in conversations with my colleague R. Hind.

LEMMA 4.8. *Suppose S is a finite subset of X. Then for any finite dimensional Euclidean space E, and any integer $d \geq 2|S|-2$ the linear map*

$$\mathbf{ev} : \mathcal{P}_d(\mathbb{R}^N, E) \to \prod_{s\in S} J_0(X, s, E), \;\; f \longmapsto \big(\, j_0(f,s)\,\big)_{s\in S}$$

is onto. In particular, for every $\varepsilon > 0$, the image of an ε-neighborhood of $0 \in \mathcal{P}_d(\mathbb{R}^N, \mathbb{R})$ is an open neighborhood of 0 in $\prod_{s\in S} J_0(X, s, E)$. □

We now specialize E to be the space $\mathrm{Sym}(N)$ of symmetric bilinear forms $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, and thus the space of functions

$$\mathbb{R}^N \to \mathrm{Sym}(N)$$

can be viewed as the space of deformations of Riemann metrics on $\mathbb{R}^N$. The metric g_0 on X is induced from the Euclidean metric δ on $\mathbb{R}^N$. If we deform δ

$$\delta \to \delta + h, \;\; h \in \mathcal{P}_d(\mathbb{R}^N, E), \;\; d > 2|S|,$$

and $|h|$ is sufficiently small, then $\delta + h$ will still be a metric on a neighborhood of X in $\mathbb{R}^n$.

Fix $s \in X$. Choose affine Euclidean coordinates $(y^\alpha)_{1\leq\alpha\leq N}$ on $\mathbb{R}^N$ such that $y^\alpha(s) = 0$, $\forall \alpha$. Choose local real analytic coordinates $(x^1, \cdots, x^n)$ on X in a neighborhood U_s of s such that $x^j(s) = 0$, $\forall j$. Along X near s the vector field ∂_{x^i} is represented by the vector field

$$\sum_\alpha \frac{\partial y^\alpha}{\partial x^i} \partial_{y^\alpha}.$$

If

$$g(\partial_{y^\alpha}, \partial_{y^\beta}) = (\delta + h)(\partial_{y^\alpha}, \partial_{y^\beta}) = \delta_{\alpha\beta} + h_{\alpha\beta}$$

then

$$g(\partial_{x^i}, \partial_{x^j}) = \sum_{\alpha,\beta} \big(\,\delta_{\alpha\beta} + h_{\alpha\beta}\,\big) \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} = g_0(\partial_{x^i}, \partial_{x^j}) + \sum_{\alpha,\beta} h_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}.$$

We think of $g_{ij}(x)$ as a real analytic map from U_s to the space of symmetric $n\times n$ matrices, of $\frac{\partial y^\alpha}{\partial x^i}$ a real analytic map Y from U_s to the space of $N \times n$ matrices, and we think of h as a real analytic map from U_s to the space of symmetric $N\times N$ matrices. Then

$$g = g_0 + Y^t h|_{U_s} Y.$$

Along U_s we write

$$Y = Y(0) + O(1), \;\; h = h(0) + O(1)$$

so that

$$g = g_0 + Y(0)^t h(0) Y(0) + O(1).$$

The map $Y(0) : \mathbb{R}^n \to \mathbb{R}^N$ is injective, since it describes the canonical injection $T_sX \hookrightarrow T_s\mathbb{R}^N$. The correspondence

$$\mathrm{Sym}(N) \ni h \longmapsto \mathcal{Y}(h) := Y^t(0) h Y(0) \in \mathrm{Sym}(n)$$

is a linear map. Intrinsically, $\mathcal{Y}$ is the restriction map, i.e.,

$$\mathcal{Y}(h)\,(u,v) = h(u,v), \;\; \forall u, v \in T_sX \subset T_s\mathbb{R}^n.$$

This shows that $\mathcal{Y}$ is onto, because any symmetric bilinear map on T_sX can be extended (in many different ways) to a symmetric bilinear map on $\mathbb{R}^N$. This concludes the proof of Theorem 4.5. □

We can refine the above existence result some more.

THEOREM 4.9. *Suppose M is a compact, real analytic manifold,* $\dim M = m$, *$f : M \to \mathbb{R}$ is a real analytic tame Morse function. For every critical point p of f we denote by $\lambda(p)$ the Morse index of f at p. For every $p \in \mathbf{Cr}_f$, we choose a vector*

$$\vec{a}(p) = \big(a_1(p), \dots, a_m(p) \big) \in \mathbb{R}^m$$

such that

$$a_1 \leq \cdots \leq a_{\lambda(p)} < 0 < a_{\lambda(p)+1} \leq \cdots \leq a_m.$$

Then, for every $\varepsilon > 0$, we can find a real analytic metric g on M, such that for every critical point p of f there exist real analytic coordinates (x^i) near p, and a vector $\vec{b} = \vec{b}(p) \in \mathbb{R}^m$ with the following properties.

(a) $x^i(p) = 0$, $\forall i = 1, \dots, m$.

(b) $|\vec{b}(p) - \vec{a}(p)| < \varepsilon$.

(c) In the coordinates (x^i) the vector field $\nabla^g f$ is described by,

$$\nabla^g f = \sum_{i=1}^m b_i x^i \partial_{x^i}.$$

PROOF. From the Morse lemma we deduce that we can find a *smooth* metric g_0 on M with the property that for every critical point p there exist *smooth* coordinates (y^i) near p with the property that

$$y^i(p) = 0, \quad \forall i = 1, 2, \dots, m$$

and

$$\nabla^{g_0} f = \sum_{i=1}^m a_i y^i \partial_{y^i}.$$

Now choose a real analytic metric g_1, sufficiently close to g_0 such that the linearization of $\nabla^{g_1} f$ at p is given by a diagonalizable operator $L_p : T_pM \to T_pM$ with eigenvalues

$$\ell_1(p) \leq \cdots \leq \ell_m(p)$$

satisfying

$$|\vec{\ell}(p) - \vec{a}(p)| < \frac{\varepsilon}{2}, \;\; \vec{\ell}(p) = (\ell_1(p), \dots, \ell_m(p)).$$

Using Theorem 4.5 we can find a real analytic metric g on M such that the gradient vector field $\nabla^g f$ can be linearized by an analytic change of coordinates in a neighborhood of every critical point, and for every critical point p the linearization of $\nabla^g f$ at p is a diagonalizable linear operator $B_p : T_pM \to T_pM$ whose eigenvalues

$$b_1(p) \leq \cdots \leq b_m(p)$$

satisfy

$$|\vec{b}(p) - \vec{\ell}(p)| < \frac{\varepsilon}{2}.$$

This completes the proof of Theorem 4.9. □

CHAPTER 5

Tame Morse-Smale flows

Suppose (ξ, f) is a tame Morse pair on the compact real analytic manifold M, $\dim M = m$, such that the flow Φ^ξ generated by ξ is tame. Then, for every critical point p of the Morse function f we denote by $W^+(p,\xi)$ (respectively $W^-(p,\xi)$) the stable (respectively the unstable) variety of p with respect to the flow Φ^ξ. Then $W^-(p,\xi)$ is a C^2-submanifold of M homeomorphic to $\mathbb{R}^{\lambda(p)}$, where $\lambda(p)$ denotes the Morse index of f at p. Similarly, $W^+(p,\xi)$ is a C^2-submanifold of M homeomorphic to $\mathbb{R}^{m-\lambda(p)}$.

We say that Φ^ξ satisfies the *Morse-Smale condition* if, for every pair of critical points p, q such that $f(p) > f(q)$, the unstable manifold of p intersects transversally the stable manifold of q.

THEOREM 5.1. *Suppose M is a compact, real analytic manifold of dimension m, and (ξ, f) is a tame Morse pair such that both f and ξ are real analytic. Denote by Φ^ξ the flow generated by ξ. Then there exists a smooth vector field η, which coincides with ξ in an open neighborhood of the critical set of f, such that the pair (η, f) is a tame Morse pair, the flow generated by η is tame and satisfies the Morse-Smale condition.*

PROOF. We follow closely the approach pioneered by S. Smale (see e.g. [**36**, Section 2.4]). For simplicity, we assume f is nonresonant, i.e., every critical level set of f contains a unique critical point. Suppose that the critical points are

$$p_0, \dots, p_\nu, \;\; f(p_0) < f(p_1) < \cdots < f(p_\nu).$$

For simplicity, we set $c_k := f(p_k)$. Define

$$\hbar = \min_{k=1,\dots,\nu} (c_k - c_{k-1}).$$

We will prove by induction that for every $k = 0, 1, \dots, \nu$, and for every $0 < \varepsilon < \frac{\hbar}{100}$, there exists a tame C^∞ vector field η_k on M with the following properties.

- $\eta_0 = \xi$.
- $\eta_k(x) = \eta_j(x)$, $\forall 0 \leq j < k$, $\forall x \in M$ such that $f(x) \notin (c_k - 2\varepsilon, c_k - \varepsilon)$.
- The pair $\big(f, \eta_k,\big)$ is a tame Morse pair.
- The flow generated by η_k is tame and

$$W^-(p_j, \eta_k) \pitchfork W^+(p_i, \eta_k), \;\; \forall 0 \leq i < j \leq k.$$

The statement is trivial for $k = 0$ so we proceed directly to the inductive step. Assume we have constructed $\eta_0, \dots, \eta_k$, and we want to produce η_{k+1}. Denote by Φ^k_t the flow generated by η_k. Set $Z := \{f = c_{k+1} - \varepsilon\}$. Then there exists $\tau > 0$ such that

$$\forall z \in Z, \;\; \forall t \in [0, t] : f(\Phi^k_t z) > c_k - 2\varepsilon.$$

Set (see Figure 4)

$$Z_\tau := \Phi^k_\tau(Z_\varepsilon),\ \ S_\tau = S_\tau = \bigcup_{t\in[0,\tau]} \Phi^k_t(Z),\ \ C_\tau = [0,\tau]\times Z,$$

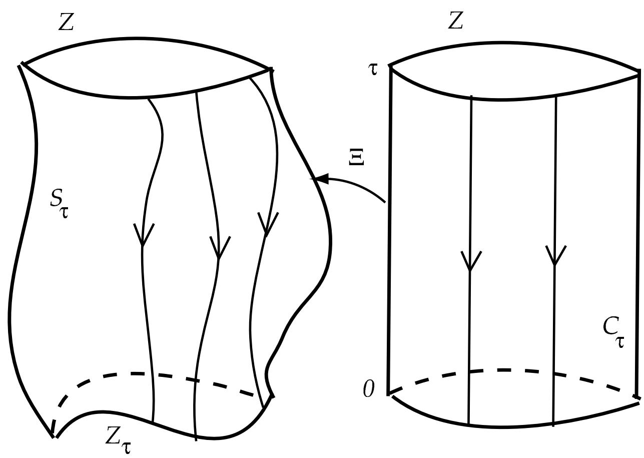


FIGURE 4. *Truncating a Morse flow.*

and define

$$X = W^-(p_{k+1},\eta_k)\cap Z,\ \ Y = \bigcup_{j\leq k} W^+(p_k,\eta_k)\cap Z.$$

Z is a real analytic manifold, X is a compact, real analytic submanifold of Z_ε, while Y is a smooth submanifold of Z_ε. From to the classical transversality results of Whitney (see [**49**] or [**25**, Chap.3, 8]) we deduce that there exists a smooth map

$$h : [0,\tau]\times Z \to Z,\ \ (t,z)\mapsto h_t(z)$$

with the following properties.

- $h_0(z) = z$, $\forall z \in Z$.
- h_t is a diffeomorphism of Z, $\forall t \in [0,\tau]$.
- $h_\tau(X)$ intersects Y transversally.

Using the approximation results in [**40**, Theorem 6] we can assume that h is real analytic. Now choose a smooth, increasing tame function $\alpha : [0,\tau] \to [0,\tau]$ such that $\alpha(t) = 0$ for all t near zero, and $\alpha(t) = \tau$, for all t near τ. Define

$$H : [0,\tau]\times Z \to Z,\ \ H_t(z) := h_{\alpha(t)}(z).$$

In other words, H is a smooth, *tame* isotopy between the identity $\boldsymbol{I}_Z$ and h_1, which is independent of t for t near 0 and τ.

The tame flow Φ^k defines a smooth tame diffeomorphism (see Figure 4),

$$\Xi : C_\tau = [0,\tau]\times Z \to S_\tau,\ \ \Xi_t(z) = \Phi^k_t(z).$$

The diffeomorphism Ξ maps η_k to the vector field ∂_t on C_τ.

Using the isotopy H we obtain a smooth tame diffeomorphism

$$\hat{H} : C_\tau \to C_\tau,\ \ \hat{H}(t,z) = H_t(z),$$

such that

$$\hat{H}_*\partial_t = \partial_t \text{ near } \{0\}\times Z \text{ and } \{1\}\times Z.$$

The pushforward of the vector field $\eta_k|_{S_\tau}$ via the diffeomorphism

$$F = \Xi\circ\hat{H}\circ\Xi^{-1} : S_\tau \to S_\tau$$

is a smooth vector field which coincides with η_k in a neighborhood of Z and in a neighborhood of Z_τ. Now define the *smooth* vector field η_{k+1} on M by

$$\eta_{k+1}(x) := \begin{cases} \eta_k(x) & x\in M\setminus S_\tau \\ F_*\eta_k(x) & x\in S_\tau. \end{cases}$$

η_{k+1} is a smooth vector field, and we denote by Φ^{k+1} the flow on M it generates. Observe that η_{k+1} coincides with the original vector field ξ in an open neighborhood of the critical set of f, and f decreases strictly on the nonconstant trajectories of η_{k+1}. By construction, we have

$$W^-(p_j,\eta_{k+1}) \pitchfork W^+(p_i,\eta_{k+1}),\ \ \forall 0\le i<j\le k+1.$$

We want to prove that it is a tame flow. We will prove that the maps

$$\Phi^{k+1} : [0,\infty)\times M\to M,\ \ \Phi^{k+1} : (-\infty,0]\times M\to M$$

are definable. We discuss only the first one, since the proof for the second map is completely similar.

Observe first that, $\hat{H}$ extends to a tame diffeomorphism

$$\mathbb{R}\times Z\to\mathbb{R}\times Z.$$

We denote by Ψ the tame flow $\mathbb{R}\times Z$ obtained by conjugating the translation flow with $\hat{H}$, i.e.,

$$\Psi_t(s,z) = \hat{H}_{t+s}H_s^{-1}(z).$$

We divide M into three definable parts

$$S_\tau,\ \ M_+ := \{f > c_{k+1}-\varepsilon\},\ \ M_- := M\setminus(S_\tau\cup M_+).$$

We now have definable functions

$$T_+ : M_+\to(0,\infty],\ \ T_0, s : S_\tau\to[0,\tau],$$

$T_+(x) :=$ the moment of time when the trajectory of Φ^{k+1} originating at x intersects Z.

$T_0(x) :=$ the moment of time when the trajectory of Φ^{k+1} originating at x intersects Z_τ,

and

$$s(x) = \tau - T_0(x).$$

We distinguish three cases.

- If $x\in M_-$ then $\Phi^{k+1}_t(x) = \Phi^k(x)$, $\forall t\ge 0$.
- If $x\in S_\tau$ then

$$\Phi^{k+1}_t(x) = \begin{cases} \Xi\circ\Psi_t\circ\Xi^{-1}(x) & t\le T_0(x) \\ \Phi^k_{t-T_0(x)}\Xi\circ\Psi_{T_0(x)}\Xi^{-1}(x) & t > T_0(x) \end{cases}$$

- If $x \in M_+$ then

$$\Phi_t^{k+1}(x) = \begin{cases} \Phi_t^k(x) & t < T_-(x) \\ \\ \Xi \circ \Psi_{t-T_-(x)} \circ \Xi^{-1} \circ \Phi^k_{T_-(x)}(x) & t \in (T_-(x), T_-(x) + \tau] \\ \\ \Phi^k_{t-\tau-T_-(x)} \circ \Xi \circ \Psi_\tau \circ \Xi^{-1} \circ \Phi^k_{T_-(x)}(x) & t > \tau + T_-(x). \end{cases}$$

This shows that $\Phi^{k+1} : [0, \infty) \times M \to M$ is definable. □

CHAPTER 6

The gap between two vector subspaces

In this section we collect a few facts about the *gap* between two vector subspaces, [**27**, IV.§2].

Suppose E is a finite dimensional Euclidean space. We denote by $(\bullet, \bullet)$ the inner product on E, and by $|\bullet|$ the associated Euclidean norm. We define as usual the norm of a linear operator $A : E \to E$ by the equality

$$\|A\| := \sup\{|Ax|; \ \ x \in E, \ \ |x| = 1\}.$$

The finite dimensional vector space $\mathrm{End}(E)$ is equipped with an inner product

$$\langle A, B\rangle := \mathrm{tr}(AB^*),$$

and we set

$$|A| := \sqrt{\langle A, A\rangle} = \sqrt{\mathrm{tr}(AA^*)} = \sqrt{\mathrm{tr}(A^*A)}.$$

Since E is finite dimensional, there exists a constant $C > 1$, depending only on the dimension of E, such that

$$\frac{1}{C}|A| \leq \|A\| \leq C|A|. \tag{6.1}$$

If U and V are two subspaces of E, then we define the *gap* between U and V to be the real number

$$\delta(U, V) := \sup\{\mathrm{dist}(u, V); u \in U, |u| = 1\} = \sup_u \inf_v \{|u-v| \ u \in U, |u| = 1, \ \ v \in V\}.$$

If we denote by $P_{V^\perp}$ the orthogonal projection onto $V^\perp$, then we deduce

$$\delta(U, V) = \sup_{|u|=1} |P_{V^\perp} u| = \|P_{V^\perp} P_U\| = \|P_U - P_V P_U\| = \|P_U - P_U P_V\|. \tag{6.2}$$

Note that

$$\delta(V^\perp, U^\perp) = \delta(U, V). \tag{6.3}$$

Indeed,

$$\begin{aligned}\delta(V^\perp, U^\perp) &= \|P_{V^\perp} - P_{U^\perp} P_{V^\perp}\| = \|\mathbb{1} - P_V - (\mathbb{1} - P_U)(\mathbb{1} - PV)\| \\ &= \|P_U - P_U P_V\| = \delta(U, V).\end{aligned}$$

We deduce that

$$0 \leq \delta(U, V) \leq 1, \ \ \forall U, V.$$

Let us point out that

$$\delta(U, V) < 1 \iff \ \dim U \leq \dim V, \ \ U \cap V^\perp = 0.$$

Note that this implies that the gap is *asymmetric* in its variables, i.e., we cannot expect

$$\delta(U, V) = \delta(V, U).$$

Set

$$\hat{\delta}(U,V) = \delta(U,V) + \delta(V,U).$$

PROPOSITION 6.1. *(a) For any vector subspaces $U, V \subset E$ we have*

$$\|P_U - P_V\| \leq \hat{\delta}(U,V) \leq 2\|P_U - P_V\|.$$

(b) For any vector subspaces U, V, W such that $V \subset W$ we have

$$\delta(U,V) \geq \delta(U,W), \;\; \delta(V,U) \leq \delta(W,U).$$

In other words, the function $(U,V) \mapsto \delta(U,V)$ is increasing in the first variable, and decreasing in the second variable.

PROOF. (a) We have

$$\begin{aligned}\hat{\delta}(U,V) &= \|P_U - P_U P_V\| + \|P_V - P_V P_U\| \\ &= \|P_U(P_U - P_V)\| + \|P_V(P_V - P_U)\| \leq 2\|P_U - P_V\|\end{aligned}$$

and

$$\begin{aligned}\|P_U - P_V\| &\leq \|P_U - P_U P_V\| + \|P_U P_V - P_V\| \\ &= \|P_U - P_U P_V\| + \|P_V - P_V P_U\| = \hat{\delta}(U,V).\end{aligned}$$

(b) Observe that for all $u \in U$, $|u| = 1$ we have

$$\operatorname{dist}(u,V) \geq \operatorname{dist}(u,W) \Longrightarrow \delta(U,V) \geq \delta(U,W).$$

Since $V \subset W$ we deduce

$$\sup_{v \in V \setminus 0} \frac{1}{|v|} \operatorname{dist}(v,U) \leq \sup_{w \in W \setminus 0} \frac{1}{|w|} \operatorname{dist}(w,U). \qquad \square$$

We denote by $\mathbf{Gr}_k(E)$ the Grassmannian of k dimensional subspaces of E equipped with the metric

$$\operatorname{dist}(U,V) = \|P_U - P_V\|.$$

$\mathbf{Gr}_k(E)$ is a compact, tame subset of $\operatorname{End}(E)$. We set

$$\mathbf{Gr}(E) := \bigcup_{k=0}^{\dim E} \mathbf{Gr}_k(E).$$

Let $\mathbf{Gr}^k(E)$ denote the Grassmannian of codimension k subspaces. For any subspace $U \subset E$ we set

$$\mathbf{Gr}(E)_U := \big\{ V \in \mathbf{Gr}(E);\;\; V \supset U \big\}, \;\; \mathbf{Gr}(E)^U := \big\{ V \in \mathbf{Gr}(E);\;\; V \subset U \big\}.$$

Note that we have a metric preserving involution

$$\mathbf{Gr}(E) \ni U \longmapsto U^\perp \in \mathbf{Gr}(E),$$

such that

$$\mathbf{Gr}_k(E)_U \longleftrightarrow \mathbf{Gr}^k(E)^{U^\perp}, \;\; \mathbf{Gr}^k(E)_U \longleftrightarrow \mathbf{Gr}_k(E)^{U^\perp}.$$

Using (2.5) we deduce that for any $1 \leq j \leq k$, and any $U \in \mathbf{Gr}_j(E)$, there exits a constant $c > 1$ such that, for every $L \in \mathbf{Gr}_k(E)$ we have

$$\frac{1}{c} \operatorname{dist}(L, \mathbf{Gr}_k(E)_U)^2 \leq |P_U - P_U P_L|^2 \leq c \operatorname{dist}(L, \mathbf{Gr}_k(E)_U)^2.$$

The constant c depends on $j, k, \dim E$, and a priori it could also depend on U. Since the quantities entering into the above inequality are invariant with respect to the action of the orthogonal group $O(E)$, and the action of $O(E)$ on $\mathbf{Gr}_j(E)$

is transitive, we deduce that the constant c is independent on the plane U. The inequality (6.1) implies the following result.

PROPOSITION 6.2. *Let $1 \leq j \leq k \leq \dim E$. There exists a positive constant $c > 1$ such that, for any $U \in \mathbf{Gr}_j(E)$, $V \in \mathbf{Gr}_k(E)$ we have*

$$\frac{1}{c}\operatorname{dist}\big(V, \mathbf{Gr}_k(E)_U\big) \leq \delta(U,V) \leq c\operatorname{dist}\big(V, \mathbf{Gr}_k(E)_U\big). \qquad \square$$

COROLLARY 6.3. *For every $1 \leq k \leq \dim E$ there exists a constant $c > 1$ such that, for any $U, V \in \mathbf{Gr}_k(E)$ we have*

$$\frac{1}{c}\operatorname{dist}(U,V) \leq \delta(U,V) \leq c\operatorname{dist}(U,V). \qquad \square$$

PROOF. In Proposition 6.2 we make $j = k$ and we observe that $\mathbf{Gr}_k(E)_U = \{U\}$, $\forall U \in \mathbf{Gr}_k(E)$. $\square$

We would like to describe a few simple geometric techniques for estimating the gap between two vector subspaces. Suppose U, V are two vector subspaces of the Euclidean space E such that

$$\dim U \leq \dim V,\ \delta(U,V) < 1.$$

As remarked earlier, the condition $\delta(U,V) < 1$ can be rephrased as $U \cap V^\perp = 0$, or equivalently, $U^\perp + V = E$, i.e., the subspace V intersects $U^\perp$ transversally. Hence

$$U \cap \ker P_V = 0.$$

Denote by S the orthogonal projection of U on V. We deduce that the restriction of P_V to U defines a bijection $U \to S$. Hence $\dim S = \dim U$, and we can find a linear map

$$h : S \to V^\perp$$

whose graph is U, i.e.,

$$U = \big\{ s + h(s);\ \ s \in S, \big\}.$$

Next, denote by T the orthogonal complement of S in V (see Figure 5), $T := S^\perp \cap V$, and by W the subspace $W := U + T$.

LEMMA 6.4.

$$T = U^\perp \cap V.$$

PROOF. Observe first that

$$(S + U) \subset T^\perp. \tag{6.4}$$

Indeed, let $t \in T$. Any element in $S + U$ can be written as a sum

$$s + u = s + s' + h(s'),\ \ s, s' \in S.$$

Then $(s + s') \perp t$ and $h(s') \perp t$, because $h(s') \in V^\perp$. Hence $T \subset U^\perp \cap S^\perp \subset U^\perp$. On the other hand, $T \subset V$ so that

$$T \subset U^\perp \cap V.$$

Since V intersects $U^\perp$ transversally we deduce

$$\dim(U^\perp \cap V) = \dim U^\perp + \dim V - \dim E = \dim V - \dim U = \dim T. \qquad \square$$

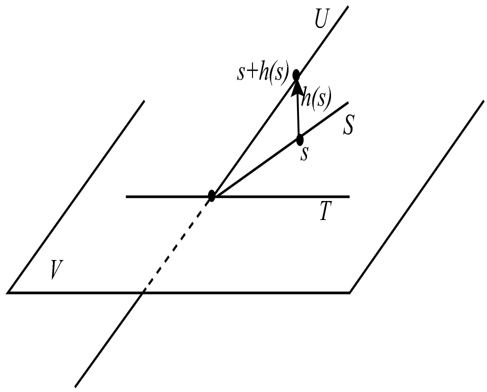


FIGURE 5. *Computing the gap between two subspaces.*

LEMMA 6.5.

$$\delta(W,V)=\delta(U,V)=\delta(U,S).$$

PROOF. The equality $\delta(U,V)=\delta(U,S)$ is obvious. Let $w_0\in W$ such that $|w_0|=1$ and

$$\operatorname{dist}(w_0,V)=\delta(W,V).$$

To prove the lemma it suffices to show that $w_0\in U$. We write

$$w_0=u_0+t_0,\ \ u_0\in U,\ \ t_0\in T,\ \ |u_0|^2+|t_0|^2=1.$$

We have to prove that $t_0=0$. We can refine the above decomposition of w_0 some more by writing

$$u_0=s_0+h(s_0),\ \ s_0\in S.$$

Then

$$P_Vw_0=s_0+t_0.$$

We know that for any $u\in U$, $t\in T$ such that $|u|^2+|t^2|$ we have

$$|u_0^2-P_Vu_0|^2=|w_0-P_Vw_0|^2\geq|(u+t)-P_V(u+t)|=|u-P_Vu|^2.$$

If in the above inequality we choose $t=0$ and $u=\frac{1}{|u|_0}$ we deduce

$$|u_0^2-P_Vu_0|^2\geq\frac{1}{|u_0|^2}|u_0^2-P_Vu_0|^2.$$

Hence $|u_0|\geq 1$ and since $|u_0|^2+|t_0|^2=1$ we deduce $t_0=0$. □

The next result summarizes the above observations.

PROPOSITION 6.6. *Suppose U and V are two subspaces of the Euclidean space E such that $\dim U\leq\dim V$ and V intersects $U^\perp$ transversally. Set*

$$T:=V\cap U^\perp,\ \ W:=U+T,$$

and denote by S the orthogonal projection of U on V. Then

$$S=T^\perp\cap V,$$

$$\dim U=\dim S,\ \ \dim W=\dim V,$$

and

$$\delta(W, V) = \delta(U, V) = \delta(U, S). \qquad \square$$

PROPOSITION 6.7. *Suppose E is an Euclidean vector space. There exists a constant $C > 1$, depending only on the dimension of E, such that, for any subspaces $U \subset E$, and any linear operator $S : U \to U^\perp$, we have*

$$\delta(\Gamma_S, U) = \|S\|\bigl(1 + \|S\|^2\bigr)^{-1/2}, \tag{6.5}$$

and

$$\frac{1}{C}\|S\|\bigl(1 + \|S\|^2\bigr)^{-1/2} \le \delta(U, \Gamma_S) \le C\|S\|\bigl(1 + \|S\|^2\bigr)^{-1/2}, \tag{6.6}$$

where $\Gamma_S \subset U + U^\perp = E$ is the graph of S defined by

$$\Gamma_S := \bigl\{\, u + Su \in E;\ \ u \in U \,\bigr\}.$$

PROOF. Observe that

$$\delta(\Gamma_S, U)^2 = \sup_{u \in U \setminus 0} \frac{|Su|^2}{|u|^2 + |Su|^2} = \sup_{u \in U \setminus 0} \frac{(S^*Su, u)}{|x|^2 + (S^*Su, u)}.$$

Choose an orthonormal basis $e_1, \dots, e_k$ of U consisting of eigenvectors of S^*S,

$$S^*Se_i = \lambda_i e_i, \ \ 0 \le \lambda_1 \le \cdots \le \lambda_k.$$

Observe that

$$\|S^*S\| = \lambda_k.$$

We deduce

$$\delta(\Gamma_S, U)^2 = \sup\bigl\{\sum_i \lambda_i u_i^2;\ \ \sum_i (1 + \lambda_i)u_i^2 = 1\,\bigr\}$$

$$= \sup\bigl\{1 - \sum_i u_i^2;\ \ \sum_i (1 + \lambda_i)u_i^2 = 1\,\bigr\}$$

$$= 1 - \inf\bigl\{\sum_i u_i^2;\ \ \sum_i (1 + \lambda_i)u_i^2 = 1\,\bigr\} = 1 - \frac{1}{1 + \lambda_k} = \frac{\|S^*S\|}{1 + \|S^*S\|} = \frac{\|S\|^2}{1 + \|S\|^2}.$$

This proves (6.5). The inequality (6.6) follows from (6.5) combined with Corollary 6.3. $\square$

Set

$$\mathcal{P}(E) = \bigl\{\,(U, V) \in \mathbf{Gr}(E) \times \mathbf{Gr}(E);\ \ \dim U \le \dim V,\ \ V \pitchfork U^\perp\,\bigr\}$$

For every pair $(U, V) \in \mathcal{P}(E)$ we denote by $\mathcal{S}_V(U)$ the *shadow* of U on V, i.e., the orthogonal projection of U on V. Let us observe that

$$U^\perp \cap \mathcal{S}_V(U) = 0.$$

Indeed, we have

$$U^\perp \cap \mathcal{S}_V(U) \subset T := U^\perp \cap V \Longrightarrow U^\perp \cap \mathcal{S}_V(U) \subset \mathcal{S}_V(U) \cap T,$$

and Proposition 6.6 shows that $\mathcal{S}_V(U)$ is the orthogonal complement of T in V. Since $\dim U = \dim \mathcal{S}_V(U)$, we deduce that $\mathcal{S}_V(U)$ can be represented as the graph of a linear operator

$$\mathcal{M}_V(U) : U \to U^\perp$$

which we will call the *slope* of the pair (U, V). From Proposition 6.6 we deduce

$$\delta(\mathcal{S}_V(U), U) = \frac{\|\mathcal{M}_V(U)\|}{\big(1+\|\mathcal{M}_V(U)\|^2\big)^{1/2}} \Longleftrightarrow \|\mathcal{M}_V(U)\| = \frac{\delta(\mathcal{S}_V(U), U)}{\big(1-\delta(\mathcal{S}_V(U), U)^2\big)^{1/2}}.$$

COROLLARY 6.8. *There exists a constant $C > 1$, which depends only on the dimension of E such that, for every pair $(U, V) \in \mathcal{P}(E)$ we have*

$$\frac{1}{C}\|\mathcal{M}_V(U)\|\big(1+\|\mathcal{M}_V(U)\|^2\big)^{-1/2} \leq \delta(U, V) \leq C\|\mathcal{M}_V(U)\|\big(1+\|\mathcal{M}_V(U)\|^2\big)^{-1/2}.$$

PROOF. Use the equality $\delta(U, V) = \delta\big(U, \mathcal{S}_V(U)\big)$ and Proposition 6.7. □

PROPOSITION 6.9. *Suppose $A : E \to E$ is an invertible symmetric operator, and U is the subspace of E spanned by the positive eigenvectors A. We denote by $m_+(A)$ the smallest positive eigenvalue of A, and by $m_-(A)$ the smallest positive eigenvalue of $-A$. Then, for every subspace $V \subset E$, such that $(U, V) \in \mathcal{P}(E)$, we have*

$$\begin{aligned}\delta(U, e^{tA}V) &\leq e^{-(m_+(A)+m_-(A))t}\|\mathcal{M}_V(U)\| \\ &= e^{-(m_+(A)+m_-(A))t}\frac{\delta(\mathcal{S}_V(U), U)}{\big(1-\delta(\mathcal{S}_V(U), U)^2\big)^{1/2}}.\end{aligned}$$

PROOF. Denote by L the intersection of V with $U^\perp$. Then we have an orthogonal decomposition

$$V = L + \mathcal{S}_V(U),$$

and if we write $\mathcal{M} := \mathcal{M}_V(U) : U \to U^\perp$ we obtain

$$V = \big\{\ell + u + \mathcal{M}u;\ \ \ell \in L,\ \ u \in U\big\}.$$

Using the orthogonal decomposition $E = U + U^\perp$ we can describe A in the block form

$$A = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix},$$

where A_+ denotes the restriction of A to U, and A_- denotes the restriction of A to $U^\perp$.

Set $V_t := e^{tA}V$, $L_t := V_t \cap U^\perp$. Since $U^\perp$ is A-invariant, we deduce that $L_t = e^{tA_-}L$, so that

$$\begin{aligned}V_t &= \big\{\, e^{tA_-}\ell + e^{tA_+}u + e^{tA_-}\mathcal{M}u;\ \ \ell \in L,\ \ u \in U\,\big\} \\ &= \big\{\, e^{tA_-}\ell + u + e^{tA_-}\mathcal{M}e^{-tA_+}u;\ \ \ell \in L,\ \ u \in U\,\big\}.\end{aligned}$$

We deduce that for every $u \in U$ the vector $u + e^{tA_-}\mathcal{M}e^{-tA_+}u$ belongs to V_t. Hence

$$\delta(U, V_t) \leq \sup_{|u|=1}|e^{tA_-}\mathcal{M}e^{-tA_+}u| = \|e^{tA_-}\mathcal{M}e^{-tA_+}\| \leq e^{-(m_+(A)+m_-(A))t}\|\mathcal{M}\|. \quad \square$$

COROLLARY 6.10. *Let A and U as above. Fix $\ell > \dim U$ and consider a compact subset $K \subset \mathbf{Gr}_\ell(E)$ such that any $V \in K$ intersects $U^\perp$ transversally. Then there exists a positive constant, depending only on K and $\dim E$ such that*

$$\delta(U, e^{tA}V) \leq Ce^{-(m_+(A)+m_-(A))t},\ \ \forall V \in K. \quad \square$$

CHAPTER 7

The Whitney and Verdier regularity conditions

For any subset S of a topological space X we will denote by $\boldsymbol{cl}(S)$ its closure.

DEFINITION 7.1. Suppose X, Y are two C^2-submanifolds of the Euclidean space E such that $X \subset \boldsymbol{cl}(Y) \setminus Y$.
(a) We say that (X, Y) satisfies *Verdier regularity condition* V at $x_0 \in X$ if there exists an open neighborhood U of x_0 in E and a positive constant C such that

$$\delta(T_x X, T_y Y) \leq C|x - y|, \quad \forall x \in U \cap X, \quad y \in U \cap Y.$$

(b) We say that (X, Y) satisfies the *Verdier regularity condition* V *along* X if it satisfies the condition V at any point $x \in X$. □

Note that if X and Y are connected and if (X, Y) satisfies V along X, then

$$\dim X \leq \dim Y.$$

As explained in [**46**], the Verdier condition is invariant under C^2-diffeomorphisms.

REMARK 7.2. The Verdier regularity condition is equivalent to the microlocal regularity condition μ of Kashiwara and Schapira, [**26**, §8.3]. For a proof of this fact we refer to [**45**]. □

The regularity condition V is intimately related to Whitney's regularity condition.

DEFINITION 7.3. Suppose X, Y are two C^1-submanifolds of the Euclidean space E such that $X \subset \bar{Y} \setminus Y$.
(a) We say that the pair (X, Y) satisfies the *Whitney regularity condition* (a) at $x_0 \in X$ if, for any sequence $y_n \in Y$ such that

- $x_n, y_n \to x_0$,
- the sequence of tangent spaces $T_{y_n} Y$ converges to the subspace T_∞,

we have $T_{x_0} X \subset T_\infty$.
(b) We say that the pair (X, Y) satisfies the *Whitney regularity condition* (b) at $x_0 \in X$ if, for any sequence $(x_n, y_n) \in X \times Y$ such that

- $x_n, y_n \to x_0$,
- the one dimensional subspaces $\ell_n = \mathbb{R}(y_n - x_n)$ converge to the line ℓ_∞,
- the sequence of tangent spaces $T_{y_n} Y$ converges to the subspace T_∞,

we have $\ell_\infty \subset T_\infty$, that is, $\delta(\ell_\infty, T_\infty) = 0$.
(c) The pair (X, Y) is said to satisfy *the regularity condition* (a) *or* (b) *along* X, if it satisfies this condition at any $x \in X$. □

The Whitney condition (a) is weaker in the sense that $(b) \Longrightarrow (a)$ and it is fairly easy to construct instances when (a) is satisfied while (b) is violated.

In applications it is convenient to use a regularity condition slightly weaker that the condition (b). To describe it suppose the manifolds X, Y are as above, $X \subset \boldsymbol{cl}(Y) \setminus Y$, and let $p \in X \cap \boldsymbol{cl}(Y)$. We can choose coordinates in a neighborhood U of p in E such that $U \cap X$ can be identified with an open subset of an affine plane $L \subset E$. We denote by P_L the orthogonal projection onto L.

We say that that (X, Y) *satisfies the condition (b') at* p if, for any sequence $y_n \to p$ such that the $T_{y_n}Y$ converges to some T_∞, and the one dimensional subspace $\ell_n \mathbb{R}(y_n - P_L y_n)$ converges to the 1-dimensional subspace ℓ_∞, we have

$$\ell_\infty \subset T_\infty, \text{ i.e., } \gamma(\ell_\infty, T_\infty) = 0.$$

It is known that (a) + (b') $\Longrightarrow$ (b).

The Whitney regularity condition (b) is equivalent with the following geometric condition, [**44**].

Proposition 7.4 (Trotman). *Suppose (X, Y) is a pair of C^1 submanifolds of the $\mathbb{R}^N$, $\dim X = m$. Assume $X \subset \bar{Y} \setminus Y$. Then the pair (X, Y) satisfies the Whitney regularity condition (b) along X if and only if, for any open set $U \subset E$, and any C^1-diffeomorphism $\Psi : U \to V$, where V is an open subset of $\mathbb{R}^N$, such that*

$$\Psi(U \cap X) \subset \mathbb{R}^m \oplus 0 \subset \mathbb{R}^N,$$

the map

$$\Psi(Y \cap U) \longrightarrow \mathbb{R}^m \times (0, \infty), \quad y \longmapsto \big(\operatorname{proj}(y), \operatorname{dist}(y, \mathbb{R}^m)^2\big),$$

is a submersion, where $\operatorname{proj} : \mathbb{R}^N \to \mathbb{R}^m$ *denotes the canonical orthogonal projection.* □

For tame objects the Verdier condition implies the Whitney condition. More precisely, we have the following result, [**33, 46**].

Proposition 7.5 (Verdier-Loi). *Suppose (X, Y) is a pair of C^2, tame submanifold of the Euclidean space E such that $X \subset \bar{Y} \setminus Y$. If (X, Y) satisfies the regularity condition V, then it also satisfies the regularity condition W.* □

Definition 7.6. Suppose X is a subset of an Euclidean space E. A *Verdier stratification* (respectively *Whitney stratification*) of X is an increasing, finite filtration

$$F_{-1} = \emptyset \subset F_0 \subset F_1 \subset \cdots \subset F_m = X$$

satisfying the following properties.

(a) F_k is closed in X, $\forall k$.
(b) For every $k = 1, \ldots, m$ the set $X_k = F_k \setminus F_{k-1}$ is a C^2 manifold of dimension k. Its connected components are called the *strata* of the stratification.
(c) (The frontier condition) For every $k = 1, \ldots, m$ we have

$$\boldsymbol{cl}(X_k) \setminus X_k \subset F_{k-1}.$$

(d) For every $0 \leq j < k \leq m$ the pair (X_j, X_k) is Verdier regular (respectively Whitney regular) along X_j.

If X is a tame set, then a Verdier (Whitney) stratification is called tame if the sets F_k are tame. □

We have the following result due to essentially to Verdier [**46**] (in the subanalytic case) and Loi [**33**] in the tame context.

THEOREM 7.7. *Suppose $S_1, \dots, S_n$ are tame subsets of the Euclidean space E. Then there exists a tame Verdier stratification of E such that each of the sets S_k is a union of strata.* □

REMARK 7.8. According to the results of Lion and Speissegger [**32**], the strata in the above Verdier stratification can be chosen to be *real analytic* submanifolds of E. □

A Whitney stratified space X has a rather restricted local structure. More precisely, we have the following fundamental result whose intricate proof can be found in [**16**, Chap,II,§5].

THEOREM 7.9. *Suppose X is a subset of a smooth manifold M of dimension m, and*

$$F_0 \subset F_1 \subset \cdots \subset F_k = M$$

is a Whitney stratification of X. Then for every stratum S of dimension j there exists

- *a closed tubular neighborhood N of S in M with projection $\pi : N \to S$,*
- *a Whitney stratified subset L_S of the sphere S^{m-j-1}*

such that $\pi : \partial N \to X$ is a locally trivial fibration with fiber homeomorphic to L_S, and $N \cap X$ is homeomorphic with the mapping cylinder of the projection $\pi : \partial N \to S$. The space L_S is called the normal link of S in X. □

CHAPTER 8

Smale transversality and Whitney regularity

Suppose M is a compact, connected real analytic manifold of dimension M, and (f,ξ) is a Morse pair on M, *not necessarily tame*. Denote by Φ^ξ the flow generated by ξ, by $W_p^-(\xi)$ (respectively $W_p^+(\xi)$) the unstable (respectively stable) manifold of the critical point p, and set

$$M_k(\xi) := \bigcup_{p\in \mathbf{Cr}_f,\ \lambda(p)\le k} W_p^-(\xi),\ \ \mathcal{S}_k^-(\xi) = M_k(\xi)\setminus M_{k-1}(\xi).$$

We say that the flow Φ^ξ satisfies the *Morse-Whitney* (respectively *Morse-Verdier*) condition if the increasing filtration

$$M_0(\xi)\subset M_1(\xi)\subset\cdots\subset M_m(\xi)$$

is a Whitney (respectively Verdier) regular stratification. In the sequel, when no confusion is possible, we will write $W_p^\pm$ instead of $W_p^\pm(\xi)$.

THEOREM 8.1. *If the Morse flow Φ^ξ satisfies the Morse-Whitney condition (a), then it also satisfies the Morse-Smale condition.*

PROOF. Let $p,q\in\mathbf{Cr}_f$ such that $p\neq q$ and $W_p^-\cap W_q^+\neq\emptyset$. Let k denote the Morse index of q, and ℓ the Morse index of q so that $\ell>k$. We want to prove that this intersection is transverse.

Let $x\in W_p^-\cap W_q^+$ and set

$$x_t := \Phi_t^\xi(x).$$

Observe that

$$T_xW_q^+ \pitchfork T_xW_p^- \Longleftrightarrow \exists t\ge 0:\ \ T_{x_t}W_q^+\pitchfork T_{x_t}W_p^-.$$

We will prove that $T_{x_t}W_q^+\pitchfork T_{x_t}W_p^-$ if t is sufficiently large.

Since (f,ξ) is a Morse pair, we can find coordinates (u^i) in a neighborhood U of q, and real numbers

$$\mu_1,\dots,\mu_m>0$$

such that

$$u^i(q)=0,\ \ \forall i,$$

$$\xi=\sum_{i=1}^k \mu_i u^i\partial_{u_i}-\sum_{\alpha>k}\mu_\alpha u^\alpha\partial_{u_\alpha}.$$

Denote by A the diagonal matrix

$$A=\mathrm{Diag}(\mu_1,\dots,\mu_k,-\mu_{k+1},\dots,-\mu_m).$$

Without any loss of generality, we can assume that the point x lies in the coordinate neighborhood U. Denote by E the Euclidean space with Euclidean coordinates (u^i). Then the path

$$t \mapsto T_{x_t} W_p^- \in \mathbf{Gr}_\ell(E)$$

is given by

$$T_{x_t} W_p^- = e^{tA} T_x W_p^-,$$

and in particular it has a limit

$$\lim_{t\to\infty} T_{x_t} W_p^- = T_\infty \in \mathbf{Gr}_\ell(E).$$

Since the pair $\big(W_q^-, W_p^- \big)$ satisfies the Whitney regularity condition (a) along W_q^-, and $x_t \to q$, as $t \to \infty$, we deduce

$$T_\infty \supset T_q W_q^-, \Longrightarrow T_\infty \pitchfork T_q W_q^+.$$

Thus, for t sufficiently large

$$T_{x_t} W_p^- \pitchfork T_{x_t} W_q^+. \qquad \square$$

Suppose (f, ξ) is a Morse pair on the compact, real analytic manifold M. Then for every critical point p of f of index k we can find local C^2-coordinates (u^i) defined in an open neighborhood U_p, and positive real numbers μ_i such that

$$u^i(p) = 0, \ \ \forall i,$$

and

$$\xi = \sum_{i\le k} \mu_i u^i \partial_{u_i} - \sum_{\alpha > k} \mu_\alpha u^\alpha \partial_{u_\alpha}.$$

If p is a hyperbolic point, i.e., $0 < k < m$, we set,

$$\gamma_u(p) = \gamma_u(\xi, p) := \min_{i\le k} \mu_i, \ \ \gamma_s(p) = \gamma_s(\xi, p) := \min_{\alpha > k} \mu_\alpha, \ \ \Gamma_s(p) = \Gamma_s(\xi, p) := \max_{\alpha > k} \mu_\alpha,$$

$$g_s(p) = g_s(\xi, p) := \Gamma_s(p) - \gamma_s(p).$$

Observe that $g_s(p)$ is the length of the smallest interval containing all the negative (or stable) eigenvalues of the linearization of ξ at p, while $\gamma_u(p)$ is the smallest positive (or unstable) eigenvalue of the linearization of ξ at p.

THEOREM 8.2. *Suppose (ξ, f) is a Morse pair on the smooth manifold M of dimension m such that the flow Φ^ξ satisfies the Morse-Smale condition. Define*

$$\nu := \min\Big\{ \frac{\gamma_u(p) + \gamma_s(p)}{\Gamma_s(p)}; \ \ p \in \mathbf{Cr}_f, \ \ 0 < \lambda(p) < \dim M \Big\}. \tag{8.1}$$

Assume ξ is at least $(\lfloor \nu \rfloor + 1)$-times differentiable. Then the following hold.
(a)(Frontier property) $\mathbf{cl}\big(M_k(\xi) \big) \setminus M_k(\xi) \subset M_{k-1}(\xi)$, $\forall k$.
(b) For every pair of critical points p, q, and every $z \in W_q^- \cap \mathbf{cl}(W_p^-)$, there exists an open neighborhood U of $z \in M$, and a positive constant C such that

$$\delta\big(T_x W_q^-, T_y W_p^-\big) \le C \operatorname{dist}(x, y)^\nu, ; \ \forall x \in U \cap W_q^-, \ \ \forall y \in U \cap W_p^-. \tag{V_ν}$$

In particular, the stratification by unstable manifolds satisfies the Whitney regularity (a).

REMARK 8.3. (a) Note that the above theorem requires no tameness assumption on the flow Φ.

(b) It is perhaps useful to visualize the condition (8.1) in which $\nu \geq 1$, as a *spectral clustering condition.*

Suppose p is an unstable critical point of f. Denote H_p is the Hessian of f at p. Using the metric g we can identify H_p with a symmetric operator. We denote by $\Sigma_p^{\pm}$ the collection of positive/negative eigenvalues of this operator. Then $\gamma_s(\xi, p)$ is the positive spectral gap,

$$\gamma_s(\xi, p) = \min \Sigma_p^+ = \operatorname{dist}(\Sigma_p^+, 0),$$

$\gamma_u(\xi, p)$ is the negative spectral gap

$$\gamma_u(\xi, p) = \operatorname{dist}(\Sigma_p^-, 0),$$

and $\Gamma_s(\xi, p)$ is the largest positive eigenvalue of H_p. The condition

$$\frac{\gamma_u(\xi, p) + \gamma_s(\xi, p)}{\Gamma_s(\xi, p)} \geq 1,$$

then says that the largest positive eigenvalue is smaller than the length of largest interval containing 0, and disjoint from the spectrum. Equivalently, this means, that the positive eigenvalues are contained in an interval whose length is not greater than the distance from the origin to the negative part of the spectrum. In particular, if the positive eigenvalues cluster in a tiny interval situated far away from the origin, this condition is automatically satisfied. □

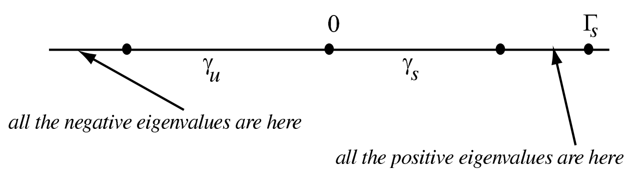


FIGURE 6. *Spectral gaps.*

PROOF. To prove part (a) it suffices to show that if

$$W_q^- \cap \boldsymbol{cl}\big(W_p^- \big) \Longrightarrow \dim W_q^- < \dim W_p^-.$$

Observe that the set $W_q^- \cap \boldsymbol{cl}\big(W_p^- \big)$ is flow invariant, and its intersection with any compact subset of $W^-(p, \xi)$ is closed. We deduce that $p \in W_q^- \cap \boldsymbol{cl}\big(W_p^- \big)$.

Fix a small neighborhood U of p in W_p^-. Then there exists a sequence $x_n \in \partial U$, and a sequence $t_n \in [0, \infty)$, such that

$$\lim_{n\to\infty} t_n = \infty, \quad \lim_{n\to\infty} \Phi_{t_n}^{\xi} x_n = q.$$

In particular, we deduce that $f(p) > f(q)$.

For every n define

$$C_n = \boldsymbol{cl}\big(\{ \Phi_t^{\xi} x_n;\;\; t \in (-\infty, t_n] \} \big).$$

Denote by $\mathbf{Cr}_q^p$ the set of critical points p' such that $f(q) < f(p') < f(p)$. For every $p' \in \mathbf{Cr}_q^p$ we denote by $d_n(p')$ the distance from p' to C_n. We can find a set

$S \subset \mathbf{Cr}_q^p$ and a subsequence of the sequence (C_n), which we continue to denote by (C_n), such that

$$\lim_{n\to\infty} d_n(p') = 0, \;\; \forall p' \in S \;\text{ and }\; \inf_n d_n(p') > 0, \;\; \forall p' \in \mathbf{Cr}_q^p \setminus S.$$

Label the points in S by $s_1, \dots, s_k$, so that

$$f(s_1) > \cdots > f(s_k).$$

Set $s_0 = p$, $s_{k+1} = q$. The critical points in S are hyperbolic, and we conclude that there exist trajectories $\gamma_0, \dots, \gamma_k$ of Φ, such that

$$\lim_{t\to-\infty} \gamma_i(t) = s_i, \;\; \lim_{t\to\infty} \gamma_i(t) = s_{i+1}, \;\; \forall i = 0, \dots, k,$$

and

$$\liminf_{n\to\infty} \operatorname{dist}(C_n, \Gamma_0 \cup \cdots \cup \Gamma_k) = 0,$$

where $\Gamma_i = \boldsymbol{cl}\big(\gamma_i(\mathbb{R})\big)$, and dist denotes the Hausdorff distance. We deduce

$$W^-_{s_i} \cap W^+_{s_{i+1}} \neq \emptyset, \;\; \forall i = 0, \dots, k.$$

Since the flow Φ^ξ satisfies the Morse-Smale condition we deduce from the above that

$$\dim W^-_{s_i} > \dim W^-_{s_{i+1}}, \;\; \forall i = 0, \dots, k.$$

In particular,

$$\dim W^-_p > \dim W^-_q.$$

To prove (b), observe first that since the map $x \mapsto \Phi_t(x)$ is $(\lfloor \nu \rfloor + 1)$-times differentiable for every t, the set of points $z \in W^-_p \cap \boldsymbol{cl}(W^-_q)$ satisfying (V_ν) is open in W^-_q and flow invariant. Since $q \in \boldsymbol{cl}(W^-_p) \cap \boldsymbol{cl}(W^-_q)$ it suffices to prove (b) in the special case $z = q$. We will achieve this using an inductive argument.

For every $0 \leq k \leq m = \dim M$ we denote by $\mathbf{Cr}_f^k$ the set of index k critical points of f. We will prove by decreasing induction over k the following statement.

> $\boldsymbol{S}(k)$**:** *For every $q \in \mathbf{Cr}_f^k$, and every $p \in \mathbf{Cr}_f$ such that $q \in \boldsymbol{cl}\big(W^-_p\big)$ there exists a neighborhood U of $q \in M$, and a constant $C > 0$ such that (V_ν) holds.*

The statement is vacuously true when $k = m$. We fix k, we assume that $\boldsymbol{S}(k')$ is true for any $k' > k$, and we will prove that the statement its is true for k as well. If $k = 0$ the statement is trivially true because the distance between the trivial subspace and any other subspace of a vector space is always zero. Therefore, we can assume $k > 0$.

Fix $q \in \mathbf{Cr}_f^k$, and $p \in \mathbf{Cr}_f^\ell$, $\ell > k$. Fix adapted coordinates (u^i) defined in a neighborhood of $\mathcal{N}$ of q such that, there exist positive real numbers $R, \mu_1, \dots, \mu_m$ with the property

$$\xi = -\sum_{i\leq k} \mu_i u^i \partial_{u^i} + \sum_{\alpha > k} \mu_\alpha u^\alpha \partial_{u_\alpha},$$

and

$$\big\{\, (\, u^1(x), \dots, u^m(x)\,) \in \mathbb{R}^m; \;\; x \in \mathcal{N} \,\big\} \supset [-R, R]^m.$$

For every $r \leq R$ we set

$$\mathcal{N}_r := \big\{\, x \in \mathcal{N}; \;\; |u^j(x)| \leq r, \;\; \forall j = 1, \dots, m \,\big\},$$

For every $x \in \mathcal{N}_R$ we define, its horizontal and vertical components,

$$\boldsymbol{h}(x) = (u^1(x), \cdots, u^k(x)) \in \mathbb{R}^k, \;\; \boldsymbol{v}(x) = (u^{k+1}(x), \dots, u^m(x)) \in \mathbb{R}^{m-k}.$$

Define (see Figure 7)

$$S_q^+(r) := \{\, x \in W_q^+ \cap \mathcal{N}_r;\ \ |\boldsymbol{v}(x)| = r \,\},\ \ Z_q^+(r) = \{\, x \in \mathcal{N}_r;\ \ |\boldsymbol{v}(x)| = r \,\}.$$

The set $Z_q^+(r)$ is the boundary of a "tube" of radius r around the unstable manifold W_q^-.

We denote by U the vector subspace of $\mathbb{R}^m$ given by $\{\boldsymbol{v}(u) = 0\}$, and by $U^\perp$ its orthogonal complement. Observe that for every $x \in W_q^- \cap \mathcal{N}_R$ we have $T_x W_p^- = U$.

Finally, for $k' > k$ we denote by $\mathcal{T}_{k'}(U^\perp) \subset \mathbf{Gr}_{k'}(\mathbb{R}^m)$ the set of k'-dimensional subspaces of $\mathbb{R}^m$ which intersect $U^\perp$ transversally.

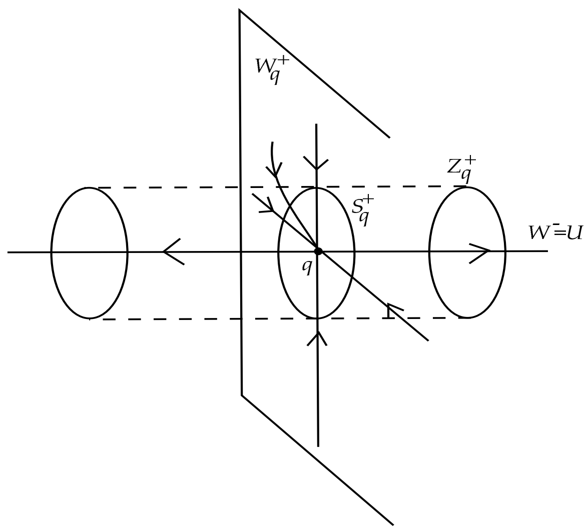


FIGURE 7. *The dynamics in a neighborhood of a hyperbolic point.*

From part (a) we deduce that there exists $r \leq R$

$$\mathcal{N}_r \cap \boldsymbol{cl}\big(W_{q'}^- \big) = \emptyset,\ \ \forall j \leq k,\ \ \forall q' \in \mathbf{Cr}_f^j,\ \ q' \neq q. \tag{8.2}$$

For every critical point p' we set

$$C(p', q)_r := C(p', q) \cap S_q^+(r).$$

Now consider the set

$$X_r(q) := C(p, q)_r \cup \bigcup_{k<\lambda(p')<\ell} C(p', q)_r.$$

For any positive number $\hbar$ we set

$$\mathcal{G}_{r,\hbar} := \boldsymbol{cl}\big(\{ T_x W_p^-;\ \ x \in Z_q^+(r);\ \ |\boldsymbol{h}(x)| \leq \hbar \} \big) \subset \mathbf{Gr}_\ell(\mathbb{R}^m). \tag{8.3}$$

LEMMA 8.4. *There exists a positive $\hbar \leq r$ such that*

$$\mathcal{G}_{r,\hbar} \subset \mathcal{T}_\ell(U^\perp).$$

Proof. We argue by contradiction. Assume that there exists sequences $\hbar_n \to 0$ and $x_n \in \mathcal{N}_r$ such that

$$|\boldsymbol{v}(x_n)| = r, \ \ |\boldsymbol{h}(x_n)| \leq \hbar_n, \ \ \delta(U, T_{x_n}W_p^-) \geq 1 - \frac{1}{n}.$$

By extracting subsequences we can assume that $x_n \to x \in S_q^+(r)$ and $T_{x_n}W_p^- \to T_\infty$ so that

$$\delta(U, T_\infty) = 1 \iff T_\infty \text{ does not intersect } U^\perp \text{ transversaly.} \tag{8.4}$$

From the frontier condition and (8.2) we deduce $x \in X_r(q)$. If $x \in C(p,q)_r$ then $x \in W_p^- \cap S_q^+(r)$, and we deduce $T_\infty = T_xW_p^-$. On the other hand, the Morse-Smale condition shows that $T_xW_p^-$ intersects transversally $T_xW_q^+ = U^\perp$ which contradicts (8.4).

Thus $x \in C(p', q)$ with $\lambda(p') = k'$, $k < k' < \ell$. Since we assume that the statement $\boldsymbol{S}(k')$ is true, we deduce $\delta(T_xW_{p'}^-, T_\infty) = 0$, i.e.,

$$T_\infty \supset T_xW_{p'}^-.$$

From the Morse-Smale condition we deduce that $T_xW_{p'}^-$ intersects $T_xW_q^+ = U^\perp$ transversally, and a fortiori, T_∞ will intersect $U^\perp$ transversally. This again contradicts (8.4). □

Fix $\hbar \in (0, r]$ such that the compact set

$$\mathcal{G}_{r,\hbar} = \big\{ T_xW_p^-; x \in W_p^- \cap Z_q^+(r), \ \ |\boldsymbol{h}(x)| \leq \hbar \big\} \subset \mathbf{Gr}_\ell(\mathbb{R}^m)$$

is a subset of $\mathcal{T}_\ell(U^\perp)$. Consider the block

$$\mathcal{B}_{r,\hbar} = \big\{ x \in \mathcal{N}_r; \ |\boldsymbol{v}(x)| \leq r, \ \ |\boldsymbol{h}(x)| \leq \hbar \big\}.$$

The set $\mathcal{B}_{r,\hbar}$ is a compact neighborhood of q. Define

$$A_u : \mathbb{R}^k \to \mathbb{R}^k, \ \ A_u = \operatorname{Diag}(\mu_1, \dots, \mu_k),$$

$$A_s : \mathbb{R}^{m-k} \to \mathbb{R}^{m-k}, \ \ A_s = \operatorname{Diag}(\mu_{k+1}, \dots, \mu_m),$$

$$A : \mathbb{R}^m \to \mathbb{R}^m, \ \ A = \operatorname{Diag}(A_u, -A_s).$$

For every $x \in \mathcal{B}_{r,\hbar} \setminus W_q^-$ we denote by I_x the connected component of

$$\{t \leq 0; \ \ \Phi_t^\xi x \in \mathcal{B}_{r,\hbar}\}$$

which contains 0. The set I_x is a closed interval

$$I_x := [-T(x), 0], \ \ T(x) \in [0, \infty].$$

If $x \in \mathcal{B}_{r,\hbar} \setminus W_q^-$ then $T(x) < \infty$. We set

$$z(x) := \Phi_{-T(x)}^\xi x, \ \ y(x) := \boldsymbol{v}(z(x)).$$

Then

$$y(x) = e^{T(x)A_s}\boldsymbol{v}(x), \ \ |y(x)| = r.$$

We deduce

$$|\boldsymbol{v}(x)| = |e^{-T(x)A_s}y(x)| \geq e^{-\Gamma_s(q)T(x)}|y(x)| = e^{-\Gamma_s(q)T(x)}r.$$

Hence

$$e^{-\Gamma_s(q)T(x)} \leq \frac{1}{r}|\boldsymbol{v}(x)|. \tag{8.5}$$

Let $x \in \mathcal{B}_{r,\hbar} \cap W_p^-$. Then

$$T_x W_p^- = e^{T(x)A} T_{z(x)} W_p^-, \;\; T_{z(x)} W_p^- \in \mathcal{G}_{r,\hbar}$$

and, we deduce

$$\delta(U, T_x W_p^-) = \delta(U, e^{T(x)A} T_{z(x)} W_p^-), \;\; U = T_q W_q^-.$$

Using Corollary 6.10 we deduce that there exists a constant $C > 0$ such that for every $V \in \mathcal{G}_{r,\hbar}$, and every $t \geq 0$ we have

$$\delta(U, e^{tA} V) \leq C e^{-(\,\gamma_s(p) + \gamma_u(p)\,)t}$$

Hence

$$\forall x \in \mathcal{B}_{r,\hbar} \cap W_p^- : \;\; \delta(U, T_x W^- p) \leq C e^{-(\,\gamma_s(q) + \gamma_u(q)\,)T(x)}.$$

Now observe that

$$-(\gamma_s(q) + \gamma_u(q)) \leq -\nu \Gamma_s(q)$$

so that

$$e^{-(\,\gamma_s(q) + \gamma_u(q)\,)T(x)} \leq e^{-\nu \Gamma_s(q) T(x)} \overset{(8.5)}{\leq} \frac{1}{r^\nu} |\boldsymbol{v}(x)|^\nu.$$

We conclude that

$$\forall x \in \mathcal{B}_{r,\hbar} \cap W_p^- : \;\; \delta(U, T_x W_p^-) \leq C \frac{1}{r^\nu} |\boldsymbol{v}(x)|^\nu = \frac{C}{r^\nu} \operatorname{dist}(x, W_q^-)^\nu.$$

Since for every $w \in \mathcal{B}_{r,\hbar} \cap W_q^-$ we have $U = T_w W_q^-$, the last inequality proves $\boldsymbol{S}(k)$. □

Corollary 8.5. *Suppose (f, ξ) is a smooth Morse pair on the real analytic manifold M such that the flow Φ^ξ generated by ξ satisfies the Morse-Smale condition, and for every hyperbolic critical point p we have*

$$\gamma_u(p) + \gamma_s(p) \geq \Gamma_s(p) \Longleftrightarrow \gamma_u(p) \geq \Gamma_s(p) - \gamma_s(p).$$

Then the filtration

$$M_0(\xi) \subset M_1(\xi) \subset \cdots \subset M, \;\; M_k(\xi) := \bigcup_{\lambda(p) \leq k} W_p^-(\xi)$$

is a Verdier stratification. In particular, if the flow Φ^ξ is also tame, then the above stratification satisfies the Whitney regularity conditions as well. □

From Theorem 4.9 and Theorem 5.1 we obtain the following result.

Corollary 8.6. *Suppose M is a compact real analytic manifold of dimension m, $f : M \to \mathbb{R}$ is a real analytic Morse function, and ν is a positive real number. Then there exist*

- *a real analytic metric g on M,*
- *a smooth vector field ξ on M,*

such that

- *(ξ, f) is a Morse pair,*
- *ξ coincides with $-\nabla^g f$ in an neighborhood of the critical set,*
- *the flow Φ^ξ generated by ξ is tame and satisfies the Morse-Smale condition,*

- *for every hyperbolic critical point p of f we have*

$$\frac{\gamma_u(\xi,p)+\gamma_s(\xi,p)}{\Gamma_s(\xi,p)} \geq \nu \tag{γ_ν}$$

In particular, if $\nu \geq 1$, then the stratification of M by the unstable manifolds of the flow Φ^ξ is both Verdier and Whitney regular. □

Remark 8.7. If the unstable manifolds of a Morse flow on a compact smooth manifold M form a Whitney stratification, then Theorem 7.9 shows that the closure of any unstable manifold is a *submanifold with conical singularities* in the sense of [**30**]. □

Remark 8.8. (a) Theorem 8.2 is not optimal. To see this, consider the projective space $\mathbb{RP}^n = \mathbf{Gr}_1(\mathbb{R}^{n+1})$. We regard it as a submanifold in the Euclidean space of symmetric operators $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.

Any symmetric operator $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defines a function

$$f_A : \mathbb{RP}^n \to \mathbb{R},\ \ L \mapsto \operatorname{tr} AP_L.$$

Suppose

$$A = \operatorname{Diag}(\lambda_0,\dots,\lambda_n),\ \ \lambda_0 < \lambda_1 < \cdots < \lambda_n.$$

Using the projective coordinates $[x_0,\dots,x_n]$ on $\mathbb{RP}^n$, we can describe the critical points of f_A as

$$\mathbf{Cr}_A = \big\{\, p_0,\dots,p_n;\ \ p_i = [\delta_{i0},\delta_{i1},\dots,\delta_{in}]\,\big\},$$

where δ_{ij} is the Kronecker symbol.

The eigenvalues of the Hessian of f at p_i are

$$\mu_j = \lambda_j - \lambda_i,\ \ j \neq i.$$

The hyperbolic critical points are $p_1,\dots,p_{n-1}$. The spectral clustering condition $(\gamma_{\nu=1})$ at p_i reads

$$\lambda_{i+1} - \lambda_{i-1} \geq \lambda_n - \lambda_i \Longleftrightarrow \lambda_i - \lambda_{i-1} \geq \lambda_n - \lambda_{i+1}.$$

This condition is satisfied if for example we choose λ_i such that

$$(\lambda_{i+1}-\lambda_i) \ll (\lambda_i - \lambda_{i-1}),\ \ \text{e.g.,}\ \ (\lambda_{i+1}-\lambda_i) < \frac{1}{i+1}(\lambda_i-\lambda_{i-1}),$$

but fails in the case $\lambda_i = i$.

However, the unstable manifolds of the critical points are *independent* of the choice of λ_i. In fact, these unstable varieties are the Schubert cells.

$$W_i = \big\{\, [x_0,\dots,x_{i-1},1,0,\dots,0];\ \ x_j \in \mathbb{R}\,\big\}.$$

By choosing λ_i so that the clustering condition is satisfied, we deduce that the unstable manifolds satisfy the Verdier regularity condition, and they do so even when the spectral clustering condition is violated.

(b) Although the clustering condition is not optimal, it is in some sense necessary. To understand this, suppose we are on a compact, real analytic 3-manifold M, and (ξ, f) is a tame Morse pair such that the flow generated by ξ satisfies the Morse-Smale condition.

Suppose $q_0 \in M$ is a critical point of f of index 1, the Hessian of f at q_0 has eigenvalues $-1, 1, 3$, and in a neighborhood of q_0 we can find real analytic coordinates (x, y, z) such that

$$x(q_0) = y(q_0) = z(q_0) = 0, \;\; \xi = x\partial_x - y\partial_y - 3z\partial_z.$$

Observe that the spectral clustering condition is violated since

$$\gamma_s(q_0) + \gamma_u(q_0) = 2 < \Gamma_s(p_0) = 3.$$

Suppose the point $q = (0, 0, 1) \in W^+_{q_0}$ also lies on the unstable variety W^-_p of a critical point p of index 2, $q \in W^+_{q_0} \pitchfork W^-_p$. Set $q_t = \Phi_t(q)$. Then $q_t = (0, 0, e^{-3t}) \in W^-_p$ so that

$$-3\partial_z = \dot{q}_0 \in T_q W^-_p.$$

Since W^-_p intersects $W^+_{q_0}$ transversally at q we deduce

$$T_q W^-_p = \operatorname{span}\{\partial_z, \partial_x + a\partial_y\}.$$

Assume $a \neq 0$. Then

$$T_0 W^-_{q_0} = \operatorname{span}\{\partial_x\},\; T_{q_t} W^-_p = \operatorname{span}\big\{e^{-3t}\partial_z, e^t\partial_x + e^{-t}a\partial_y\big\} = \operatorname{span}\big\{\partial_z, \partial_x + e^{-2t}a\partial_y\big\}.$$

We deduce that

$$\delta\big(\,T_0 W^-_{q_0}, T_{q_t} W^-_p\,\big) \sim |a|e^{-2t}, \;\; \text{as} \;\; t \to \infty,$$

so that

$$\lim_{t\to\infty} \frac{\delta\big(\,T_0 W^-_{q_0}, T_{q_t} W^-_p\,\big)}{\operatorname{dist}(0, q_t)} = \lim_{t\to\infty} e^t = \infty.$$

(c) The last example raises a natural question. Can we still conclude that a Morse-Smale flow satisfies the weaker Morse-Whitney condition, without assuming the clustering condition? We describe below a simple situation which suggests that this need not be the case.

Suppose we are in a 3-dimensional situation, and near a critical point q of index 1 we can find coordinates (x, y, z) such that $x(q) = y(q) = z(q) = 0$, and the (descending) Morse flow has the description

$$\Phi_t(x, y, z) = (e^{at}, x, e^{-bt}y, e^{-ct}z), \;\; a > 0, \;\; c > b > 0.$$

The infinitesimal generator of this flow is described by the (linear) vector field

$$\xi = ax\partial_x - by\partial_y - cz\partial_z.$$

The stable variety is the plane $x = 0$, while the unstable variety is the x-axis. We assume that the spectral clustering condition is violated, i.e.,

$$c > a + b.$$

We set $g := c - b$ so that $g > a > 0$. Consider the arc

$$(-1, 1) \ni s \mapsto \gamma(s) := (s, s, 1).$$

Observe that the arc γ is a straight line segment that intersects transversally the stable variety of q at the point $\gamma(0) = (0, 0, 1)$. Suppose that γ is contained in the unstable variety W^-_p of a a critical point p of index 2. We deduce that an open neighborhood of $\gamma(0)$ in W^-_p can be obtained by flowing the arc γ along the flow Φ. More precisely, we look th the open subset of W^-_p given by the parametrization

$$(-1, 1) \times \mathbb{R} \ni (s, t) \mapsto \Phi_t(\gamma(s)) = \big(\,se^t, e^{-t}s, e^{-4t}\,\big).$$

The left half of the *Maple* generated Figure 8 depicts a portion of this parameterized surface corresponding to $|s| \leq 0.2$, $t \in [0, 2]$, $a = b = 1$, $c = 8$, so that the spectral clustering condition is violated. It approaches the x-axis in a rather dramatic way, and we notice a special behavior at the origin. This is where the condition (b') is be violated. The right half of Figure 8 describes the same parametrized situation when $a = 1$, $b = 1$, and $c = 1.5$, so that the spectral clustering condition is satisfied. The asymptotic twisting near the orgin is less pronounced in this case.

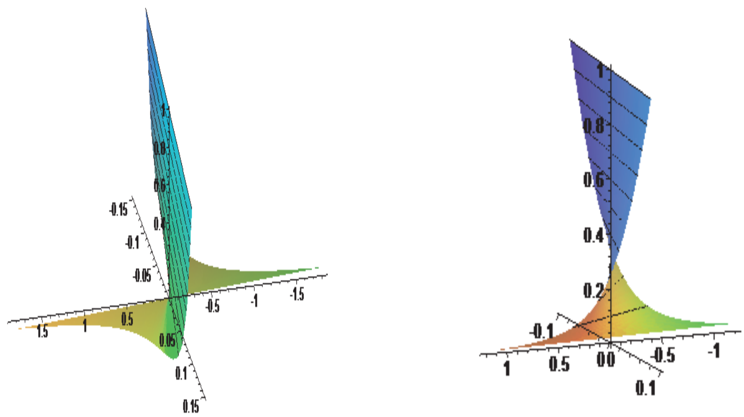


FIGURE 8. *Different behaviors of 2-dimensional unstable manifolds.*

Fix a nonzero real number m, define $s_t := me^{-gt}$, and consider the point

$$p_t := \Phi_t\big(\,\gamma(s_t)\,\big) = (e^{at}s_t, e^{-bt}s_t, e^{-ct}\,) = (me^{(a-g)t}, me^{-ct}, e^{-ct}) \in W_p^-.$$

Observe that since $b < c$ we have $\lim_{t\to\infty} s_t = 0$, and since the clustering condition is violated we have $a - g < 0$ so that

$$\lim_{t\to\infty} p_t = q = (0, 0, 0).$$

The tangent space of W_p^- at the point $\gamma(s_t)$ is spanned by

$$\gamma'(s_t) = (1, 1, 0) \ \text{ and } \ \xi(\gamma(s_t)\,) = (as_t, -bs_t, -c).$$

Denote by L_t the tangent plane of W_p^- at p_t. It is spanned by

$$\Xi_t := \xi(p_t) = \big(ae^{at}s_t, -be^{-bt}s_t, -ce^{-ct}\,\big) = \big(\,mae^{(a-g)t}, -mbe^{-ct}, -ce^{-ct}\,\big),$$

and by

$$u_t := D\Phi_t\gamma'(s_t) = \big(\,e^{at}, e^{-bt}, 0\,\big).$$

Observe that L_t is also spanned by

$$mae^{-at}u_t = (ma, mae^{-(a+b)t}, 0) \ \text{ and } \ e^{(g-a)t}\Xi_t = \big(ma, -mbe^{(g-a-c)t}, -ce^{(g-a-c)t}\big)$$

Noting that $g - a - c = -(b + a)$ we deduce that L_t is also spanned by the pair of vectors $e^{-at}u_t$ and

$$X_t := mae^{-at}u_t - e^{(g-a)t}\Xi_t = \big(0, e^{-(b+a)t}m(a+b), ce^{-(a+b)t}\big).$$

Now observe that

$$e^{(a+b)t}X_t = (0, m(a+b), c),$$

which shows that L_t converges to the 2-plane L_∞ spanned by

$$(1,0,0) = \frac{1}{ma}\lim_{t\to\infty} e^{-at}u_t = (1,0,0) \ \text{ and } \ (0, m(a+b), c).$$

On the other hand, if we denote by π the projection onto the x-axis, the unstable variety of q, then

$$p_t - \pi(p_t) = (0, me^{-ct}, e^{-ct})$$

and the line ℓ_t spanned by the vector $p_t - \pi(p_t)$ converges to the line ℓ_∞ spanned by the vector $(m, 1)$. The vectors $(\,m(a+b), c\,)$ and $(m, 1)$ are colinear if and only if $c = (a + b)$. We know that this is not the case because the spectral clustering condition is violated.

Hence $\ell_\infty \not\subset L_\infty$, and this shows that the pair (W_q^-, W_p^-) does not satisfy Whitney's regularity condition (b') at the point $q = \lim_t p_t$. □

CHAPTER 9

The Conley index

In this section we want to investigate the Conley indices of the isolated stationary points of gradient like tame flows. We begin with a fast introduction to Conley theory. For more details we refer to [**7, 42**].

Suppose X is a compact metric space, and $\Phi : \mathbb{R} \times X \to X$, $(t, x) \mapsto \Phi_t(x)$ is a continuous flow. Thinking of this flow as an action of $\mathbb{R}$ on X, we will denote $\Phi_t(x)$ by $t \cdot x$. For any set $W \subset X$ we define

$$I^{\pm}(W) := \{ x \in W;\ \ t \cdot x \in W,\ \ \forall t,\ \ \pm t \geq 0 \},\ \ I(W) := I^+(W) \cap I^-(W).$$

An *isolated* invariant set of the flow is a closed, flow invariant subset $S \subset X$ such that there exists a compact neighborhood W of S in X with the property that $S = I(W)$. The set W is called an *isolating neighborhood* of S.

Suppose $W \subset X$ is compact. Then the subset $A \subset W$ is said to be *positively invariant* with respect to W if

$$x \in A,\ \ t \geq 0,\ \ [0, t] \cdot x \subset W \Longrightarrow [0, t] \cdot x \subset A.$$

Suppose W is an isolating neighborhood of S. An *index pair in W* (or *index pair rel W*) for the isolated invariant set S is a pair of compact sets (N, N^-), $N^- \subset N \subset W$, with the following properties.

($\mathbf{I}_0$) N is positively invariant in W.
($\mathbf{I}_1$) $N \setminus N^-$ is a neighborhood of S, and $S = I(\,\boldsymbol{cl}(N \setminus N^-)\,)$.
($\mathbf{I}_2$) N^- is positively invariant in N.
($\mathbf{I}_3$) If $x \in N$, and $[0, \infty) \cdot x \not\subset N$, then there exists $t \geq 0$ such that $[0, t] \cdot x \subset N$, and $t \cdot x \in N^-$.

A pair of compact sets (N, N^-) satisfying the conditions $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ will be called an *index pair* of S. Note that the definition of an index pair assumption $\mathbf{I}_0$ is not required because we do not specify any isolating neighborhood W.

THEOREM 9.1 (Existence of index pairs). *Suppose S is an isolating invariant set of the flow Φ, W is an isolated neighborhood of S and U is a neighborhood of S. Then there exists an index pair (N_U, N_U^-) of S in W such that*

$$\boldsymbol{cl}(N_U \setminus N_U^-) \subset U. \qquad \square$$

Suppose S is an isolated invariant set. To any index pair (N, N^-) we associate the *pointed space* N/N^-. When $N^- \neq \emptyset$, then the equivalence class of N^- serves as basepoint in N/N^-. When $N^- = \emptyset$, then N/N^- is defined to be the disjoint union between N and a point $*$ which serves as basepoint.

For the reader's convenience we outline below the proof of the fact that the homotopy type of N/N^- is independent of the choice of index pair (N, N^-). For more details we refer to [**7**, III.4] and [**42**, Thm. 4.10].

For every $t \geq 0$ we define

$$ {}^tN := \{\ x \in N;\ \ [-t,0]\cdot x \subset N\ \}, $$

$$ {}^{-t}N^- := \{x \in N;\ \ \exists t' \in [0,t] : [0,t']\cdot x \subset N,\ \ t'\cdot x \in N^-\ \} $$

Then ${}^tN \subset N$, ${}^{-t}N^- \supset N^-$. The inclusion induced map

$$ i_t : {}^tN/\,{}^tN \cap N^- \to N/N^- $$

is a homotopy equivalence with homotopy inverse $f_t : N/N^- \to {}^tN/\,{}^tN \cap N^-$ given by (see [**7**, III.4.2])

$$ f_t([x]) = \begin{cases} [t\cdot x] & [0,t]\cdot x \subset N \\ [{}^tN \cap N^-] & \text{otherwise.} \end{cases} $$

Similarly, the inclusion induced map

$$ j_t : N/N^- \to N/{}^{-t}N^- $$

is a homotopy equivalence with homotopy inverse g_t given as the composition $i_t \circ h_t$, where $h_t : N/{}^{-t}N^- \to {}^tN/{}^tN \cap N^-$ is the *homeomorphism* given by

$$ h_t([x]) = \begin{cases} [t\cdot x] & [0,t] \subset N \setminus N^- \\ [{}^tN \cap N^-] & \text{otherwise.} \end{cases} $$

Suppose (N_0, N_0^-) and (N_1, N_1^-) are two index pairs in W for S. Then there exists $T = T(N_0, N_1) > 0$ such that for any $t > T$ we have

$$ ({}^tN_0, {}^tN_0 \cap N_0^-) \subset (N_1, {}^{-t}N_1^-),\ \ ({}^tN_1, {}^tN_1 \cap N_1^-) \subset (N_0, {}^{-t}N_0^-). $$

Fix $t > T(N_1, N_0)$, denote by α_t the inclusion induced map

$$ \alpha_t : {}^tN_0/{}^tN_0 \cap N_0^- \to N_1/{}^{-t}N_1^-, $$

and by β_t the inclusion induced map

$$ \beta_t : {}^tN_1/{}^tN_1 \cap N_1^- \to N_0/{}^{-t}N_0^-. $$

Define $\mathcal{C}^t_{N_1,N_0} : N_0/N_0^- \to N_1/N_1^-$ as the composition

$$ N_0/N_0^- \xrightarrow{f_t^0} {}^tN_0/{}^tN_0 \cap N_0^- \xrightarrow{\alpha_t} N_1/{}^{-t}N_1^- \xrightarrow{g_t^1} N_1/N_1^-. $$

For any $t, t' > T(N_1, T_0)$, the maps $\mathcal{C}^t_{N_1,N_0}$ and $\mathcal{C}^{t'}_{N_1,N_0}$ are homotopic. We denote by

$$ \mathcal{C}_{N_1,N_0} \in [N_0/N_0^-, N_1/N_1^-], $$

the homotopy class determined by this family of maps, and we will refer to it as the *connector* from N_0 to N_1.

If (N_0, N_0^-), (N_1, N_1^-) and (N_2, N_2^-) are three index pairs, and

$$ t > \max\{\ T(N_2, N_1),\ T(N_1, N_0),\ \ T(N_2, N_0)\ \}, $$

then we have a homotopy

$$ \mathcal{C}^t_{N_2,N_0} \simeq \mathcal{C}^t_{N_2,N_1} \circ \mathcal{C}^t_{N_1,N_0}. $$

In particular, if $N_2 = N_0$ we deduce

$$ \mathcal{C}^t_{N_0,N_1} \circ \mathcal{C}^t_{N_1,N_0} \simeq \mathcal{C}^t_{N_0,N_0} \simeq \mathbb{1}, $$

so that all the connectors are homotopy equivalences.

The homotopy type of the pointed space $[N/N^-]$ is therefore independent of the index pair (N, N^-) of S. It is called the *Conley index* of S and it is denoted by $\boldsymbol{h}(S)$, or $\boldsymbol{h}(S, \Phi)$.

Consider now a compact tame set X embedded in some Euclidean space E. Denote by $|\bullet|$ the Euclidean norm on E. Suppose Φ is a tame flow on X.

DEFINITION 9.2. A stationary point p of Φ is called *Morse like* if there exists a tame continuous function $f : X \to \mathbb{R}$ with the following properties.

- $f(p) = 0$.
- There exists $c_0 > 0$ such that
$$\mathbf{Cr}_\Phi \cap \{0 < |f| < c_0\} = \emptyset.$$
- The set $\mathbf{Cr}^0 := \mathbf{Cr}_\Phi \cap \{f = 0\}$ is finite.
- The function f decreases, along the trajectories of the flow, not necessarily strictly.
- The function f decreases *strictly* along any portion of nonconstant trajectory situated in the region $\{|f| < c_0\}$

The function f is called a *local*[1] *Lyapunov* function adapted to the stationary point p. □

Suppose $p \in X$ is a Morse like stationary point of the flow, and f is a local Lyapunov function adapted to p. For every $c \in \mathbb{R}$ we denote by X_c the level set $\{f = c\}$. Denote by W_p^+ and respectively W_p^- the stable and respectively unstable varieties of the point p, and set

$$\mathcal{L}_p^-(\varepsilon) := W_p^- \cap X_{-\varepsilon}, \;\; \mathcal{L}_p^+(\varepsilon) := W_p^+ \cap X_\varepsilon.$$

LEMMA 9.3. *Suppose $\varepsilon \in (0, c_0)$. Then the following hold.*
(a) The link $\mathcal{L}_p^\pm(\varepsilon)$ is a compact subset of $X_{\pm\varepsilon}$.
(b) The tame set $W_p^\pm(\varepsilon) = W_p^\pm \cap \{|f| \le \varepsilon\}$ is tamely homeomorphic to a cone on $\mathcal{L}_p^\pm(\varepsilon)$.

PROOF. (a) We prove only the case $\mathcal{L}_p^+(\varepsilon)$ since the other case is obtained from this by time reversal. We argue by contradiction. Suppose

$$x_0 \in \boldsymbol{cl}(\,\mathcal{L}_p^+(\varepsilon)\,) \setminus \mathcal{L}_p^+(\varepsilon)$$

Then there exists a tame continuous path $(0, 1] \ni s \mapsto x_s \in \mathcal{L}_p^+(\varepsilon)$ such that

$$\lim_{s \to 0^+} x_s = x_0.$$

Since $f(t \cdot x_s) \in [0, \varepsilon]$, $\forall s, t > 0$ we deduce $f(t \cdot x_0) \in [0, \varepsilon]$, $\forall t \ge 0$. If we set

$$q = \lim_{t \to \infty} t \cdot x_0$$

we deduce that q is a stationary point of Φ such that $f(q) \in [0, \varepsilon]$. Since $\varepsilon < c_0$ we deduce $q \in \mathbf{Cr}^0$, and since $x_0 \notin W_p^+$ we deduce $q \ne p$.

Consider the family of paths (see Figure 9)

$$g_t : [0, 1] \to X, \;\; g_t(s) = t x_s.$$

[1]Here the attribute *local* is abusively used to remind us that f behaves like a Lyapunov function only on an open neighborhood of p, namely $\{|f| < c_0\}$. This neighborhood could be quite large.

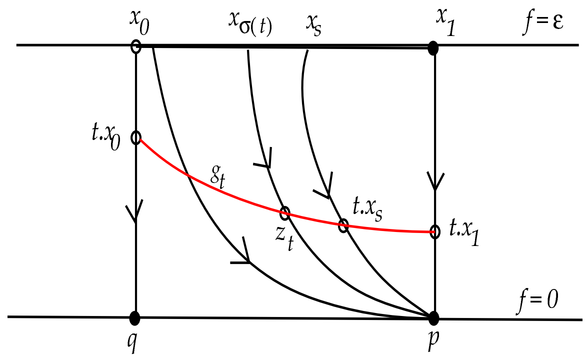


FIGURE 9. *The stable variety of* p *is arbitrarily close to that of* q.

Let

$$\delta := \min\{|q' - q''|;\ \ q', q'' \in \mathbf{Cr}^0,\ \ q' \neq q''\},\ \ d_t := |t \cdot x_0 - t \cdot x_1|,$$

and consider the definable family of closed subsets of the unit interval

$$I_t := \Big\{\, s \in [0,1];\ \ |t \cdot x_s - t \cdot x_0| = \frac{1}{2}\min(\delta, d_t)\,\Big\}.$$

Note that $I_t \neq \emptyset$, $\forall t > 0$. We can then find a definable function

$$\sigma : [0, \infty) \to [0, 1]$$

such that $\sigma(t) \in I_t$, $\forall t > 0$. Set $z_t := t \cdot x_{\sigma(t)}$ so that

$$|z_t - t \cdot x_0| = \frac{1}{2}\min\{\delta, d_t\},\ \ \forall t > 0.$$

The function σ is continuous for t sufficiently large and the limit

$$\sigma_\infty := \lim_{t\to\infty} \sigma(t)$$

exists and it is finite. Observe that the definable path

$$t \mapsto t \cdot x_{\sigma(t)} \in \{0 \leq f \leq \varepsilon\},$$

has a limit as $t \to \infty$ which we denote by z_∞. Since $d_t \to |q - p| \geq \delta$ we deduce

$$|z_\infty - q| = \frac{1}{2}\delta.$$

In particular, we deduce that z_∞ is not a stationary point of the flow.

Consider now the function

$$e : X \to (-\infty, 0],\ \ e(x) = f(x) - f(\Phi_1(x)),$$

where Φ_1 denotes the time-1 map determined by the flow Φ. Since z_∞ is not a stationary point we deduce

$$e(z_\infty) < 0.$$

Because the time-1 map Φ_1 is continuous, we deduce that, for every positive $\hbar$ such that $\hbar \leq |e(z_\infty)|$, there exists an open neighborhood $U_\hbar$ of z_∞ in X such that

$$e(x) < \hbar,\ \ \forall z \in U_\hbar.$$

In particular, for sufficiently large t, we have $z_t \in U_\hbar$, and thus

$$0 \leq f(\,\Phi_1(z_t)\,) < f(z_t) - \hbar.$$

If we let $t \to \infty$ we deduce

$$0 \leq f(\,\Phi_1(z_\infty)\,) \leq f(z_\infty) - \hbar = -\hbar.$$

This contradiction proves the compactness of $\mathcal{L}_p^+(\varepsilon)$.
(b) From part (a) we deduce easily that $W_p^+(\varepsilon)$ is compact. Consider the tame homeomorphism

$$[0,1) \ni t \mapsto t(s) = \frac{s}{1-s} \in [0,\infty).$$

Now consider the map

$$[0,1] \times \mathcal{L}_p^+(\varepsilon) \to W_p^+(\varepsilon), \;\; (s,x) \mapsto t(s)\cdot x.$$

This maps the slice $\{1\} \times \mathcal{L}_p^+(\varepsilon)$ to p and it induces a tame continuous bijection from the cone on $\mathcal{L}_p^+(\varepsilon)$ to $W_p^+(\varepsilon)$. Since $W_p^+(\varepsilon)$ is compact we deduce that this map is a homeomorphism. □

The (tame) topological type of $\mathcal{L}_p^+(\varepsilon)$ and respectively $\mathcal{L}_p^-(\varepsilon)$ is independent of ε if ε is sufficiently small because the tame continuous map

$$f : W_p^\pm(\varepsilon) \setminus \{p\} \to \mathbb{R}$$

is locally trivial for $\varepsilon > 0$. We will refer to this tame homeomorphism class as the *stable* and respectively *unstable link* of p, and we will denote it by $\mathcal{L}_p^\pm$.

Observe that for $\varepsilon > 0$ sufficiently small the tame set $W_p^\pm \cap \{|f| \leq \varepsilon\}$ is tamely homeomorphic to the cone on $\mathcal{L}_p^\pm$, and that the links $\mathcal{L}_q^\pm(\varepsilon)$, $q \in \mathbf{Cr}^0$ are mutually disjoint compact subsets of $X_{\pm\varepsilon}$.

Proposition 9.4. *Let $\varepsilon \in (0, c_0)$ and let K be a tame compact neighborhood of $\mathcal{L}_p^-(\varepsilon)$ in the level set $X_{-\varepsilon}$ such that*

$$K \cap W_q^- = \emptyset, \;\; \forall q \in \mathbf{Cr}^0, \;\; q \neq p,$$

and set

$$N = N_{\varepsilon,K} := \big(\, W_p^- \cup W_p^+ \cup (-\infty, 0]\cdot K\,\big) \cap \big\{ |f| \leq \varepsilon \,\big\}.$$

Then the pair (N,K) is an index pair for p.

Proof. The conditions I_2 and I_3 in the definition of an index pair are clearly satisfied due to the existence of the Lyapunov function f, so it suffices to show that N is a *compact, isolating, neighborhood* of p. In the proof we will need several auxiliary results.

Lemma 9.5. *Suppose*

$$(0,1] \ni s \mapsto x_s \in X_{-\varepsilon}, \;\; (0,1) \ni s \mapsto t_s \in (0,\infty)$$

are tame continuous paths such that

$$\lim_{s\to 0^+} t_s = \infty \;\; \text{and} \;\; f(\,(-t_s)\cdot x_s) \leq 0, \;\; \forall s \in (0,1).$$

Then there exists $q \in \mathbf{Cr}^0$ such that $x_0 \in \mathcal{L}_q^-(\varepsilon)$ and $\lim_{s\to 0^+}(-t_s)\cdot x_s = q$.

Proof. Observe that

$$(-T)\cdot x_0 \in \{-\varepsilon \leq f \leq 0\}, \;\; \forall T > 0$$

so that there exists $q \in \mathbf{Cr}^0$ such that $x_0 \in \mathcal{L}_q^-(\varepsilon)$. Set $z_s = (-t_s)\cdot x_s$.

The definable path $s \mapsto z_s$ has a limit $z_0 = \lim_{s\to 0+} z_s$. Since

$$T \cdot z_0 \in \{-\varepsilon \le f \le 0\}, \;\; \forall T > 0,$$

the point z_0 must be a stationary point. We claim $z_0 = q$. We argue by contradiction, so we assume $z_0 \neq q$.

Set $y_s := (-t(s)) \cdot x_0$. For every $s \in (0, 1]$ consider the definable continuous path

$$g_s : [0, 1] \to X, \;\; g_s(\lambda) = (-t(s)) \cdot x_{\lambda \cdot s}.$$

Observe that $g_s(0) = y_s$ and $g_s(1) = z_s$. Arguing as in the proof of Lemma 9.3 we can find a definable function

$$(0, 1) \ni s \mapsto \lambda_s \in [0, 1]$$

such that

$$\operatorname{dist}\big(g_s(\lambda_s), q\big) = \frac{1}{2}\min\{\delta, |z_s - y_s|\}, \;\; \delta := \min\{\, |q' - q''|; \;\; q', q'' \in \mathbf{Cr}^0, \;\; q' \neq q''\,\}.$$

We set

$$\gamma_s := g_s(\lambda_s) = (-t(s)) \cdot x_{\lambda_s s}.$$

Then, as $s \searrow 0$, the point γ_s converges to a point γ_0 such that

$$\gamma_0 \in \{-\varepsilon \le f \le 0\}, \;\; \operatorname{dist}(\gamma_0, q) = \frac{1}{2}\min\{\delta, |z_0 - q|\} = \frac{1}{2}\delta.$$

Thus γ_0 *is not a stationary point* of Φ. We claim that

$$f(T \cdot \gamma_0) \ge -\varepsilon, \;\; \forall T > 0. \tag{9.1}$$

Indeed, for every $T > 0$, and for every $\hbar > 0$ there exists a small neighborhood $U = U_{T,\hbar}$ of γ_0 such that for every $x \in U$ we have

$$|f(T \cdot x) - f(T \cdot \gamma_0)| < \hbar.$$

We can now find $s > 0$ such that $\gamma_s \in U_{T,\hbar}$ and $t(s) > T$, from which we deduce

$$f(T \cdot \gamma_0) \ge f(T \cdot \gamma_s) - \hbar \ge f(t(s) \cdot \gamma_s) - \hbar = f(x_{\lambda_s \cdot s}) - \hbar = -\varepsilon - \hbar.$$

This proves the claim (9.1) which in turn implies that γ_0 *has to be a stationary point*. This contradiction completes the proof of Lemma 9.5. □

Observe that for every $x \in X_{-\varepsilon}$ we have

$$\Phi_{-\infty}(x) \in \{f \ge 0\}.$$

Define $T = T_{-\varepsilon} : X_{-\varepsilon} \to [0, \infty]$ by setting $T(x) = \infty$ if $\Phi_{-\infty}x \in \mathbf{Cr}^0$, and otherwise, we let $T(x)$ to be the unique positive real number such that

$$(-T(x)) \cdot x \in X_0.$$

Using the definable homeomorphism

$$\sigma : [0, \infty) \to [0, 1), \;\; t \mapsto \sigma(t) = \frac{t}{1+t}.$$

we obtain a compactification $[0, \infty]$ of $[0, \infty)$ tamely homeomorphic to $[0, 1]$.

LEMMA 9.6 (Deformation Lemma). *(a) The tame function*

$$X_{-\varepsilon} \ni x \mapsto T_{-\varepsilon}(x) \in [0, \infty]$$

is continuous.
(b) The tame function

$$\mathcal{D}_{\Phi}^{-\varepsilon} : \{(x,t) \in X_{-\varepsilon} \times [0,\infty];\ \ t \leq T_{-\varepsilon}(x)\} \to \{-\varepsilon \leq f \leq 0\},\ \ (x,t) \mapsto (-t)\cdot x$$

is continuous.

PROOF. For simplicity, during this proof, we will write $T(x)$ instead of $T_{-\varepsilon}(x)$.
(a) By invoking the closed graph theorem it suffices to show that for any continuous definable path

$$(0,1) \ni s \mapsto (x_s, T(x_s)) \in X_{-\varepsilon} \times [0, \infty]$$

such that

$$x_s \to x_0,\ \ T(x_s) \to T_0 \in [0, \infty],$$

then $T_0 = T(x_0)$. Observe that if $T(x_s) = \infty$, for all s sufficiently small, then there exists $q \in \mathbf{Cr}^0$ such that $x_s \in \mathcal{L}_q^-((\varepsilon)$, and since $\mathcal{L}_q^-(\varepsilon)$ is compact, we deduce $x_0 \in \mathcal{L}_q^-(\varepsilon)$. Thus, we can assume that $T(x_s) < \infty$, for all s.

If $T_0 < \infty$, the conclusion follows from the continuity of the flow. Thus, we can assume $T_0 = \infty$, and $T(x_s) \nearrow \infty$ as $s \searrow 0$, and we have to prove that there exists $q \in \mathbf{Cr}^0$ such that $x_0 \in \mathcal{L}_q^-(\varepsilon)$. This follows immediately from the fact that

$$(-T)\cdot x_0 \in \{-\varepsilon \leq f \leq 0\},\ \ \forall T > 0,$$

so that x_0 must belong to the unstable variety of a stationary point situated in the region $\{-\varepsilon \leq f \leq 0\}$.
(b) Again we rely on the closed graph theorem. We have to show that for every tame continuous paths

$$(0,1) \ni s \mapsto (x_s, t_s) \in X_{-\varepsilon} \times [0, \infty],$$

such that

$$0 \leq t_s \leq T(x_s),\ \ \lim_{s\to 0^+} x_s = x_0,\ \ \lim_{s\to 0^+} t_s = t_0,\ \ \lim_{s\to 0^+} (-t_s)\cdot x_s = y_0 \in X_0,$$

we have $y_0 = (-t_0)\cdot x_0$.

Arguing as in (a), we see that the only nontrivial situation is when $t_s \nearrow \infty$ as $s \searrow 0$. In this case, we have to prove that $y_0 \in \mathbf{Cr}^0$ and $x_0 \in \mathcal{L}_{y_0}^-(\varepsilon)$. This follows from Lemma 9.5. □

The Deformation Lemma has many useful corollaries.

COROLLARY 9.7. *The continuous tame map*

$$\mathcal{T}_{\Phi}^{-\varepsilon} : X_{-\varepsilon} \to X_0,\ \ x \to \mathcal{T}_{\Phi}^{-\varepsilon}(x) = (-T_{-\varepsilon}(x))\cdot x \in X_0$$

induces a tame homeomorphism

$$X_{\varepsilon}^* = X_{-\varepsilon} \setminus \bigcup_{q\in\mathbf{Cr}^0} \mathcal{L}_q^-(\varepsilon) \to X_0^* = X_0 \setminus \mathbf{Cr}^0 .$$

PROOF. The map $\mathcal{T}_{\Phi}^{-\varepsilon}: X_{-\varepsilon}^{*} \to X_{0}^{*}$ is continuous and bijective. Its inverse is continuous because its graph is closed. □

Consider the strip

$$\mathcal{S}_{-\varepsilon} := \Big\{ (x,s) \in X_{-\varepsilon} \times [0,1];\;\; s \leq \sigma_{-\varepsilon}(x) = \frac{T_{-\varepsilon}(x)}{1+T_{-\varepsilon}(x)},\;\; \forall x \in X_{-\varepsilon} \Big\}.$$

Observe that we have a tame homeomorphism

$$\mathcal{A}_{-\varepsilon}: X_{-\varepsilon} \times [0,1] \to \mathcal{S}_{-\varepsilon},\;\; (x,\lambda) \mapsto (x, \sigma_{-\varepsilon}(x)\cdot\lambda)$$

and a tame homeomorphism

$$S: \mathcal{S}_{-\varepsilon} \to \big\{ (x,t) \in X_{-\varepsilon} \times [0,\infty];\;\; t \leq T_{-\varepsilon}(x) \big\},$$

given by

$$S(x,s) \mapsto (x, \frac{s}{1-s}).$$

The composition

$$\hat{\mathcal{D}}_{\Phi}^{-\varepsilon} := \mathcal{D}_{\Phi}^{-\varepsilon} \circ S \circ \mathcal{A}_{-\varepsilon}: X_{-\varepsilon} \times [0,1] \to \{-\varepsilon \leq f \leq 0\}$$

is a tame continuous map, which along $X_{-\varepsilon} \times \{1\}$ it coincides with the map $\mathcal{T}_{\Phi}^{-\varepsilon}: X_{-\varepsilon} \to X_0$.

The natural deformation retraction of $X_{-\varepsilon} \times [0,1]$ onto $X_{-\varepsilon} \times \{1\}$ determines a deformation retraction of

$$\mathcal{R}_{\Phi}^{-\varepsilon}: \{-\varepsilon \leq f \leq 0\} \times [0,1] \to \{-\varepsilon \leq f \leq 0\}$$

of $\{-\varepsilon \leq f \leq 0\}$ onto $\{f=0\}$. The next result summarizes the above observations.

COROLLARY 9.8. *The deformation $\hat{\mathcal{D}}_{\Phi}^{-\varepsilon}$ induces a homeomorphism between the mapping cylinder of $\mathcal{T}_{\Phi}^{-\varepsilon}: X_{-\varepsilon} \to X_0$ and the region $\{-\varepsilon \leq f \leq 0\}$.* □

REMARK 9.9. The maps $T_{-\varepsilon}$, $\mathcal{D}_{\Phi}^{-\varepsilon}$, $\mathcal{T}_{\Phi}^{-\varepsilon}$, $\mathcal{R}_{\Phi}^{-\varepsilon}$ have "positive" counterparts $T_{\varepsilon}(x): X_{\varepsilon} \to [0,\infty)$,

$$\mathcal{D}_{\Phi}^{\varepsilon}: \big\{ (x,t) \in X_{\varepsilon} \times [0,\infty];\;\; t \leq T_{\varepsilon}(x) \big\} \to \{\, 0 \leq f \leq \varepsilon \,\},$$

$$\mathcal{T}_{\Phi}^{\varepsilon}: X_{\varepsilon} \to X_0,$$

and

$$\mathcal{R}_{\Phi}^{\varepsilon}: \{\, \varepsilon \geq f \geq 0 \,\} \times [0,1] \to \{\, \varepsilon \geq f \geq 0\},$$

and their similar properties follow by time reversal from their "negative" counterparts. □

Now set

$$K_0 := \mathcal{T}_{\Phi}^{-\varepsilon}(K) \subset X_0,\;\; K^+ = (\mathcal{T}_{\Phi}^{\varepsilon})^{-1}(K_0) \subset X_{\varepsilon}.$$

Then K_0 is a compact neighborhood of p in X_0, K^+ is a compact neighborhood of $\mathcal{L}_p^+(\varepsilon)$ in X_{ε}, and we have the equality

$$N = \underbrace{\hat{\mathcal{D}}_{\Phi}^{-\varepsilon}\big(\, K \times [0,1] \,\big)}_{N_{\leq 0}} \cup \underbrace{\hat{\mathcal{D}}_{\Phi}^{\varepsilon}\big(\, K^+ \times [0,1] \,\big)}_{N_{\geq 0}}. \tag{9.2}$$

Now observe that $N_{\leq 0}$ is a *compact* neighborhood of p in $\{f \leq 0\}$ and $N_{\geq 0}$ is a *compact* neighborhood of p in $\{f \geq 0\}$.

The fact that N is an isolating neighborhood of p follows from (9.2). This completes the proof of Proposition 9.4. □

THEOREM 9.10. *Suppose Φ is a tame flow on a tame compact set X, p is a Morse like stationary point of Φ, and f is a local Lyapunov function adapted to p. We denote by W_p^- the unstable variety of p, and for every $\varepsilon > 0$ we set*

$$W_p^-(\varepsilon) := W_p^- \cap \{ -\varepsilon \le f \le 0 \}, \;\; \mathcal{L}_p^-(\varepsilon) := W_p^- \cap \{ f = -\varepsilon \}.$$

Then the Conley index $h_\Phi(p) := h(\{p\}, \Phi)$ is homotopy equivalent to the pointed space $W_p^-(\varepsilon)/\mathcal{L}_p^-(\varepsilon)$, for all sufficiently small $\varepsilon > 0$.

PROOF. We continue to use the same notations as in the proof of Proposition 9.4. Fix $\varepsilon > 0$ sufficiently small so that the only stationary points of Φ in $\{|f| \le \varepsilon\}$ lie on the level set X_0.

Because both $X_{-\varepsilon}$ and $\mathcal{L}_p^-(\varepsilon)$ are tame compact tame sets we can find a triangulation of $X_{-\varepsilon}$ so that $\mathcal{L}_p^{-\varepsilon}$ is a subcomplex of the triangulation of $X_{-\varepsilon}$. From the classical results of J.H.C. Whitehead [**48**] we deduce that for any neighborhood U of $\mathcal{L}_p^-(\varepsilon)$ we can find triangulations of the pair $(X_{-\varepsilon}, \mathcal{L}_p^-(\varepsilon))$ such that the simplicial neighborhood of $\mathcal{L}_p^-(\varepsilon)$ in $X_{-\varepsilon}$ is contained in U and collapses onto $\mathcal{L}_p^-(\varepsilon)$. Fix such a simplicial neighborhood K which is disjoint from W_q^-, $\forall q \in \mathbf{Cr}_\Phi^0$, $q \ne p$. Because K collapses onto $\mathcal{L}_p^-(\varepsilon)$ we can find a *tame* deformation retraction onto $\mathcal{L}_p^-(\varepsilon)$.

Form the index pair $(N, K) = (N_{\varepsilon,K}, K)$. Let us point out that both N and K are compact sets, and in particular the inclusion $K \hookrightarrow N$ is a cofibration.

Using the deformation retraction $\hat{\mathcal{D}}_\Phi^\varepsilon$ we see that the pair (N, K) is homotopy equivalent to the pair $(N_{\le 0}, K)$, $N_{\le 0} = N \cap \{f \le 0\}$. Corollary 9.8 implies that $N_{\le 0}$ is homeomorphic to the mapping cylinder of the tame map

$$\mathcal{T}_\Phi^{-\varepsilon} : K \to K_0 = \mathcal{T}_\Phi^{-\varepsilon}(K) \subset X_0.$$

Corollary 9.7 shows that $\mathcal{T}_\Phi^{-\varepsilon}$ induces a tame homeomorphism

$$K^* = K \setminus \mathcal{L}_p^-(\varepsilon) \to K_0^* = K_0 \setminus \{p\}.$$

Now observe that $W_p^-(\varepsilon)$ is also homeomorphic to the mapping cylinder of the map $\mathcal{T}_\Phi^{-\varepsilon} : \mathcal{L}_p^-(\varepsilon) \to \{p\}$. We deduce that $N_{\le 0}$ is homeomorphic to the mapping cylinder of the natural projection

$$\pi : K \to K/\mathcal{L}_p^-(\varepsilon).$$

A tame deformation retraction of K onto $\mathcal{L}_p^-(\varepsilon)$ extends to a deformation of the mapping cylinder of π to the mapping cylinder of $\pi|_{\mathcal{L}_p^-(\varepsilon)}$ which is homeomorphic to $W_p^-(\varepsilon)$. □

REMARK 9.11. The Conley index computation in this section bares a striking resemblance with the computation of Morse data in the Goresky-MacPherson stratified Morse theory, [**17**]. We believe this resemblance goes beyond the level of accidental coincidence, but we will pursue this line of thought elsewhere. □

Here is a simple application of the above result. Suppose X is a tame space, and Φ is a *Morse like* tame flow on X. This means that Φ has finitely many stationary points, and admits a tame Lyapunov function f. Observe that the local minima are stationary points of Φ. They are characterized by the condition $W_p^- = \{p\}$. We denote by $\mathbf{Cr}_\Phi \subset X$ the set of stationary points.

For every compact tame space $Y \neq \emptyset$ we denote by $\mathcal{P}_Y(t)$ the Poincaré polynomial of Y

$$\mathcal{P}_Y(t) = \sum_{k\geq 0}(\dim H_k(Y,\mathbb{R}))t^k \in \mathbb{Z}[t].$$

If A is a compact tame subset of Y we denote by $\mathcal{P}_{Y,A}(t) \in \mathbb{Z}[t]$ the Poincaré polynomial of the pair (Y,A) defined in a similar fashion. In particular, for every $p \in \mathbf{Cr}_\Phi$, we denote by $\mathbb{M}_{\Phi,p}(t)$ the Poincaré polynomial of the Conley index of p, and we will refer to it as the *Morse polynomial of the stationary point* p. As in [**36**], we define an order relation $\succ$ on $\mathbb{Z}[t]$ by declaring $A \succeq B$ if there exists a polynomial $Q \in \mathbb{Z}[t]$ with nonnegative coefficients such that

$$A(t) = B(t) + (1+t)Q(t).$$

Corollary 9.12 (Morse inequalities). *Let Φ be a Morse like flow on X with Lyapunov function f be as above.*

$$\sum_{p\in\mathbf{Cr}_\Phi} \mathbb{M}_{\Phi,p}(t) \succeq \mathcal{P}_X(t).$$

Proof. Define the discriminant set,

$$\Delta_f := f(\mathbf{Cr}_\Phi).$$

Δ_Φ is a finite set of real numbers

$$\Delta_\Phi = \{c_0 < c_1 < \cdots < c_n\}.$$

For $k = 0, \dots, n$ we set

$$\mathbf{Cr}_\Phi^k := \mathbf{Cr}_\Phi \cap \{f = c_k\}.$$

Now choose $r_0 = c_0 < r_1 < c_1 \cdots < c_{n-1} < r_n < c_n = r_{n+1}$ and set $X^k := \{f \leq r_k\}$. For each $k = 0, 1, \dots, n$ the pair $[X^{k+1}, X^k]$ is an index pair for the isolated invariant set $\mathbf{Cr}_\Phi^k$. We deduce that

$$h_\Phi(\mathbf{Cr}_\Phi^k) = \bigvee_{p\in\mathbf{Cr}_\Phi^k} h_\Phi(p).$$

Hence

$$\mathcal{P}_{X^{k+1},X^k}(t) = \sum_{p\in\mathbf{Cr}_\Phi^k} \mathbb{M}_{\Phi,p}(t).$$

Using [**36**, Remark 2.16] we deduce

$$\sum_k \mathcal{P}_{X^{k+1},X^k}(t) \succeq \mathcal{P}_X(t)$$

from which the Morse inequality follow immediately. □

Remark 9.13. Let $p \in \mathbf{Cr}_\Phi^k$. If $\mathcal{L}_p^-(\varepsilon) = \emptyset$ then $\mathbb{M}_{\Phi,p}(t) = 1$. Otherwise

$$h_\Phi(p) \simeq (C\mathcal{L}_p^-(\varepsilon), \mathcal{L}_p^-(\varepsilon)),$$

where CA denotes the cone on the topological space A. From the long exact sequence of the pair $(C\mathcal{L}_p^-(\varepsilon), \mathcal{L}_p^-(\varepsilon))$ we deduce that if $\mathcal{L}_P^-(\varepsilon) \neq \emptyset$ then

$$\dim H_0(C\mathcal{L}_p^-(\varepsilon), \mathcal{L}_p^-(\varepsilon)) = 0, \quad \dim H_1(C\mathcal{L}_p^-(\varepsilon), \mathcal{L}_p^-(\varepsilon)) = \dim H_0(\mathcal{L}_p^-(\varepsilon)) - 1$$

$$\dim H_{k+1}(C\mathcal{L}_p^-(\varepsilon), \mathcal{L}_p^-(\varepsilon)\,) = \dim H_k(\mathcal{L}_p^-(\varepsilon)\,), \ \ \forall k > 0.$$

If we denote by $\tilde{\mathbb{M}}_{\Phi,p}(t)$ the Poincaré polynomial of the reduced homology of $\mathcal{L}_p^-(\varepsilon)$ we deduce

$$\mathbb{M}_{\Phi,p}(t) = t\tilde{\mathbb{M}}_{\Phi,p}(t).$$

If we define for uniformity

$$\tilde{\mathbb{M}}_{\Phi,p}(t) = t^{-1}, \ \text{if} \ \ \mathcal{L}_p^-(\varepsilon) = \emptyset$$

then the previous equality holds in all the cases. We can rephrase the Morse inequalities as

$$\sum_{p \in \mathbf{Cr}_\Phi} t\tilde{\mathbb{M}}_{\Phi,p}(t) \succeq \mathcal{P}_X(t). \tag{9.3}$$

□

CHAPTER 10

Flips/flops and gradient like tame flows

The results in the previous section allow us to give a more detailed picture of the gradient like tame flows on compact tame sets. For any compact tame space X we denote by $\mathbf{Fl}_{\mathbf{grad}}(X)$ the set of gradient-like tame flows on X with finitely many stationary points.

Definition 10.1. (a) A *tame blowdown* is a continuous tame map $\beta : Y \to X$, such that X and Y are compact tame sets, and there exists a finite nonempty subset $L = L_\beta \subset X$ such that the induced map

$$\beta : Y \setminus \beta^{-1}(L_\beta) \to X \setminus L_\beta$$

is a homeomorphism. The set L_β is called the *blowup locus* of β. The compact set $\beta^{-1}(L_\beta)$ is called the *exceptional locus* of β and it is denoted by E_β. We will also say that Y is a *tame blowup* of X. A *weight* for the blowdown map β is a tame continuous function $w : X \to [0, \infty)$ such that $w^{-1}(0) = L_\beta$. We will refer to a pair (blowdown, weight) as a *weighted blowdown*.

(b) A *tame flop* is a diagram of the form

$$\begin{array}{ccc} & Y & \\ {\scriptstyle\beta_-}\swarrow & & \searrow{\scriptstyle\beta_+} \\ X_- & & X_+ \end{array}$$

where $\beta_\pm : Y \to X_\pm$ are tame blowdowns. The *connector* associated to the flop is obtained by gluing the mapping cylinder of β_- to the mapping cylinder of β_+ along Y using the identity map $\mathbb{1}_Y$. We will denote the connector by $(\stackrel{\beta_-}{\longleftarrow} Y \stackrel{\beta_+}{\longrightarrow})$.

(c) A *tame flip*[1] is a diagram

$$\begin{array}{ccc} Y_- & & Y_+ \\ {\scriptstyle\beta_-}\searrow & & \swarrow{\scriptstyle\beta_+} \\ & X & \end{array}$$

where $\beta_\pm : Y_\pm X$ are blowdown maps. The *connector* of the flip is the tame space obtained by gluing the mapping cylinder of β_- to the mapping cylinder of β_+ along X using the identity map $\mathbb{1}_X$. We will denote it by $(\stackrel{\beta_-}{\longrightarrow} X \stackrel{\beta_+}{\longleftarrow})$. □

Remark 10.2. In the above definition of a blowdown map $\beta : Y \to X$ we allow for the possibility that the exceptional locus E_β is empty. For example, the map

$$\{0\} \to \{0, 1\}, \quad 0 \mapsto 0,$$

[1]The "o" in flop indicates that the arrows arrow coming *out* of the middle of the diagram, while the "i" in flip indicate that the arrows are coming *into* the middle of the diagram.

is a blowdown map, with blowup locus $\{1\}$, and empty exceptional locus. □

Suppose that Φ is a gradient-like tame flow on a compact tame space X, and that the set $\mathbf{Cr}_\Phi$ of stationary points is finite. Fix a Lyapunov function $f : X \to \mathbb{R}$, and let

$$\{c_0 < c_1 < \cdots < c_\nu\}$$

be the set $f(\mathbf{Cr}_\Phi)$. For $i = 1, \dots, \nu$ we set $d_i := \frac{c_{i-1}+c_i}{2}$, and we define

$$Y_i = \{f = d_i\},$$

and

$$\mathbf{Cr}^j_\Phi := \Bigl\{x \in \mathbf{Cr}_\Phi;\ \ f(x) = c_j\Bigr\},\ \ X_j := \{f = c_j\},\ \ \forall j = 0, \dots, \nu.$$

For every point $x \in X$ we denote by $\Phi(x)$ the trajectory of Φ through x,

$$\Phi(x) := \bigl\{\Phi_t(x);\ \ t \in \mathbb{R}\bigr\}.$$

In the previous section we have proved that the flow defines tame blowdowns

$$\lambda_i = \lambda_i^\Phi : Y_{i+1} \to X_i,\ \ \lambda_i(y) = \Phi(y) \cap X_i,$$

$$\rho_i = \rho_i^\Phi : Y_i \to X_i, ,\ \ \rho_i(y) = \Phi(y) \cap X_i,$$

and $\mathbf{Cr}^i_\Phi = L_{\rho_i} = L_{\lambda_i}$. The exceptional loci are the (un)stable links. The space X is obtained via the attachments

$$\mathbf{Cyl}_{\lambda_0} \cup_{Y_1} (\stackrel{\rho_1}{\longrightarrow} X_1 \stackrel{\lambda_1}{\longleftarrow}) \cup_{Y_2} \cdots \cup_{Y_{n-2}} (\stackrel{\rho_{n-1}}{\longrightarrow} X_{n-1} \stackrel{\lambda_{n-1}}{\longleftarrow}) \cup_{Y_n} \mathbf{Cyl}_{\rho_n},$$

where $\mathbf{Cyl}_g$ denotes the mapping cylinder of a tame continuous map g.

The tame blowdowns λ_i and ρ_i carry natural weights. To define them we first need to define the tame maps

$$T_i^+ : X_i \to (0, \infty],\ \ T_i^- : X_i \to (0, \infty]$$

where for every $x \in X_i$, we denote by $T_i^+(x)$ the moment of time when the flow line through x intersects Y_i, and by $T_i^-(x)$ the moment of time when the backwards flow line trough x intersects Y_{i+1}. Equivalently,

$$T_i^+(x) = \sup\bigl\{\, t > 0;\ \ f(\Phi_t(x)) \geq d_i \,\bigr\},\ \ T_i^-(x) = \sup\bigl\{\, t > 0;\ \ f(\Phi_{-t}(x)\,) \leq d_{i+1} \,\bigr\}.$$

Observe that

$$T_i^\pm(x) = \infty \Longleftrightarrow x \in \mathbf{Cr}^i_\Phi\,.$$

In Section 9 we have proved that the tame functions $T_i^\pm$ are continuous. Now define

$$w_i^\pm := \frac{1}{T_i^\pm}.$$

The functions $w_i^\pm$ are continuous, nonnegative and

$$w_i^\pm(x) = 0 \Longleftrightarrow x \in \mathbf{Cr}^i_\Phi\,.$$

In other words, w_i^+ is a weight for ρ_i, and w_i^- is a weight for λ_i.

Definition 10.3. A *weighted chain* of tame flips is a sequences $\boldsymbol{\Xi}_\Phi = \boldsymbol{\Xi}_\Phi(\lambda_i, \rho_i, w_i^\pm)$ of flips

$$Y_{-1} \stackrel{\rho_0}{\longrightarrow} X_0 \stackrel{\lambda_0}{\longleftarrow} Y_1 \stackrel{\rho_1}{\longrightarrow} X_1 \stackrel{\lambda_1}{\longleftarrow} \cdots \stackrel{\rho_{n-1}}{\longrightarrow} X_{n-1} \stackrel{\lambda_{n-1}}{\longleftarrow} Y_n \stackrel{\rho_n}{\longrightarrow} X_n \stackrel{\lambda_n}{\longleftarrow} Y_{n+1},$$

and weights w_i^+ for ρ_i, w_i^- for λ_i such that X_0 and X_n are finite sets, $Y_{-1} = Y_{n+1} = \emptyset$, and $L_{\rho_i} = L_{\lambda_i}$, $\forall i$. The tame space associated to a weighted chain is defined as

$$\mathbf{Cyl}_{\lambda_0} \cup_{Y_1} \mathbf{Cyl}_{\rho_1} \cup_{X_1} \cup \mathbf{Cyl}_{\lambda_1} \cup_{Y_2} \cdots \cup_{Y_n} \mathbf{Cyl}_{\rho_n} X_n.$$

We denote by $\mathcal{C}_w$ the set of weighted tame chains and, for every compact tame set X, we denote by $\mathcal{C}_w(X)$ the set of weighted chains whose associated space is X. □

The discussion preceding the above definition shows that we have a natural map

$$\mathbf{\Xi} : \mathbf{Fl_{grad}}(X) \to \mathcal{C}_w(X), \quad \Phi \longmapsto \mathbf{\Xi}_\Phi.$$

Under this map, the stationary points of Φ correspond bijectively to the points in the blowup loci $L_{\rho_i} = L_{\lambda_i}$. The exceptional loci of ρ_i correspond to unstable links of stationary points, while the exceptional loci of λ_i correspond to the stable links.

THEOREM 10.4. *The map* $\mathbf{\Xi} : \mathbf{Fl_{grad}}(X) \to \mathcal{C}_w(X)$ *is surjective.*

PROOF. The strategy is simple: we will construct a right inverse for $\mathbf{\Xi}$. More precisely, given a weighted chains of flips $\Xi(\lambda_i, \rho_i, w_i^{\pm}, 0 \le i \le n) \in \mathcal{C}_w(X)$ we will construct local flows and Lyapunov functions on the various mapping cylinders associated to this chain, and then we concatenate them. It suffices to do this for a single blowdown map $\beta : Y \to X$, with weight w.

For us, a local tame flow on a tame set S will be a tame continuous map $\Psi : \mathbb{R}_S \to S$ where $\mathbb{R}_S \subset \mathbb{R} \times S$ is a tame subset such that

- $\{0\} \times S \subset \mathbb{R}_S$,
- for every $s \in S$, the set $I_s := \{t \in \mathbb{R};\ \ (t, s) \in \mathbb{R}_S\}$ is an interval of positive length, and
- for every $s \in S$ and $t_0, t_1 \in I_s$ such that $t_0 + t_1 \in I_s$ we have

$$\Phi_{t_0+t_1}(s) = \Phi_{t_0}\big(\Phi_{t_1}(s)\big).$$

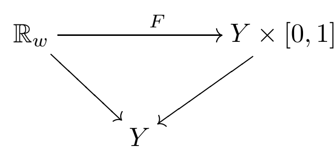
Define

$$T : Y \to (0, \infty], \ \ T(y) = \begin{cases} \frac{1}{w(y)} & w(y) \neq 0 \\ \infty & w(y) = 0 \end{cases},$$

and set

$$\mathbb{R}_w := \big\{(y, t) \in Y \times [0, \infty];\ \ t \le T(y)\big\}.$$

Fix a tame homeomorphism $F : \mathbb{R}_w \to Y \times [0, 1]$ such that, $F(y, 0) = (y, 0)$, and the diagram below is commutative



where the maps $\mathbb{R}_w, Y \times [0, 1] \to Y$ are the natural projections (see Figure 10).

Consider the translation flow on $Y \times [-\infty, \infty]$, whose stationary points are $(y, \pm\infty)$, $y \in Y$. It restricts to a (local) flow on $\mathbb{R}_w$ whose trajectories are the vertical lines $[0, T(y)] \ni t \mapsto (y, t)$. Via F we obtain a tame local flow on $Y \times [0, 1]$, whose orbits are the vertical segments $\{0\} \times [0, 1]$. The bottom point $(y, 0)$ will reach the top point $(y, 1)$ in $T(y)$ units of time.

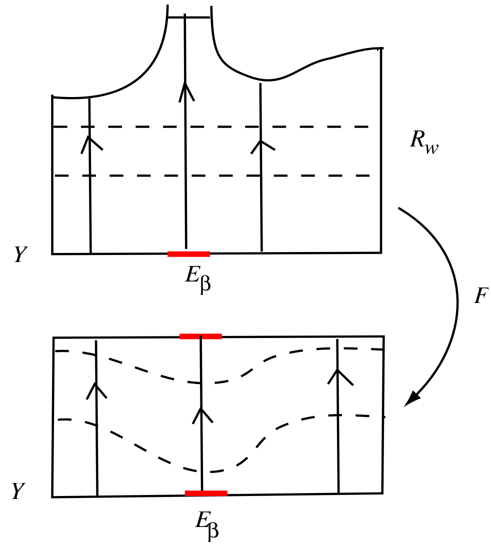


FIGURE 10. *Constructing a gradient flow on the mapping cylinder of a tame map.*

The natural map $Y \times [0,1] \to [0,1]$ decreases strictly along the trajectories of this local flow, and thus defines a Lyapunov function. The points in $E_\beta \times \{1\}$ are stationary points. This local flow descends to a local flow on

$$\mathbf{Cyl}_\beta = Y \times [0,1] \cup_\beta X,$$

where the points in the singular locus $L_\beta \subset X$ are stationary points. □

REMARK 10.5. To transform the above theorem into a useful technique for producing gradient like flows, we need to explain how to construct weighted blow-down/up maps.

Note that given a compact tame space Y and $E \subset Y$ a compact tame subset, then X/E is a compact tame space, and the natural projection $Y \to Y/E$ is a blowdown map. We would like to investigate the opposite process.

Suppose we are given a compact tame space X, a point $p_0 \in X$, and a continuous tame function $w : X \to [0, \infty)$ such that $w^{-1}(0) = \{x_0\}$.

We can then find $r_0 > 0$ such that the induced map

$$w : \{\, 0 < w < r_0\} \to (0, r_0]$$

is a (tamely) locally trivial fibration. The level sets $\{w = \varepsilon\}$, $\varepsilon \in (0, r_0)$ are all (tamely) homeomorphic. We will refer to any one of them as the w-link[2] of p_0, and we will denote it by $\mathcal{L}_w(p_0)$.

Observe that the closed neighborhood $\{w \le r_0\}$ of p_0 is tamely homeomorphic to the cone on $\mathcal{L}_w$, or equivalently, the mapping cylinder of the constant map $\mathcal{L}_w \to \{p_0\}$.

Consider now an arbitrary, tame continuous map $\mu : \mathcal{L}_w \to E$, where E is a tame compact set. Observe that the canonical map from the mapping cylinder of

[2]We do not know if the homeomorphism type of the w-link depends on the weight w, or that it is homeomorphic to the link of the point x_0 in X as defined in Appendix B.

μ to the mapping cylinder of the constant map $\mathcal{L}_w \to \{p_0\}$ is a blowdown map $\mathbf{Cyl}_\mu \to \{w \le r_0\}$ with blowup locus $\{x_0\}$, and exceptional locus E. We can now define the blowup space $\widehat{X}_{w,\mu}$ to be

$$\widehat{X}_{w,\mu} = \{w \ge r_0\} \cup_{\mathcal{L}_w} \mathbf{Cyl}_\mu . \qquad \square$$

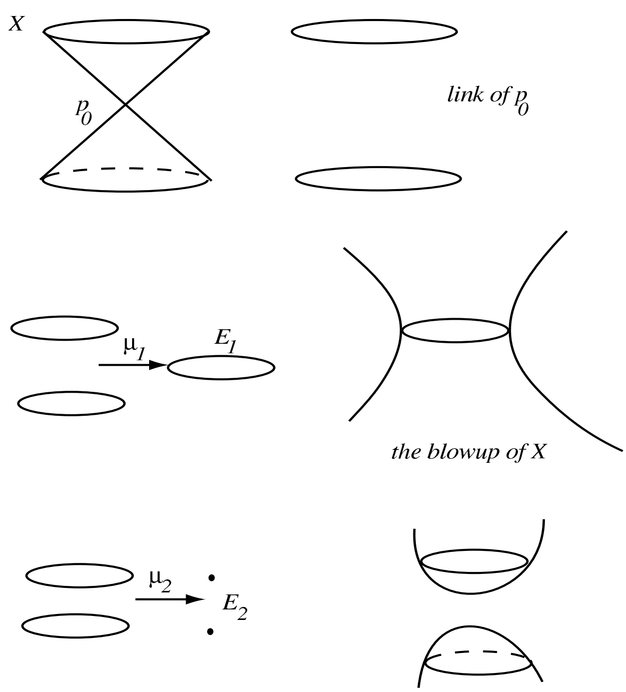


FIGURE 11. *Blowing up the vertex of a cone in two different ways.*

EXAMPLE 10.6. (a) Suppose X is the Euclidean space $\mathbb{R}^n$, p_0 is the origin, and w denotes the Euclidean norm. The w-link of p_0 is the round sphere S^{n-1}. If we denote by $\mu : S^{n-1} \to \mathbb{RP}^{n-1}$ the canonical double covering, then the blowup $\widehat{X}_{\mu,w}$ is the usual blowup in algebraic geometry.

(b) Suppose X is the semialgebraic cone (see Figure 11)

$$X = \{ (x, y, z) \in \mathbb{R}^3; \;\; z^2 = x^2 + y^2, \;\; |z| \le 1 \}$$

and p_0 is the origin. Assume $w(x, y, z) = |z|$. Then the link of p_0 consists of two circles.

We can choose μ in many different ways. For example, we can choose $\mu = \mu_1 : S^1 \sqcup S^1 \to S^1$ to be the natural identification map, or we can choose $\mu = \mu_2 : S^1 \sqcup S^1 \to \{0, 1\}$ to be the map which collapses each of the two circles to a different point. The resulting blowup spaces $\widehat{X}_{w,\mu}$ are depicted in Figure 11.

These types of blowups appear in Morse theory, when we cross a level set of a 3-dimensional Morse function containing saddle point. $\square$

CHAPTER 11

Simplicial flows and combinatorial Morse theory

In this section we want to apply our Conley index computations to investigate a special class of tame flows on triangulated tame spaces. It will turn out that Forman's combinatorial Morse theory is a special case of this investigation.

We define a *simplicial scheme* (or *simplicial set*) to be *finite* collection $\mathcal{K}$ of nonempty finite sets with the property that

$$A \in \mathcal{K},\ \ B \subset A \Longrightarrow B \in \mathcal{K}.$$

The sets in $\mathcal{K}$ are called the *open faces* of K. The union of all the sets in K is called the vertex set of K and will be denoted by $\mathcal{V}(\mathcal{K})$. The dimension of an open face $A \in \mathcal{K}$ is the nonnegative integer

$$\dim A := \#A - 1.$$

We set

$$\dim K := \max\big\{ \dim A;\ \ A \in \mathcal{K} \big\}.$$

A vertex is a 0-dimensional face.

For every subset $\mathcal{A} \subset \mathcal{K}$ we define its *combinatorial closure* to be

$$\boldsymbol{cl}_c(\mathcal{A}) = \big\{ B \in \mathcal{K};\ \ \exists A \in \mathcal{A}:\ \ B \subset A, \big\}.$$

A *subscheme* of $\mathcal{K}$ is a subset $\mathcal{A} \subset \mathcal{K}$ such that $\mathcal{A} = \boldsymbol{cl}_c(\mathcal{A})$. The *$\ell$-th skeleton* of $\mathcal{K}$ is the subscheme

$$\mathcal{K}_\ell = \big\{ A \in K;\ \ \dim A \leq \ell \big\}.$$

For any subset $S \subset \mathcal{V}(\mathcal{K})$ we denote by $\mathcal{F}(S)$ the subscheme of $\mathcal{K}$ spanned by the faces with vertices in S,

$$\mathcal{F}(S) := \big\{ A \in \mathcal{K};\ \ A \subset S \big\}.$$

For any vertex v of $\mathcal{K}$, we denote by $L(v) = L(v, \mathcal{K})$ the set of vertices adjacent to c in $\mathcal{K}$, and we set

$$S(v) = S(v, \mathcal{K}) := \{v\} \cup L_v.$$

The *combinatorial star* of v in $\mathcal{K}$ is then the subscheme

$$\mathcal{S}(v) = \mathcal{S}(v, \mathcal{K}) := \mathcal{F}(S(v)),$$

while the *combinatorial link* of v in $\mathcal{K}$ is the subscheme

$$\mathcal{L}(v) = \mathcal{L}(v, \mathcal{K}) := \mathcal{F}(L(v)).$$

For every finite set S we denote by $\mathbb{R}^S$ the vector space of functions $S \to \mathbb{R}$. $\mathbb{R}^S$ has a canonical basis consisting of the Dirac functions $(\delta_s)_{s\in S}$, where

$$\delta_s(s') = \begin{cases} 1 & s' = s \\ 0 & s' \neq s. \end{cases}$$

For any subset $A \subset S$ we denote by $\Delta(A)$ the convex hull of the set

$$\{\delta_a;\ \ a \in A\} \subset \mathbb{R}^S.$$

If $\mathcal{K}$ is a simplicial scheme, then the *geometric realization* of $\mathcal{K}$ is the closed subset

$$|\mathcal{K}| = \bigcup_{A \in \mathcal{K}} \Delta(A) \subset \mathbb{R}^{\mathcal{V}(\mathcal{K})}.$$

The sets $\Delta(A)$ are called the *faces* of the geometric realization. Observe that $\Delta(A)$ is an affine simplex of dimension $\dim A$. We denote by $\mathbf{St}(v)$ the geometric realization of $\mathcal{S}(v)$ and by $\mathbf{Lk}(v)$ the geometric realization of $\mathcal{L}(v)$.

EXAMPLE 11.1. (a) Suppose $(P, \leq)$ is a finite poset (partially ordered set). Then the *nerve* of $(P, \leq)$ is the simplicial scheme $\mathcal{N}(P, \leq)$, with vertex set P, and open faces given by the chains of P, i.e., the linearly ordered subsets of P. For any poset P we will denote by $|P|$ the geometric realization of its nerve

$$|P| := |\mathcal{N}(P)|.$$

We say that two posets are homeomorphic or homotopic if the geometric realizations of their nerves are such.
(b) Suppose $\mathcal{K}$ is a simplicial scheme. Then $\mathcal{K}$ is a finite poset, where the order relation is given by inclusion. The nerve of $(\mathcal{K}, \subset)$ is called the *first barycentric subdivision of* $\mathcal{K}$, and it is denoted by $D\mathcal{K}$. We define inductively

$$D^{n+1}\mathcal{K} := D(D^n\mathcal{K})$$

We say that $D^n\mathcal{K}$ is the n-th barycentric subdivision of $\mathcal{K}$.
(c) Suppose $\mathcal{K}_1$ and $\mathcal{K}_2$ are two simplicial schemes with disjoint vertex sets $\mathcal{V}_1, \mathcal{V}_2$. We define the *join* of $\mathcal{K}_1$ and $\mathcal{K}_2$ to be the simplicial scheme $\mathcal{K}_1 * \mathcal{K}_2$ with vertex set $\mathcal{V}_1 \cup \mathcal{V}_2$, and faces $F_1 \cup F_2$, $F_i \in \mathcal{K}_i$. The join of a simplicial scheme and a point which is not a vertex of $\mathcal{K}$ is called the cone on $\mathcal{K}$ and it is denoted be $\mathrm{Cone}\,(\mathcal{K})$.
(d) If $\mathcal{K}$ is a simplicial scheme, then the suspension of $\mathcal{K}$ is the simplicial scheme $\Sigma\mathcal{K}$ defined as the join of $\mathcal{K}$ with the simplicial scheme $S^0 = \{\{N\}, \{S\}\}$, where $N, S \notin \mathcal{V}(\mathcal{K})$. The n-th iterated suspension of $\mathcal{K}$ is defined inductively as

$$\Sigma^n\mathcal{K} := \Sigma(\Sigma^{n-1}\mathcal{K}).$$

□

If $\mathcal{K}_0$ and $\mathcal{K}_1$ are two simplicial schemes, then a *simplicial map* from $\mathcal{K}_0$ to $\mathcal{K}_1$ is a map

$$f : \mathcal{V}(\mathcal{K}_0) \to \mathcal{V}(\mathcal{K}_1),$$

such that

$$A \in \mathcal{K}_0 \Longrightarrow f(A) \in \mathcal{K}_1.$$

A morphism $f : \mathcal{K}_0 \to \mathcal{K}_1$ induces a morphism $Df : S\mathcal{K}_1 \to S\mathcal{K}_1$ between the first barycentric subdivisions, and a continuous, piecewise linear map $f_\sharp : |\mathcal{K}_0| \to |\mathcal{K}_1|$.

DEFINITION 11.2. (a) A *dynamical orientation* on the simplicial scheme $\mathcal{K}$ is a binary relation $\rightsquigarrow$ on $\mathcal{V}(\mathcal{K})$ with the following properties.

- If $u \rightsquigarrow v$ then $\{u, v\}$ is a one dimensional face of $\mathcal{K}$.
- For any open face $A \in \mathcal{K}$, the restriction of $\rightsquigarrow$ to A is a linear order.

(b) A *combinatorial flow* is a pair $(\mathcal{K}, \rightsquigarrow)$, where $\mathcal{K}$ is a simplicial scheme and $\rightsquigarrow$ is a dynamical orientation on $\mathcal{K}$. □

If $(\mathcal{K}, \rightsquigarrow)$ is a combinatorial flow, and $p \in \mathcal{V}(K)$ then we set

$$L(p \rightsquigarrow) := \big\{ u \in \mathcal{V}(\mathcal{K});\;\; p \rightsquigarrow u \big\},\;\; \mathcal{L}(p, \rightsquigarrow) := \mathcal{F}(\, L(p \rightsquigarrow)\,),$$

$$W(p \rightsquigarrow) := L(p \rightsquigarrow) \cup \{p\},\;\; \mathcal{W}(p \rightsquigarrow) := \mathcal{F}(W(p \rightsquigarrow)\,).$$

The sets $W(\rightsquigarrow p)$, $L(\rightsquigarrow p)$ etc., are defined in a similar fashion. We will say that $\mathcal{L}(p \rightsquigarrow)$ is the *unstable combinatorial link.*

Using the argument in the proof of Proposition 2.11 based on the canonical tame flow on an affine simplex described in Example 2.10 we can associate to any combinatorial flow $(\mathcal{K}, \rightsquigarrow)$ a tame flow $\Phi = \Phi^{\rightsquigarrow}$ on the geometric realization $|\mathcal{K}|$. The faces of the geometric realization are invariant subsets of the flow. Moreover, if $u \rightsquigarrow v$, then along the edge $[u, v]$, the flow runs from u to v. We will say that $\Phi^{\rightsquigarrow}$ is the *simplicial flow* determined by the dynamical orientation $\rightsquigarrow$.

THEOREM 11.3. *Suppose* $(\mathcal{K}, \rightsquigarrow)$ *is a combinatorial flow, and* Φ *is the simplicial flow on* $|\mathcal{K}|$ *associated to* $\rightsquigarrow$. *Then the following hold.*
(a) The map

$$\mathcal{V}(\mathcal{K}) \ni v \mapsto \delta_v \in |\mathcal{K}|$$

is a bijection from the vertex set of $\mathcal{K}$ *to the set of stationary points of* Φ.
(b) For every vertex v *of* $\mathcal{K}$, *the Conley index of* $\delta_v \in |\mathcal{K}|$ *is homotopy equivalent to the pointed space*

$$|\mathrm{Cone}\,(\mathcal{L}(v \rightsquigarrow)|/|\mathcal{L}(v \rightsquigarrow)|.$$

PROOF. Part (a) is obvious. To prove (b) observe that the star $\mathbf{St}(v)$ is a compact, flow invariant neighborhood of δ_v. Thus, the Conley index of δ_v in $|\mathcal{K}|$ is homotopy equivalent to the Conley index of δ_v in $\mathbf{St}(v)$.

Observe that we have a partition

$$S(v) = \{p\} \sqcup \mathcal{L}(\rightsquigarrow v) \sqcup \mathcal{L}(v \rightsquigarrow).$$

Now define

$$f : S(v) \to \{-1, 0, 1\}$$

by setting

$$f(u) := \begin{cases} 0 & u = v \\ 1 & u \in \mathcal{L}(\rightsquigarrow v) \\ -1 & u \in \mathcal{L}(v \rightsquigarrow). \end{cases}$$

The function f induces a piecewise linear function $\mathbf{St}(v) \to [-1, 1]$ which, for simplicity, we continue to denote by f.

From the explicit description in Example 2.10 of the canonical tame flow on an affine simplex we deduce that δ_v is a Morse like stationary point of the flow Φ on $\mathbf{St}(v)$, and f is a tame local Lyapunov function adapted to δ_v. The result now follows from Theorem 9.10. □

EXAMPLE 11.4. The cheapest way of producing a dynamical orientation on a simplicial scheme $\mathcal{K}$ is to choose an injection

$$f : \mathcal{V}(\mathcal{K}) \to \mathbb{R}.$$

Then we define

$$x \overset{f}{\rightsquigarrow} y \Longleftrightarrow f(x) > f(y),\;\; \{x, y\} \in \mathcal{K}.$$

Then f defines a piecewise linear function $f : |\mathcal{K}| \to \mathbb{R}$ which is a tame Lyapunov function for the simplicial flow determined by $\overset{f}{\rightsquigarrow}$.

Alternatively, the restriction of a generic linear map $f : \mathbb{R}^{\mathcal{V}(K)} \to \mathbb{R}$ to the affine realization $|\mathcal{K}|$ is injective on the vertex set. This function is a stratified Morse function in the sense of Goresky-Macpherson, and in this case, the Conley index computations also follow from the computaions in [**17**] of the local Morse data of a stratified Morse function. □

Let us present a few applications of this result to the homotopy theory of posets. We need to introduce some terminology

Suppose $(P, \leq)$ is a finite poset. Recall that for any $x, y \in P$ we define the order intervals

$$[x, y] := \{\, z \in P;\ \ x \leq z \leq y \,\},\ \ (x, y) = \{ z \in P;\ \ x < z < y \,\},$$

and we say that y *covers* x if $[x, y] = \{x, y\}$. We write this $y \gtrdot x$. We define

$$P_{<x} := \{ x \in P;\ \ x < y \,\}.$$

An *order ideal* of P is a subset $I \subset P$ such that

$$x \in I \Longrightarrow P_{\leq x} \subset I.$$

For every chain $x_0 < x_1 < \cdots < x_k$ in P, we will refer to the integer k as the length of the chain. Given $x \leq y$, we define $\ell(x, y)$ the be the maximal length of a chain originating at x and ending at y. Observe that

$$x \lessdot y \Longleftrightarrow \ell(x, y) = 1.$$

Finally, we will say that a poset is contractible, if it is homotopic to the poset consisting of a single point.

A map between two posets $F : (P, <_P) \to (Q, <_Q)$ is called *isotone* if

$$x \leq_P y \Longrightarrow F(x) \leq_Q F(y).$$

Note that an isotone map induces a simplicial map between the nerve of P and the nerve of Q.

A function $f : P \to \mathbb{R}$ on a poset P is called *admissible* if

$$f(x) = f(y) \Longrightarrow \ \ x \text{ and } y \text{ are not comparable.}$$

Suppose $f : P \to \mathbb{R}$ is admissible. For every $x \in P$ we set

$$V^+(x) = V^+(x, f) := \{\, y > x;\ \ f(x) > f(y) \,\},\ \ \mathcal{S}^+(x) = \mathcal{S}^+(x, f) := \{x\} \cup V^+(x),$$

$$V^-(x) = V^-(x, f) = \{ z < x;\ \ f(z) > f(x) \,\},\ \ \mathcal{S}^-(x) = \mathcal{S}^-(x, f) := \{x\} \cup V^-(x).$$

REMARK 11.5. Here is the intuition behind the sets $V^{\pm}(x, f)$. Note that these sets are empty for every $x \in P$ if and only if f is a strictly increasing function. In other words, the sets $V^{\pm}(x)$ collect the "*violations*" at x of the strictly increasing condition. □

The admissible function f defines a partial order

$$x \prec_f y \Longleftrightarrow f(x) > f(y) \ \text{ and } \ x < y,$$

so that

$$V^-(x) = \{ y \in P;\ \ y \prec_f x \,\},\ \ V^+(x) = \{ z \in P;\ \ x \prec_f z \,\}.$$

If $f : P \to \mathbb{R}$ is an admissible function, then we have a simplicial flow Φ^f on the nerve of P given by the dynamical orientation

$$x \overset{f}{\rightsquigarrow} y \Longleftrightarrow f(x) > f(y) \text{ and } x \text{ and } y \text{ are comparable elements of } P.$$

The function f induces a piecewise linear Lyapunov function of this flow. Every point $x \in P$ is a stationary point of this flow. We denote by $h_f(x)$ its Conley index.

The unstable combinatorial link $\mathcal{L}(x \overset{f}{\rightsquigarrow})$ of x is the nerve of the poset

$$V^+(x) \cup (P_{<x} \setminus V^-(x)),$$

which is the join

$$\mathcal{N}\big(V^+(x)\big) * \mathcal{N}\big(P_{<x} \setminus V^-(x)\big).$$

Above we use the convention that

$$\emptyset * Y = Y, \text{ for any topological space } Y.$$

DEFINITION 11.6. Suppose f is a real valued admissible function on the poset P. A point $x \in P$ is called a *regular* point of f if one of the posets $V^+(x)$ or $P_{<x} \setminus V^-(x)$ is contractible. Otherwise the vertex x is called a *critical* point of f. We denote by $\mathbf{Cr}_f$ the set of critical points of f. □

COROLLARY 11.7. *If x is a regular point of f then its Conley index is trivial.* □

DEFINITION 11.8. Suppose $f : P \to \mathbb{R}$ is a real valued admissible function on a poset P.

(a) The order of f is the nonnegative integer

$$\omega(f) := \max\big\{\ell(x,y);\ \ x \leq y \ \text{ and } \ f(x) \geq f(y)\,\big\}.$$

(b) We say that f is *coherent* if

$$x \preccurlyeq_f y \Longrightarrow \ f \text{ is strictly decreasing on the interval } [x,y].$$

In other words, if $x < z < y$ and $f(x) > f(y)$ then $f(x) > f(z) > f(y)$.

(b) We say that f satisfies the condition $\boldsymbol{\mu}_+$ if there exists a map

$$C_+ = C_+^f : P \to P$$

such that $C_+(x)$ is the unique maximal element of $\mathcal{S}^+(x)$. In particular $x \leq C_+(x)$. The map C_+ is called the *upper projector* associated to f.

(c) We say that f satisfies the condition $\boldsymbol{\mu}_-$ if there exists a map $C_- = C_-^f : P \to P$ such that $C_-(x)$ is the unique minimal element of $\mathcal{S}^-(x)$. The map C_- is called the *lower projector* associated to the $\boldsymbol{\mu}_-$-function f.

(d) We say that f satisfies the condition $\boldsymbol{\mu}$ if it satisfies both $\boldsymbol{\mu}_+$ and $\boldsymbol{\mu}_-$. A *Morse-Forman* function is an admissible function of order ≤ 1 satisfying the condition $\boldsymbol{\mu}$.

□

EXAMPLE 11.9. (a) Any strictly decreasing function on a finite poset P is a coherent function of order zero.

(b) Suppose $\mathcal{K}$ is a simplicial scheme with vertex set V, i.e., an ideal of the poset 2^V_* of nonempty subsets of V. Then a discrete Morse function $f : \mathcal{K} \to \mathbb{R}$ of the

type introduced by R. Forman in [**14**] is a Morse-Forman function on the poset of faces of a simplicial scheme.
(c) If $f : P \to \mathbb{R}$ satisfies $\boldsymbol{\mu}_-$, and $I \subset P$ is an ideal, then $f|_I$ satisfies $\boldsymbol{\mu}_-$.
(d) In Figure 12 we have depicted a coherent function of order two on the poset of faces of the two dimensional simplex. The arrows indicate the dynamical orientation determined by this function. This function also satisfies condition $\boldsymbol{\mu}$.

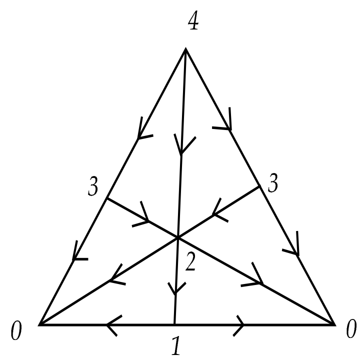


FIGURE 12. *A coherent function of order* 2.

□

COROLLARY 11.10. *If $f : P \to \mathbb{R}$ is a function satisfying condition $\boldsymbol{\mu}_+$, then any point $x \in P$ such that $x \neq C_+(x)$ is a regular point.*

PROOF. If $x \neq C_\pm(x)$, then the nerve of $V_+(x)$ is a cone with vertex $C_+(x)$, hence contractible. □

Observe that the Conley indices of the critical point of a $\boldsymbol{\mu}_-$-function do not depend on the function but only on the projector C_- associated to it. We want to investigate a few properties of this projector.

Suppose $f : P \to \mathbb{R}$ satisfies the condition $\boldsymbol{\mu}_-$, and let $C_- : P \to P$ be the associated projector. The map C_- is an idempotent, i.e.,

$$C_- \circ C_- = C_-.$$

We denote by Fix_{C_-} the set of fixed points of C_-, and we regard C_- as a map $P \to \mathrm{Fix}_{C_-}$. Each fiber of this map contains a unique minimal element. The function f is strictly decreasing on each fiber, and if $x < y$ and $C_-(x) \neq C_-(y)$ then $f(x) < f(y)$.

Denote by $[f]$ the restriction of f to Fix_{C_-}. We define a binary relation $\to$ on Fix_{C_-} by declaring $x \to y$ if and only if $x \neq y$ and there exist $x', y' \in P$ such that $x' < y'$, $C_-(x') = x$, $C_-(y') = y$.

LEMMA 11.11. *If $x, y \in \mathrm{Fix}_{C_-}$, and $x \to y$, then $[f](x) < [f](y)$.*

PROOF. There exists $x', y' \in P$ such that

$$x = C_-(x') \leq x' < y' \geq C_-(y') = y$$

Since $x < y'$, and $C_-(x) \neq C_-(y')$ we deduce

$$[f](x) = f(x) < f(y') \leq f(C_-(y')) = [f](y).$$ □

We denote by $\prec$, or $\prec_f$, the transitive extension of the binary relation $\to$ on Fix_{C_-}. From Lemma 11.11 we deduce that $\prec_f$ is a partial order on Fix_{C_-}, the natural projection $C_- : P \to \mathrm{Fix}_{C_-}$ is isotone, and the function $[f]$ is strictly increasing with respect to this order $\prec_f$.

The partial order $\prec$ on Fix_{C_-} can be given a more explicit description. More precisely $x \prec y$, $x, y \in \mathrm{Fix}_{C_-}$, if and only if, $x \neq y$ and there exists a sequence $x_0, x_1, x_2, \dots, x_n \in P$ such that $x_0 = x$, $C_-(x_n) = y$, and

$$x_0 < x_1 \geq C_-(x_1) < x_2 \geq C_-(x_2) < \ldots \geq C_-(x_{n-1}) < x_n \geq C_-(x_n) = y.$$

The next existence result generalizes a result of M. Chari [**5**] in the context of Morse-Forman functions, and shows that the above process can be reversed.

PROPOSITION 11.12. *Suppose $(P, <)$ and $(Q, \prec)$ are finite posets and $\pi : P \to Q$ is an isotone map such that every fiber of π contains a unique minimal element. Then for every injective increasing function $[f] : Q \to \mathbb{R}$ there exists an injective function $f : P \to \mathbb{R}$ with the following properties.*

(a) *The function f is decreasing on the fibers of π.*
(b) $\max_{x \in \pi^{-1}(\alpha)} f(x) = [f](\alpha)$, *$\forall \alpha \in \pi(P)$.*
(c) *If $x < y$ and $\pi(x) \neq \pi(y)$, then $f(x) < f(y)$.*

The function f satisfies the condition $\boldsymbol{\mu}_-$, and if C_- denotes the lower projector associated to f then the induced map

$$\pi : (\mathrm{Fix}_{C_-}, \prec_f) \to (Q, \prec)$$

is an isotone injection.

PROOF. For every $\alpha \in Q$ we denote by $\alpha_- \in P$ the unique minimal element in the fiber $\pi^{-1}(\alpha)$, and for every $x \in P$ we set $x_- := \pi(x)_-$, i.e., x_- the unique minimal element in the fiber of π containing x.

Suppose $[f] : Q \to \mathbb{R}$ is an injective increasing function. For $\alpha \in Q$ we set $r_\alpha := [f](\alpha)$, and we choose open intervals I_α containing r_α such that

$$\alpha \neq \beta \Longrightarrow I_\alpha \cap I_\beta = \emptyset.$$

Such a choice is possible since $[f]$ was chosen to be injective.

For every $\alpha \in Q$ we construct a strictly decreasing injective function $f_\alpha : \pi^{-1}(\alpha) \to I_\alpha$ such that $f_\alpha(\alpha_-) = r_\alpha$. Now define $f : P \to \mathbb{R}$ by setting $f(x) := f_{\pi(x)}(x)$. By construction, f is strictly decreasing on each equivalence class. Moreover, if $x < y$ and $\pi(x) \neq \pi(y)$ then several things happen.

- $\pi(x) \prec \pi(y)$.
- $r_{\pi(x)} = [f](\pi(x))) < [f](\pi(y)) = r_{\pi(y)}$.
- $f(x) \in I_{\pi(x)}$, $\forall x \in P$.

On the other hand, the intervals $I_{\pi(x)}$ and $I_{\pi(y)}$ are disjoint, and since $r_{\pi(x)} < r_{\pi(y)}$, we deduce that any number in $I_{\pi(x)}$ is smaller than any number in $I_{\pi(y)}$ so that $f(x) < f(y)$. This shows that f satisfies all the required properties. □

DEFINITION 11.13. Suppose $(P, <)$ and $(Q, \prec)$ are finite posets and $\pi : P \to Q$ is an isotone map.

(a) The π is called *lower acyclic* if every nonempty fiber of π has a unique minimal element.

(b) The map π is called *coherent* if

$$x < y \text{ and } \pi(x) = \pi(y) \Longrightarrow \pi(x) = \pi(z), \ \forall z \in [x, y].$$

(c) A function $f : P \to \mathbb{R}$ is called *compatible* with π if it is strictly decreasing on the fibers of π, and if $x < y$ and $\pi(x) \neq \pi(y)$ then $f(x) < f(y)$. We denote by $F_\pi(P)$ the set of functions compatible with π. □

Denote by $\mathcal{M}_P$ the set of lower acyclic and isotone maps $(P, <) \to (Q, \prec)$, and by $\mathcal{M}_P^c$ the subset consisting of the coherent ones. Note that $\mathcal{M}_P^c \neq \emptyset$ because $\mathbb{1}_P \in \mathcal{M}_P$. In this case the set $F_{\mathbb{1}_P}(P)$ consists of the strictly increasing functions on P.

Proposition 11.12 can be rephrased as saying that

$$\pi \in \mathcal{M}_P \Longrightarrow F_\pi(P) \neq \emptyset.$$

Moreover, if f is a $\boldsymbol{\mu}_-$-function then the associated projector C_- is lower acyclic and isotone with respect to the order $\prec_f$ on Fix_{C_-}, and $f \in F_{C_-}(P)$. We have obtained the following result generalizing [**29**, Thm. 11.2, 11.4].

Corollary 11.14. *A function $f : P \to \mathbb{R}$ satisfies property $\boldsymbol{\mu}_-$ if and only if there exists a map $\pi \in \mathcal{M}_P$ such that $f \in F_\pi(P)$. Moreover, for a fixed map $\pi \in \mathcal{M}_P$, and any $f, g \in F_\pi(P)$ we have and equality of simplicial flows, $\Phi^f = \Phi^g$. In particular*

$$h_f(x) = h_g(x), \ \forall x \in P,$$

where $h_f(x)$ denotes the Conley index of the stationary point x of the tame flow induced by f. □

Observe that if $\pi : (P, <) \to (Q, \prec)$ is a lower acyclic and isotone map, and $g : (Q, \prec) \to \mathbb{R}$ is an injective increasing map, then $g \circ p$ is a lower acyclic and isotone map from P to a finite linearly ordered set. Moreover,

$$F_\pi(P) = F_{g\circ\pi}(P).$$

Thus, to produce functions $f : P \to \mathbb{R}$ satisfying the condition $\boldsymbol{\mu}_-$ it suffices to produce isotone maps $f : P \to \mathbb{R}$ such that for every $r \in f(P)$ the fiber $f^{-1}(r)$ contains a unique minimal element.

The condition $\boldsymbol{\mu}_-$ and the coherence condition are particularly useful for a special class of posets, namely the posets of faces of a regular CW decomposition of a space.

In the remainder of this section, we will assume that P is the poset $\mathcal{F}(X)$ of faces of a regular CW-decomposition of a compact space X. All the functions will be assumed coherent

Observe that the intersection of two faces is either empty, or a face of X, i.e., $\mathcal{F}(X)$ is a meet semilattice. By [**34**, Thm. III.1.7], geometric realization of the nerve of the poset $\mathcal{F}(X)$ is PL homeomorphic to X. In particular, if $F \in \mathcal{F}(X)$ is a closed face, then $\mathcal{F}(X)_{<F}$ is the union of all the proper faces of F so that the geometric realization of the nerve of $\mathcal{F}(X)_{<F}$ is PL homeomorphic to the PL space $\partial F \cong S^{\dim F - 1}$. Similarly, the geometric realization of $\mathcal{F}(X)_{\leq F}$ is PL homeomorphic to the closed ball $F \cong \mathbb{D}^{\dim F}$ equipped with its the natural PL structure.

Suppose $f : \mathcal{F}(X) \to \mathbb{R}$ is a coherent function. For any face F we denote by $V^+_{max}(F)$ the maximal elements in $V^+(F)$. Since f is coherent, we deduce

$$V^+(F) = \bigcup_{T \in V^+_{max}(F)} (F, T].$$

If $V^+_{max}(F) = \emptyset$, we define $M^+(F) = \emptyset$.

If $V^+_{max}(F) \neq \emptyset$, we define $M^+(F)$ to be the simplicial scheme with vertex set $V^+_{max}(F)$ such that $\{T_1, \dots, T_k\} \subset V^+_{max}(F)$ is a face if and only if $T_1 \cap \cdots \cap T_k \neq \emptyset$. In other words, $M^+(F)$ is the nerve of the cover $\bigcup_{T \in V^+_{max}(F)} (F, T]$. Observe that

$$(F, T_1] \cap (F, T_2] = \begin{cases} \emptyset & T_1 \cap T_2 = \emptyset \\ (F, T_1 \cap T_2] & T_1 \cap T_2 \neq \emptyset. \end{cases}$$

The order intervals $(F, G]$ are contractible, and we deduce from the Nerve Theorem [**2**, Thm. 10.6] that the nerve of $V^+(F)$ and $M^+(F)$ have the same homotopy type. We obtain the following consequence.

COROLLARY 11.15. *Suppose $f : \mathcal{F}(X) \to \mathbb{R}$ is a coherent function. If $M^+(F)$ is a non-empty contractible simplicial scheme, then F is a regular point of f.* □

REMARK 11.16. Observe that the coherent function $f : \mathcal{F}(X) \to \mathbb{R}$ satisfies condition $\boldsymbol{\mu}_+$ if and only if, for every face F, the simplicial complex $M^+(F)$ is either empty, or consists of a single point. □

Suppose now that f satisfies $\boldsymbol{\mu}_-$, and denote by C_- the associated projector. We denote by $\mathcal{F}_-(X)$ the set of faces F such that $F = C_-(F)$. The set $\mathcal{F}_-(X)$ can be identified with the set of $\sim_f$-equivalence classes, and thus is equipped with the quotient order $\prec$.

Given $F, G \in \mathcal{F}_-(X)$ we have $F \prec G$ if and only if there exists a sequence of faces $F_0, F_1, \dots, F_n \in \mathcal{F}_-(X)$, and a sequence of faces $F'_1, \dots F'_n \in \mathcal{F}(X)$ such that the following hold.

- $F_0 = F$, $F_n = G$.
- F_{i-1} and F_i are faces of F'_i, $\forall i = 1, \dots, n$.
- $f(F_{i-1}) < f(F'_i) \leq f(F_i)$, $\forall i = 1, \dots, n$.

Fix a closed face F, and set $F_- = C_-(F)$. In other words

$$B \leq F,\ \ f(B) \geq f(F) \iff B \in [F_-, F] \iff V^-(F) = [F^-, F).$$

For any $B < F$ set

$$\mathcal{C}_B(F) := \mathcal{F}_{<F}(X) \setminus [B, F).$$

LEMMA 11.17. *For any $B < F$, the geometric realization of the nerve of the poset $\mathcal{C}_B(F)$ is homeomorphic to the ball $\mathbb{D}^{\dim F - 1}$.*

PROOF. Denote by Y the union of proper faces of F which do not contain B, i.e.,

$$Y = \bigcup_{G \in \mathcal{C}_B(F)} G.$$

The space Y is a PL space, and the geometric realization of the nerve of $\mathcal{C}_B(F)$ is PL homeomorphic to Y.

We set $n := \dim F$, and we assume $F \subset \mathbb{R}^n$. Choose a point b_0 a point in the relative interior of B, and for every $r > 0$ denote by L_r the intersection of F with the sphere of radius r in $\mathbb{R}^n$ centered at b_0.

For r sufficiently small, L_r is homeomorphic to a closed ball of dimension $n-1$. For every $x \in F \setminus \{b_0\}$ we denote by $\sigma_r(x)$ the intersection of the line $[b_0, x]$ with the link L_r. For $r > 0$ sufficiently small, the map σ_r defines a homeomorphism $Y \to L_r$. □

Note that $\mathcal{C}_B(F)$ consists of all closed faces of ∂F which do not contain B. Denote by $\partial F \setminus \mathbf{St}_B$ the union of all the closed faces $F' \in \mathcal{C}_B(F)$.

THEOREM 11.18. *Suppose $f : \mathcal{F}(X) \to \mathbb{R}$ is coherent and satisfies $\boldsymbol{\mu}_-$. If $F \neq C^-(F)$ then F is a regular point of f, while if $F = C_-(F)$ then the Conley index of F with respect to the simplicial flow defined by f is*

$$\begin{aligned} h_f(F) &= \big[\, |\mathrm{Cone}(S^{\dim F-1} * M^+(F)\,)|,\, |S^{\dim F-1} * M^+(F)|\,\big] \\ &\simeq \big[\,\mathrm{Cone}\,\Sigma^{\dim F}|M^+(F))|,\, \Sigma^{\dim F}|M^+(F))|\,\big], \end{aligned}$$

where we use the convention $\Sigma^n\emptyset := S^{n-1}$.

PROOF. If $F \neq C^-(F)$, then Lemma 11.17 shows that the poset

$$\mathcal{F}(X)_{<F} \setminus V^-(F) = \mathcal{F}(X)_{<F} \setminus [C_-(F), F)$$

is contractible, and thus F is a regular point.

If $F = C^-(F)$, then $\mathcal{F}(X)_{<F} \setminus V^-(F) = \mathcal{F}(X)_{<F}$, and the poset $\mathcal{F}_{<F}$ is PL homeomorphic to the sphere $S^{\dim F-1}$. The poset $V^+(F)$ is homotopic to $|M^+(F)|$ and thus

$$|\mathcal{L}(F \overset{f}{\rightsquigarrow})| \simeq S^{\dim F-1} * |M^+(F)| \simeq \Sigma^{\dim F}|M^+(F))|.$$

The result now follows from Theorem 11.3. □

Suppose $f : \mathcal{F}(X) \to \mathbb{R}$ is coherent and satisfies the condition $\boldsymbol{\mu}_-$. For any face $F \in \mathcal{F}(X)$ we denote by $\tilde{\mathcal{P}}_{M^+(F)}(t)$ the Poincaré polynomial of the reduced homology of $|M^+(F)|$, with the convention that

$$\tilde{\mathcal{P}}_{M^+(F)}(t) = t^{-1} \;\text{ if }\; M^+(F) = \emptyset.$$

Denote by $\tilde{M}_{F,t}(t)$ the Poincaré polynomial of the reduced homology of $|\mathcal{L}(F \overset{f}{\rightsquigarrow})|$. Then

$$\tilde{M}_{F,t}(t) = \mathcal{P}_{\Sigma^{\dim F}|M^+(F))|}(t) = t^{\dim F}\tilde{\mathcal{P}}_{M^+(F)}(t),$$

and from (9.3) we deduce the Morse inequalities

$$\sum_{F=C_-(F)} t^{\dim F+1}\tilde{\mathcal{P}}_{M^+(F)}(t) \succeq \mathcal{P}_X(t). \tag{11.1}$$

Observe that if f satisfies the condition $\boldsymbol{\mu}$, then for any face F the simplicial complex $M^+(F)$ is either empty, i.e.,$C_+(F) = F$, or consists of a single point, and $F \neq C_+(F)$. In this case, the Morse inequalities are very similar to the classical ones

$$\sum_{F=C_-(F)=C_+(F)} t^{\dim F} \succeq \mathcal{P}_X(t). \tag{11.2}$$

We denote by $\mathcal{M}_\nu(X)$ the set of coherent functions $f : \mathcal{F}(X) \to \mathbb{R}$ satisfying the condition $\boldsymbol{\mu}_-$ and of order $\leq \nu$. Observe that any function in $\mathcal{M}_0$ is a strictly decreasing function.

Note also that

$$\mathcal{M}_0(X) \subset \mathcal{M}_1(X) \subset \cdots .$$

We define

$$\mathcal{M}(X) := \bigcup_{\nu \geq 0} \mathcal{M}_\nu(X).$$

Given $f \in \mathcal{M}_\nu(X)$, and $F \in \mathcal{F}(X)$ such that $F = C_-(F)$, then $M^+(F)$ is a simplicial complex of dimension $\leq \nu - 1$.

To construct a coherent $\boldsymbol{\mu}_-$ function $f : \mathcal{F}(X) \to \mathbb{R}$ it suffices to construct a lower acyclic, coherent, isotone map $\Phi : \mathcal{F}(X) \to \mathbb{R}$. For every r in the range of Φ we denote by F_r the unique minimal face in $\Phi^{-1}(r)$, by V_r^+ the set of maximal elements of $\Phi^{-1}(r)$, and by M_r^+ the nerve of the cover $\big\{ (F_r, F] \big\}_{F \in V^+ r}$. Then every function $f \in F_\Phi(P)$ satisfies condition $\boldsymbol{\mu}_-$, it is coherent, its critical set is contained in the set $\{F_r;\ \ r \in \Phi\big(\mathcal{F}(X)\big)\}$. Moreover

$$h_f(F_r) \simeq \big[\operatorname{Cone} \Sigma^{\dim F} |M_r^+)|,\, \Sigma^{\dim F} |M_r^+)| \big].$$

If the fibers of Φ are intervals, so that f satisfies the condition $\boldsymbol{\mu}$, then $h_f(F_r)$ is trivial if the fiber $\Phi^{-1}(r)$ contains more than one face. When $\Phi^{-1}(r) = \{F_r\}$ we have

$$h_f(F_r) \simeq [F_r, \partial F_r].$$

A Morse-Forman function is induced by an isotone map $\Phi : \mathcal{F}(X) \to \mathbb{R}$ whose fibers are intervals of length at most 1. Using Theorem 10.4, or rather its proof, we obtain the following result.

Corollary 11.19. *Suppose $\mathcal{F}(X)$ is the poset of faces of a regular CW-decomposition of a compact space X, $(P, <)$ is a poset and $\pi : \mathcal{F}(X) \to P$ is an isotone map whose fibers are order intervals of $\mathcal{F}(X)$. Then X has the homotopy type of a cell complex where the cells of dimension k are in bijection with the k-dimensional faces $F \in \mathcal{F}(X)$ such that F is the only point in the fiber of π containing F, i.e., $\{F\} = \pi^{-1}\big(\pi(F)\big)$.*

In particular, if all the fibers of π are intervals of positive length then X is weakly contractible. □

Example 11.20. In the left-hand side of Figure 13 we have depicted a coherent function f of order two on the poset of faces of a 2-dimensional (affine) simplicial complex X. It satisfies the condition $\boldsymbol{\mu}_-$, but it does not satisfy the condition $\boldsymbol{\mu}_+$. The simplicial flow determined by this function is depicted the right-hand side of the figure.

The vertices labelled by 4, 1 and -1 correspond to the faces F satisfying the condition

$$F = C_-(F),$$

so these are the only stationary points of the flow which could have nontrivial Conley index, and thus could potentially affect the topology of X. For a vertex v such that $f(v) = 1$, the simplicial complex $M^+(v)$ is contractible (it corresponds to the barycenter of an edge labelled 0) and thus the Conley index is trivial.

If v is a vertex such that $f(v)=4$, then the simplicial complex $M^+(v)$ consists of two points (labelled A, B in the figure) and we deduce that the Conley index of such a point is $[S^1, *]$. In this case we observe that the Morse inequalities become equalities, and we see that we can use the flow to collapse X to a wedge of 3 circles. □

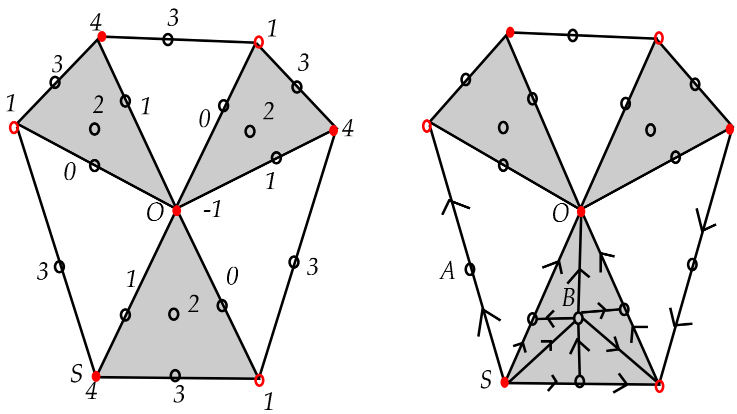


FIGURE 13. *A $\boldsymbol{\mu}_-$ function of order 2 and its associated "gradient" flow.*

REMARK 11.21. (a) It is not hard to prove that if f is a coherent function on the poset $\mathcal{F}(X)$ of faces of a polytopal decomposition of PL space X, then we can modify f to a coherent *injective* function $g : \mathcal{F}(X) \to \mathbb{R}$ such that $\overset{f}{\rightsquigarrow} = \overset{g}{\rightsquigarrow}$.
(b) If $f : \mathcal{F}(X) \to \mathbb{R}$ satisfies the condition $\boldsymbol{\mu}$, then we can use the flow determined by f to extract information about the *simple* homotopy type of X. If the order of f is ≤ 1, then f is a discrete Morse function of the type introduced by R. Forman, and many of the results [**14**] follow from general properties of the Conley index and tame flows. We will not pursue this point of view. □

CHAPTER 12

Tame currents

In this final section we will describe a natural tame generalization of the subanalytic currents introduced by R. Hardt in [**21**]. Our terminology concerning currents closely follows that of Federer [**13**] (see also the more accessible [**35**]). We will then use the finite volume flow technique of Harvey-Lawson [**23**] for certain tame flows on compact real analytic manifolds to produce interesting deformations of the DeRham complex.

Suppose X is a C^2, oriented manifold of dimension n. We denote by $\Omega_k(X)$ the space of k-dimensional *currents* in X, i.e., the topological dual space of the space $\Omega^k_{cpt}(X)$ of smooth, compactly supported k-forms on M. We will denote by

$$\langle \bullet, \bullet \rangle : \Omega^k_{cpt}(X) \times \Omega_k(X) \to \mathbb{R}$$

the natural pairing. The boundary of a current $T \in \Omega_k(X)$ is the $(k-1)$-current defined via the Stokes formula

$$\langle \alpha, \partial T \rangle := \langle d\alpha, T \rangle, \ \ \forall \alpha \in \Omega^{k-1}_{cpt}(X).$$

For every $\alpha \in \Omega^k(M)$, $T \in \Omega_m(X)$, $k \leq m$ define $\alpha \cap T \in \Omega_{m-k}(X)$ by

$$\langle \beta, \alpha \cap T \rangle = \langle \alpha \wedge \beta, T \rangle, \ \ \forall \beta \in \Omega^{n-m+k}_{cpt}(X).$$

We have

$$\langle \beta, \partial(\alpha \cap T) \rangle = \langle \, d\beta, (\alpha \cap T), \rangle = \langle \alpha \wedge d\beta, T \rangle$$

$$= (-1)^k \langle d(\alpha \wedge \beta) - d\alpha \wedge \beta, T \rangle = (-1)^k \langle \beta, \alpha \cap \partial T \rangle + (-1)^{k+1} \langle \beta, d\alpha \cap T \rangle$$

which yields the *homotopy formula*

$$\partial(\alpha \cap T) = (-1)^{\deg \alpha} \big(\, \alpha \cap \partial T - (d\alpha) \cap T \, \big). \tag{12.1}$$

The pair $(X, \boldsymbol{or}_X)$, $\boldsymbol{or}_X$ orientation on X, defines a current $[X, \boldsymbol{or}_X] \in \Omega_n(X)$, called the *the current of integration along* X. The current $[X, \boldsymbol{or}_X]$ defines an inclusion

$$\Omega^k(X) \to \Omega_{n-k}(X), \ \ \alpha \mapsto \alpha \cap [X, \boldsymbol{or}_X].$$

If X_0, X_1 are oriented C^2-manifolds of dimensions n_0 and respectively n_1, and $f : X_0 \to X_1$ is a C^2-map, then to every current $T \in \Omega_k(X_0)$ such that the restriction of f to $\operatorname{supp} T$ is proper, we can associate a current $f_*T \in \Omega_{k-(n_1-n_0)}(X_1)$ defined by

$$\langle \beta, f_*T \rangle = \langle f^*\beta, T \rangle, \ \ \forall \beta \in \Omega^{k-(n_1-n_0)}_{cpt}(X_1).$$

If $D \subset \mathbb{R}^n$ is a tame C^1 submanifold of $\mathbb{R}^n$ of dimension k then any orientation $\boldsymbol{or}_D$ on D determines a k-dimensional current $[D, \boldsymbol{or}_D]$ via the equality

$$\langle \alpha, [D, \boldsymbol{or}_D] \rangle := \int_D \alpha, \ \ \forall \alpha \in \Omega^k_{cpt}(\mathbb{R}^n).$$

The integral in the right-hand side is well defined because any compact, k-dimensional tame set has finite k-dimensional Hausdorff measure. We denote by $\mathcal{T}_k(\mathbb{R}^n)$ the Abelian subgroup of $\Omega_k(\mathbb{R}^n)$ generated by currents of the form $[D, \boldsymbol{or}_D]$ as above, and by $\mathcal{T}_k^{\mathbb{R}}(\mathbb{R}^n)$ the vector space spanned by such currents. We will refer to the currents in $\mathcal{T}_k(\mathbb{R}^n)$ as (integral) *tame currents*. The support of a tame current is a tame closed set.

For every closed tame set $S \subset \mathbb{R}^n$ we define

$$\mathcal{C}_k(S) := \big\{\, T \in \mathcal{T}_k(\mathbb{R}^n);\ \ \operatorname{supp} T,\ \ \operatorname{supp} \partial T \subset S \,\big\}.$$

Observe that we obtain a chain complex $(\mathcal{C}_\bullet(S), \partial)$

$$\cdots \to \mathcal{C}_k(S) \xrightarrow{\partial} \mathcal{C}_{k-1}(S) \to \cdots .$$

Suppose C^1-map $f : \mathbb{R}^n \to \mathbb{R}^m$ whose restriction to the tame set $S \subset \mathbb{R}^n$ is proper. Then f induces a morphism of chain complexes $f_\# : \mathcal{C}_\bullet(S) \to \mathcal{C}_\bullet(f(S))$. Arguing as in the proof of [**21**, Lemma 4.3] we obtain the following result.

LEMMA 12.1 (Lifting Lemma). *Suppose f is a tame C^1-map of an open neighborhood of a tame set S such that the induced map $S \mapsto f(S)$ is a homeomorphism. Then the induced map $f_\# : \mathcal{C}_\bullet(S) \to \mathcal{C}_\bullet(f(S))$ is an isomorphism of chain complexes.* □

We can use the lifting lemma as in R. Hardt in [**21**] to show the following result.

PROPOSITION 12.2. *Suppose $S_i \in \mathbb{R}^{n_i}$, $i = 0, 1$ are tame sets. Then any proper, continuous tame map $f : S_0 \to S_1$ induces a morphism of chain complexes*

$$f_\# : \mathcal{C}_\bullet(S) \to \mathcal{C}_\bullet(S_1).$$ □

We recall the construction of this map. Denote by $\Gamma_f \subset \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ the graph of f. We obtain a "roof"

$$\begin{array}{ccc} & \Gamma_f & \\ {\scriptstyle l}\swarrow & & \searrow{\scriptstyle r} \\ S_0 & & S_1 \end{array}$$

where the left map ℓ and the right map r are induced by the canonical projections $\mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \to \mathbb{R}^{n_i}$. Observe that ℓ is a *homeomorphism* and the restriction of r to Γ_f is proper. If $T \in \mathcal{C}_k(S)$ we define using the Lifting Lemma

$$f_\# T := r_\# \ell_\#^{-1} T.$$

We would like to explain how to geometrically describe the boundary of a tame current. This would require the notion of tame tube around a tame submanifold of $\mathbb{R}^n$.

Suppose $M \subset \mathbb{R}^n$ is a C^p-manifold, $p \geq 2$. We denote by $\mathcal{N}(M)$ the normal bundle of M in $\mathbb{R}^n$, i.e.,

$$\mathcal{N}(M) := \big\{ (v, x) \in \mathbb{R}^n \times M;\ \ v \perp T_x M \,\big\}.$$

Observe that if M is tame, so is $\mathcal{N}(M)$. We let $\boldsymbol{p} = \boldsymbol{p}_M : \mathcal{N}(M) \to M$ denote the natural projection, and $\boldsymbol{r} = \boldsymbol{r}_M : \mathcal{N}(M) \to [0, \infty)$ denote the radial distance function defined by

$$r(v, x) = |v|,$$

where $|v|$ denotes the Euclidean length of v. Observe that $\boldsymbol{p}$ and $\mathbb{R}$ are tame if M is tame.

We denote by $\exp : \mathcal{N}(M) \to \mathbb{R}$ the *exponential map*

$$\exp(v, x) = x + v.$$

Observe that if M is tame, then so is exp.

A *tube* around M in $\mathbb{R}^n$ is an open neighborhood U of M such that the exponential map induces a C^2-diffeomorphism

$$\exp : \exp^{-1}(U) \to U.$$

To each tube we can associate a *projection* $\pi = \pi_U : U \to M$, and a *radial distance* function $\rho = \rho_U : U \to [0, \infty)$ defined by the commutative diagrams

$$\begin{array}{ccc} \exp^{-1}(U) & \xrightarrow{\exp} & U \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & M & \end{array}, \qquad \begin{array}{ccc} \exp^{-1}(U) & \xrightarrow{\exp} & U \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathbb{R}_{\geq 0} & \end{array}$$

A tube U around U is called *tame* if U is tame, and there exists a tame C^p function

$$\varepsilon : M \to (0, \infty)$$

such that

$$\exp^{-1}(U) = \big\{ (v, x) \in \mathcal{N}(M);\ \ |v| < \varepsilon(x) \big\}.$$

We will refer to ε as the *width function* of the tame tube. From [**8**, Thm. 6.11, Lemma 6.12] we obtain the following result.

THEOREM 12.3 (Abundance of tame tubes). *Suppose M is a tame C^p submanifold of $\mathbb{R}^n$, $p \geq 2$. Then any tame open neighborhood $\mathcal{O}$ of M contains a tame tube with width function strictly smaller than < 1.* □

Fix $p \geq 2$, and suppose D is a tame, connected, orientable C^p-submanifold of $\mathbb{R}^n$ of dimension m. Fix an orientation $\boldsymbol{or} = \boldsymbol{or}_D$ on D, and a Verdier stratification of D such that D is a stratum. Recall that this implies that the Whitney regularity condition is satisfied as well. Denote by $(\dot{D}^i_w)_{1 \leq i \leq \nu}$ the $(m-1)$-dimensional strata of this stratification. Set

$$\dot{D} := \boldsymbol{cl}(D) - D, \quad \dot{D}_w := \bigcup_{i=1}^{n} \dot{D}^i_w.$$

Then $\dot{D} \setminus \dot{D}_w$ is a tame set of dimension $< m - 1$.

Choose a tube U_i (not necessarily tame) around $\dot{D}^i_w$ in $\mathbb{R}^n$ with projection π_i, and radial distance ρ_i with the following properties.

- The map
$$\pi_i \times \rho_i : U_i \cap D \to \dot{D}^i_w \times (0, \infty),$$
is submersive.
- There exists a smooth function $d_i : \dot{D}^i_W \to (0, \infty)$ such that the restriction of π to the set $D \cap \{\rho_i = d_i\}$ is a locally trivial fibration, and the set $D \cap \{\rho_i \leq d_i\}$ is homeomorphic to the mapping cylinder of $\pi_i : D \cap \{\rho_i = d_i\} \to \dot{D}^i_w$.

The existence of such a tube is guaranteed by the normal equisingularity of strata of a Whitney stratification (see [**16**, Lemma II.2.3, Thm. II.5.4].

Using Theorem 12.3 we deduce that there exists a *tame* tube $W_i \subset U_i$ around $\dot{D}^i_w$. Using [**8**, Lemma 6.12] we can even arrange that the width function of W_i satisfies $\varepsilon_i(x) < \frac{1}{2}d_i(x)$, $\forall x \in \dot{D}^i_w$. We will say that W_i is a *Whitney tube* of $\dot{D}^i_w$ (relative to D).

Fix $x_i \in \dot{D}^i_w$, and set

$$S_i := \bigl\{ y \in D;\ \ \pi_i(y) = x_0,\ \ \rho_i(y) = \varepsilon_i(\pi_i y) \bigr\} = (\pi_i \times \rho_i)^{-1}\bigl(x_i, \varepsilon_i(x_i)\bigr).$$

S_i is a tame zero dimensional set so that it is finite.

The restriction of π_i to $L_i := D \cap \{\rho_i(y) = \varepsilon_i(\pi_i y)\}$ is a locally trivial fibration over $\dot{D}^i_w$ with fiber S_i, and the set $D \cap \{\rho_i \leq \varepsilon_i(\pi_i y)\}$ is homeomorphic to the mapping cylinder of $\pi_i : L_i \to \dot{D}^i_w$.

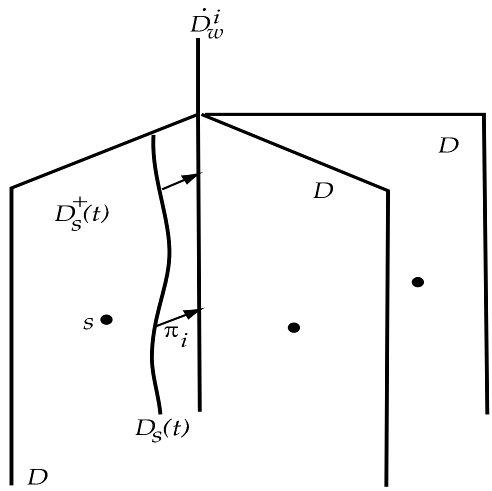


FIGURE 14. *Normal equisingularity in codimension* 1.

For $t \in (0,1)$, and $s \in S_i$, we denote by $D_s(t)^+$ the component of $D \cap \bigl\{ t\varepsilon_i \leq \rho_i < \varepsilon_i, \bigr\}$ containing the point s, and we denote by $D_s(t)$ its boundary (see Figure 14),

$$D_s(t) := \bigl\{ y \in D^+_s(t);\ \ \rho_i(y) = t\varepsilon_i(\pi y) \bigr\}.$$

The orientation $\boldsymbol{or}$ on D induces orientations on the components $D^+_s(t)$, and in turn, these define orientations on their boundaries $D_s(t)$ via the outer-normal-first convention. The projection π_i induce diffeomorphisms $\pi_i : D_s(t) \to \dot{D}^i_w$, and thus orientations $\boldsymbol{or}_s$ on M. We have the following result.

THEOREM 12.4 (Generalized Stokes formula).

$$\partial[D, \boldsymbol{or}] = \sum_{i=1}^{\nu} \sum_{s \in S_i} [\dot{D}^i_w, \boldsymbol{or}_s].$$

PROOF. We choose a triangulation of D such that all the open faces are tame C^3-manifolds and each one of them is contained in a stratum of the Verdier stratification. This reduces the problem to the following special case.

Denote by Δ_m the *standard m-simplex*

$$\Delta_k := \big\{ (t_0, \dots, t_m) \in \mathbb{R}^{m+1}_{\geq 0};\ \sum_{j=0}^m t_j = 1 \big\}$$

We denote by $e_0, \dots, e_m$ the vertices of Δ_m, and for every $I \subset \{0, \dots, m\}$ we denote by $\mathcal{O}_I$ the *open* face spanned by the vertices e_i, $i \in I$.

We define a *tame m-simplex* to be a pair $D = (\Delta_m, f)$, where $f : \Delta_m \to \mathbb{R}^n$ is a tame continuous map with the following properties.

- The map f is a homeomorphism onto its image.
- The images of the open faces are C^3-submanifolds.
- The collection of images of the open faces of Δ_m form a Verdier stratification of $f(\Delta_k)$.

For a tame m-simplex $D = (\Delta_m, f)$ and $I \subset \{0, \dots, m\}$ we will write

$$D_f(I) := f\big(\mathcal{O}_I\big).$$

For simplicity, we will write

$$D_f := D_f(\{0, \dots, m\}),\ \ D_f^k = D(\{0, \dots, \hat{k}, \dots, m\}),\ \ \boldsymbol{bd}(D) := D \setminus D_f.$$

f induces orientations $\boldsymbol{or}_f$ on D_f, and $\boldsymbol{or}_k$ on D_f^k. The theorem is then a consequence of the following equality

$$\partial[D_f, \boldsymbol{or}_f] = \sum_{k=0}^m (-1)^k [D_f^k, \boldsymbol{or}_k]. \tag{12.2}$$

Denote by $[\Delta_m]$ the tame current defined by the standard m-simplex equipped with the orientation defined by the frame $(\boldsymbol{e}_1 - \boldsymbol{e}_0, \dots, \boldsymbol{e}_m - \boldsymbol{e}_0)$, where $(\boldsymbol{e}_i)$ is the canonical basis of $\mathbb{R}^{m+1}$. Using Proposition 12.2 (or rather its proof) we deduce

$$[D_f, \boldsymbol{or}_f] = f_\#[\Delta_m].$$

The equality (12.2) follows from the fact that $f_\#$ is a morphism of chain complexes

$$\partial[D_f, \boldsymbol{or}_f] = f_\#\partial[\Delta_m]. \qquad \square$$

REMARK 12.5. (a) To detect the boundary contributions $[\dot{D}_w^i, \boldsymbol{or}_s]$ *we do not need to know precisely a Whitney stratification of D*. We look at the $(m-1)$-dimensional stratum $\dot{D}$, and orient it in some fashion using an orientation $\boldsymbol{or}_\partial$.

Next, find a tube $(T, \pi, \rho, \varepsilon)$ around $\dot{D}$ and consider the shrinking tubes

$$T_s := \big\{ z \in T;\ \ \rho(z) \leq s\varepsilon(\pi z) \big\},\ \ s \in (0,1).$$

We denote by $\partial T_s \pitchfork D$ the subset of ∂T_s where the intersection is transversal.

Suppose that, for all s, the set $\partial T_s \pitchfork D$ projects properly via π onto a dense open subset $\dot{D}_{reg}$ of $\dot{D}$. We denote by $\dot{D}^i_{reg}$ the components of $\dot{D}_{reg}$, and by $\partial T_s^i \cap D$ the preimage of $\dot{D}^i_{reg}$ in $\partial T_s \cap D$ via π. Then the components of $\partial T_s^i \cap D$ are equipped with orientations as boundary components of $D \setminus T_s$, and we denote by n_i the degree of the map

$$\pi : \partial T_s^i \cap D \to \dot{D}^i_{reg}.$$

Then

$$\partial[D] = \sum_i n_i [\dot{D}^i_{reg}, \boldsymbol{or}_\partial].$$

The reason for this equality is that the set of points x on $\dot{D}^i_{reg}$ where the Whitney condition for the pair $(D, \dot{D}^i_{reg})$ is violated forms a tame set of dimension $< \dim \dot{D}^i_{reg}$ so it does not affect the current defined by $\dot{D}^i_{reg}$.

(b) The proof of the very simple and natural statement of the Lifting Lemma requires quite sophisticated results in geometric measure theory. In Appendix A we present a proof of (12.2) which does not use the Lifting Lemma so that the reader could appreciate the subtlety of this result, and the strength of tame geometric techniques. □

We want to apply the above facts concerning tame currents to the study of asymptotics of certain simple tame flows.

Consider the standard simplex Δ_m, with vertices $\boldsymbol{e}_0, \dots, \boldsymbol{e}_m \in \mathbb{R}^{m+1}$. This labelling of the vertices defines a tame flow Φ_t on Δ_m and a flow Φ^∂_t on its boundary $\boldsymbol{bd}(\Delta_m)$. Consider the tame, increasing homeomorphism

$$\tau : \mathbb{R} \to (-1, 1),\ \ t \mapsto \frac{t}{\sqrt{1+t^2}}.$$

For every tame subset of $S \subset \mathbb{R}$ (i.e., a finite union of open intervals and singletons) we define

$$\Gamma^S := \Big\{ \big(\tau(t), x, \Phi_t x\big);\ \ x \in \Delta_m,\ \ t \in S \Big\} \subset [-1, 1] \times \Delta_m \times \Delta_m.$$

The projection

$$[-1, 1] \times \Delta_m \times [-1, 1] \times \Delta_m \to \Delta_m,\ \ (\tau, x, y,) \mapsto (\tau, x)$$

defines homeomorphisms

$$\Gamma^S \to S \times \Delta_m.$$

We orient S in the canonical way as a tame subset of $\mathbb{R}$. *We fix an orientation* $\boldsymbol{or}_m$ *on* $\boldsymbol{Int}\,\Delta_m$. Using the above homeomorphism and the orientation $\boldsymbol{or}_m$ we obtain and orientation on the top dimensional part of Γ^S, and thus a tame current $[\Gamma^S]$.

For simplicity we will set

$$\Gamma^+ = \Gamma^{[0,\infty)},\ \ \Gamma^- = \Gamma^{(-\infty,0]},\ \ \Gamma^t = \Gamma^{\{t\}}.$$

The boundary of Δ_m is Φ invariant, and we denote by Φ^∂ the flow induced by Φ on the boundary. We orient the boundary using the orientation induced from $\boldsymbol{or}_m$. Using the flow Φ_∂ and the orientation $\partial\boldsymbol{or}_m$ we define in a similar way the currents $[\Gamma^S_\partial]$, S tame subset of $\mathbb{R}$.

Every tame subset $S \subset \mathbb{R}$ canonically defines a tame current $[S] \subset \mathcal{T}_\bullet(\mathbb{R})$. To avoid notational overload we will continue to denote the current $[S]$ simply by S. We can extend by linearity the maps

$$S \mapsto [\Gamma^S],\ \ [\Gamma^S_\partial]$$

to the maps

$$\mathcal{T}_\bullet(\mathbb{R}) \ni [S] \mapsto [\Gamma^S],\ \ [\Gamma^S_\partial] \in \mathcal{T}_\bullet([-1, 1] \times \Delta_m \times \Delta_M.$$

If $S \in \mathcal{T}_\bullet(\mathbb{R})$ is a *compactly supported* tame current, then

$$\partial[\Gamma^S] = \Gamma^{\partial S} + (-1)^{\dim S}[\Gamma^S_\partial].$$

In particular, we have

$$\partial[\,\Gamma^{[0,T]}\,] = [\Gamma^T] - [\Gamma^0] - [\,\Gamma^{[0,T]}_\partial\,], \tag{12.3a}$$

$$\partial[\,\Gamma^{[-T,0]}\,] = [\Gamma^0] - [\Gamma^T] - [\,\Gamma_\partial^{[-T,0]}\,]. \tag{12.3b}$$

We denote by $\mathcal{H}^d$ the d-dimensional Haudorff measure. If $S \subset \mathbb{R}$ is a compact tame set then both Γ^S and Γ_∂^S have finite Hausdorff measures of dimensions $m + \dim S$ and $m - 1 + \dim S$ respectively. Arguing exactly as in the proof of Lemma A.5 we obtain the following result.

LEMMA 12.6. *(a) As $T \to \infty$ the current $[\,\Gamma^{[0,T]}\,]$ converges in the mass norm to $[\,\Gamma^+\,]$, and the current $[\,\Gamma^{[-T,0]}\,]$ converges in the mass norm to $[\,\Gamma^-\,]$, i.e.,*

$$\lim_{T\to\infty} \mathcal{H}^{m+1}\big(\,\Gamma^{[T,\infty)}\,\big) = 0 = \lim_{T\to\infty} \mathcal{H}^{m+1}\big(\,\Gamma^{(-\infty,-T]}\,\big).$$

(b) Similarly, as $T \to \infty$ the current $[\,\Gamma_\partial^{[0,T]}\,]$ converges in the mass norm to $[\Gamma_\partial^+]$ and the current $[\,\Gamma_\partial^{[-T,0]}\,]$ converges in the mass norm to $[\Gamma_\partial^-]$. □

If we let $T \to \infty$ in the equalities (12.3a) and (12.3b) we obtain

$$\partial\Gamma^+ = [\Gamma^\infty] - [\Gamma^0] - [\,\Gamma_\partial^+\,], \tag{12.4a}$$

$$\partial\Gamma^- = [\Gamma^0] - [\Gamma^{-\infty}] - [\,\Gamma_\partial^-\,], \tag{12.4b}$$

where $[\Gamma^\infty]$ is a tame current supported in $\boldsymbol{cl}(\Gamma^+) \setminus \Gamma^+$ and $[\Gamma^{-\infty}]$ is a tame current supported in $\boldsymbol{cl}(\Gamma^-) \setminus \Gamma^-$. We will use the generalized Stokes formula to obtain a very explicit description of the currents $[\Gamma^{\pm\infty}]$. This will require some more terminology.

For every $k \in \{0, \dots, m\}$, denote by $W_k^\pm$ the stable/unstable variety of the stationary point $\boldsymbol{e}_k$ of the flow Φ. If $(t_0, \dots, t_m)$ denote the barycentric coordinates on Δ_m then

$$\begin{aligned} W_k^+ &= \big\{(t_0, \dots, t_m);\ \ t_j = 0,\ \ \forall j > k,\ \ t_i < 1,\ \ \forall i < k\,\big\} \\ &= [\boldsymbol{e}_k, \dots, \boldsymbol{e}_m] \setminus [\boldsymbol{e}_{k+1}, \dots, \boldsymbol{e}_m], \\ W_k^- &= \big\{(t_0, \dots, t_m);\ \ t_i = 0,\ \ \forall i < k,\ \ t_j < 1,\ \ \forall j > k\,\big\} \\ &= [\boldsymbol{e}_k, \dots, \boldsymbol{e}_m] \setminus [\boldsymbol{e}_0, \dots, \boldsymbol{e}_{k-1}]. \end{aligned}$$

PROPOSITION 12.7.

$$\operatorname{supp}\Gamma^\infty \subset \bigcup_{\ell \geq k} W_\ell^+ \times W_k^-, \tag{12.5a}$$

$$\operatorname{supp}\Gamma^{-\infty} \subset \bigcup_{k \leq \ell} W_k^- \times W_\ell^+, \tag{12.5b}$$

PROOF. The inclusion (12.5b) follows from (12.5a) by time reversal so it suffices to prove (12.5a). Suppose $(x_\infty, y_\infty) \in \Gamma^\infty$. From the curve selection property, we can find continuous definable paths

$$[0, \infty) \ni s \longmapsto t_s \in \mathbb{R},\ \ x_s \in \Delta_m,$$

such that as $s \to \infty$ we have

$$t_s \to \infty,\ \ x_s \to x_\infty,\ \ \Phi_{t_s} x_s \to y_\infty.$$

If x_s is a stationary point for all sufficiently large s then $x_\infty = y_\infty$ and the conclusion is immediate. We assume that x_s is not a stationary point for any $s \geq 0$.

Denote by C_s the portion of trajectory

$$C_s = \{ \Phi_t x_s;\ \ t \in [0, t_s] \}$$

and form the strip

$$\Sigma = \bigcup_{s \geq 0} C_s.$$

We set

$$C_\infty := \boldsymbol{cl}(\Sigma) \setminus \Sigma.$$

Observe that C_∞ is a *compact, Φ-invariant, tame* subset of Δ_m. Moreover $x_\infty \in C_\infty$.

Denote by $f : \Delta_m \to \mathbb{R}$ the affine function uniquely determined by the conditions

$$f(\boldsymbol{e}_i) = i,\ \ \forall i = 0, \dots, m.$$

For $\varepsilon > 0$ sufficiently small define

$$E_i := \{ p \in \Delta_m;\ \ |f(p) - f(\boldsymbol{e}_i)| < \varepsilon \}.$$

E_i is an open tame neighborhood of $\boldsymbol{e}_i$ and if $\varepsilon < \frac{1}{2}$ we have

$$E_i \cap E_j = \emptyset,\ \ \forall i \neq j.$$

For every $i = 0, \dots, m$ we set

$$A_i(s) := \{ t \in [0, t_s];\ \ \Phi_t x_s \in E_i \}.$$

Note that because f is a Lyapunov function for f the set $A_i(s)$ is a (possible empty) connected subset, for every i and s. We have $(m+1)$ definable families of definable sets

$$\big(A_0(s) \big)_{s \in [0,1)}, \dots, \big(A_m(s) \big)_{s \in [0,1)}.$$

For every $i = 0, \dots, m$ and every $s \geq 0$ we denote by $L_i(s)$ the length of the interval $A_i(s)$ Define the *relevant set*

$$R := \{ i = 0, \dots, m;\ \ \lim_{s \to \infty} L_i(s) = \infty \}.$$

Note that $R \neq \emptyset$. Indeed, if $R = \emptyset$, using the fact that x_s is not a stationary point, we deduce

$$C_\infty \cap \{ \boldsymbol{e}_0, \dots, \boldsymbol{e}_m \} = \emptyset.$$

This is impossible since C_∞ is a compact invariant subset so it must contain stationary points of Φ.

Fix $s_0 > 0$ such that

$$A_r(s) \neq \emptyset,\ \ \forall s > s_0,\ \ r \in R.$$

Since the flow Φ admits a Lyapunov function f we deduce that, for every $s > s_0$ and $r_1, r_2 \in R$ such that $r_2 > r_1$, the interval $A_{r_2}(s)$ is situated to the left of the interval $A_{r_1}(s)$ (see Figure 15).

More precisely, this means

$$t_2 < t_1,\ \ \forall t_1 \in A_{r_1}(s),\ \ t_2 \in A_{r_2}(s),\ \ s > s_0,\ \ r_2 > r_1.$$

Now define

$$\ell = \max R,\ \ k = \min R.$$

We deduce that $\Phi_\infty x_\infty = \boldsymbol{e}_\ell$, i.e. $x_\infty \in W_\ell^+$, and $\Phi_{-\infty} y_\infty = \boldsymbol{e}_k$, i.e. $y_\infty \in W_k^-$. □

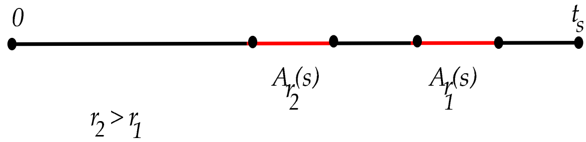


FIGURE 15. *The relevant intervals*

Observe that

$$\dim W_\ell^+ \times W_k^- = (m-\ell)+k$$

to that the m-dimensional strata of Γ^∞ are contained in

$$\bigcup_{k=0}^m W_k^+ \times W_k^-.$$

Hence, if we fix orientations $\boldsymbol{or}_k$ on $W_k^+ \times W_k^-$ we obtain an equality of the form

$$[\Gamma^\infty] = \sum_k \epsilon_{m,k}[W_k^+ \times W_k^-, \boldsymbol{or}_k] \tag{12.6}$$

where $\epsilon_{m,k}$ are some integers. Our next goal will be to show that we can choose the orientations $\boldsymbol{or}_k$ in a natural way so that all the integers $\epsilon_{m,k}$ are equal to 1. This will require a few more additional steps.

The key step towards achieving our goal is a remarkable property of the simplicial flow Φ_t. Denote by $\mathcal{P}_m$ the projection

$$\mathcal{P} : \Delta_m \setminus \{\boldsymbol{e}_m\} \to [\boldsymbol{e}_0, \dots, \boldsymbol{e}_{m-1}]$$

defined by

$$\mathcal{P}(x) := \text{the intersection of the line } \boldsymbol{e}_m x \text{ with the face } [\boldsymbol{e}_0, \dots, \boldsymbol{e}_{m-1}].$$

LEMMA 12.8 (Conservation of parallelism). *Suppose the two distinct points $x_0, x_1 \in \boldsymbol{Int}\,\Delta_m$ determine a line parallel to the face $[\boldsymbol{e}_0, \dots \boldsymbol{e}_{m-1}]$. i.e., they lie in a hyperplane $\{t_m = const\}$. Then for every $t \in \mathbb{R}$, the line determined by the points $\Phi_t(x_0)$ and $\Phi_t(x_1)$ is parallel with the line determined by the points x_0, x_1 and with the line determined by $\mathcal{P}_m(\Phi_t(x_0))$ and $\mathcal{P}_m(\Phi_t(x_1))$.*

PROOF. We argue by induction over m. For $m = 0, 1$ the statement is trivially true. We assume it is true for Δ_m and we prove its validity for Δ_{m+1}. We denote by S the set $\{x_0, x_1\}$, and we set for simplicity $\mathcal{P} = \mathcal{P}_{m+1}$.

The set $S \subset \boldsymbol{Int}\,\Delta_{m+1} \setminus \{\boldsymbol{e}_{m+1}\}$ is contained in a hyperplane $\{t_{m+1} = c\}$, where $c \in [0, 1)$. The restriction of $\mathcal{P}$ to $\boldsymbol{Int}\,\Delta_{m+1} \cap \{t_{m+1} = c\}$ defines an affine map

$$\boldsymbol{Int}\,\Delta_{m+1} \cap \{t_{m+1} = c\} \to \boldsymbol{Int}[\boldsymbol{e}_0, \dots, \boldsymbol{e}_m],$$

such that for any $y_0, y_1 \in \boldsymbol{Int}\,\Delta_{m+1} \cap \{t_{m+1} = c\}$, the line determined by y_0, y_1 is parallel with the line determined by $\mathcal{P}(y_0)$ and $\mathcal{P}(y_1)$, and

$$\operatorname{dist}(y_0, y_1) = (1-c)\operatorname{dist}\big(\mathcal{P}(y_0), \mathcal{P}(y_1)\big). \tag{12.7}$$

From the iterated cone description of Φ we deduce that $\mathcal{P} \circ \Phi_t = \Phi_t \circ \mathcal{P}$, $\forall t \in \mathbb{R}$. The lemma now follows from the inductive assumption. $\square$

For $\varepsilon \in (0, 1)$ we define an ε-neighborhood of $\boldsymbol{e}_k \in W_k^\pm$ (see Figure 16)

$$W_k^\pm(\varepsilon) := \big\{\, w \in W_k^\pm;\ \ |t_k(w) - 1| < \varepsilon \,\big\}.$$

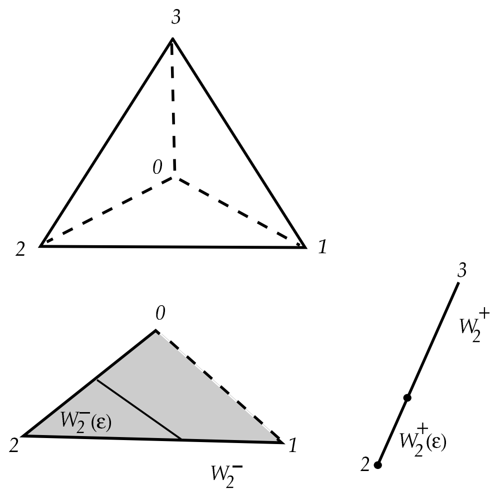


FIGURE 16. *Organizing the (un)stable varieties of a simplicial flow.*

Let $k = 0, \dots, m$, and consider a point w_+ in the relative interior of the stable variety of $\boldsymbol{e}_k$,

$$w_+ \in \boldsymbol{Int}\, W_k^+ = \boldsymbol{Int}[\boldsymbol{e}_k, \dots, \boldsymbol{e}_m]$$

For $\varepsilon_- > 0$ we denote by $\mathcal{N}_k^-(w_+, \varepsilon_+)$ the translate of $W_k^-(\varepsilon_-)$ at w

$$\mathcal{N}_k^-(w_+, \varepsilon_-) := (w_+ - \boldsymbol{e}_k) + W_k^-(\varepsilon_-).$$

For $\varepsilon_- > 0$ sufficiently small, this set is contained in Δ_m. We denote by $\mathcal{N}_k^-(w_+, \varepsilon_-)_{reg}$ the regular (top dimensional) part of $\mathcal{N}_k^-(w_+)$.

If we denote by $V_k^-(w_+)$ the affine k-plane through w_+, and parallel to the face $[\boldsymbol{e}_0, \dots, \boldsymbol{e}_k]$, then $V_k^-(w_+)$ intersects $\boldsymbol{Int}\, W_k^+$ transversally at w_+, and for $\varepsilon > 0$ sufficiently small, $\mathcal{N}_k^-(w_+, \varepsilon_-)$ is a neighborhood of w_+ in $V_{k+1}(w) \cap \Delta_m$.

Similarly, for $w_- \in \boldsymbol{Int}\, W_k^-$, and $\varepsilon_+ > 0$ we denote by $\mathcal{N}_k^+(w_-, \varepsilon_+)$ the translate of $W_k^+(\varepsilon_+)$ at w_-,

$$\mathcal{N}_k^+(w_-, \varepsilon_+) := (w_- - \boldsymbol{e}_k) + W_k^+(\varepsilon_+).$$

If we denote by $V_k^+(w_-)$ the affine $(m-k)$-plane through w and parallel to the face $[\boldsymbol{e}_k, \dots, \boldsymbol{e}_m]$, then $V_k^+(w_-)$ intersects $\boldsymbol{Int}\, W_k^-$ transversally at w, and for $\varepsilon_+ > 0$ sufficiently small, $\mathcal{N}_k^+(w_-, \varepsilon_+)$ is a neighborhood of w_- in $V_k^+(w_-) \cap \Delta_m$. We denote by $\mathcal{N}_k^+(w_-, \varepsilon_+)_{reg}$ the regular (top dimensional) part of $\mathcal{N}_k^+(w_-, \varepsilon_+)$.

PROPOSITION 12.9. *Let $k \in \{1, \dots, m-1\}$. Then there exist a definable function*

$$T_k : \boldsymbol{Int}\, W_k^+ \times \boldsymbol{Int}\, W_k^- \times (0,1) \times (0,1) \to \mathbb{R},$$

$$(w_+, w_-, \varepsilon_-, \varepsilon_+) \mapsto T_k(w_+, w_-, \varepsilon_-, \varepsilon_+),$$

such that, for all $(w_+, w_-, \varepsilon_-, \varepsilon_+) \in \boldsymbol{Int}\, W_k^+ \times \boldsymbol{Int}\, W_k^- \times (0,1) \times (0,1)$, and all $t > T_k(w_+, w_-, \varepsilon_-, \varepsilon_+)$, the normal slice

$$\mathcal{N}_k(w_+, w_-, \varepsilon) := \mathcal{N}_k^-(w_+, \varepsilon_-)_{reg} \times \mathcal{N}_k^+(w_-, \varepsilon_+)_{reg}$$

intersects Γ^t_{reg}, the regular part of the graph of Φ_t, at a unique point. Moreover, the intersection at that point is transversal in $\boldsymbol{Int}\,\Delta_m \times \boldsymbol{Int}\,\Delta_m$.

PROOF. Observe that

$$(x, y) \in \Gamma_t \cap \mathcal{N}^-_k(w_+, \varepsilon_-)_{reg} \times \mathcal{N}^+_k(w_-, \varepsilon_+)_{reg}$$

if and only if

$$y \in \Phi_t\big(\mathcal{N}^-_k(w_+, \varepsilon_-)_{reg}\big) \cap \mathcal{N}^+_k(w_+, \varepsilon_+)_{reg}, \;\; x = \Phi_{-t}y.$$

Moreover

$$\Gamma_t \pitchfork \mathcal{N}^-_k(w_+, \varepsilon_-)_{reg} \times \mathcal{N}^+_k(w_-, \varepsilon_+)_{reg} \Longleftrightarrow \Phi_t\big(\mathcal{N}^-_k(w_+, \varepsilon_-)_{reg}\big) \pitchfork \mathcal{N}^+_k(w_-, \varepsilon_+)_{reg}.$$

Set $w_+(t) := \Phi_t w_+$. From the conservation of parallelism we deduce that the set $\Phi_t\big(\mathcal{N}^-_k(w_+, \varepsilon)_{reg}\big)$ is an open subset of the affine plane $V^-_k\big(w_+(t)\big)$. In particular, if $\Phi_t\big(\mathcal{N}^-_k(w_+, \varepsilon)_{reg}\big)$ intersects $\mathcal{N}^+_k(w_+, \varepsilon)_{reg}$, it does so transversally.

To understand the region $\Phi_t\big(\mathcal{N}^-_k(w_+, \varepsilon_-)\big)$ better, consider the projections

$$\mathcal{P}_j : [\boldsymbol{e}_0, \ldots, \boldsymbol{e}_j] \setminus \{\boldsymbol{e}_j\} \to [\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{j-1}],$$

$$\mathcal{P}_j(x) := \text{the intersection of the line } \boldsymbol{e}_j x \text{ with the face } [\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{j-1}].$$

We obtain a sequence of points

$$w^m_+, \ldots, w^{k+1}_p, \;\; w^k_+$$

defined inductively as

$$w^m_+ = w^+, \;\; w^{j-1}_+ = \mathcal{P}_j(w^j_+).$$

Observe that (see Figure 17)

$$w^j_+ \in \boldsymbol{Int}[\boldsymbol{e}_k, \ldots, \boldsymbol{e}_j], \;\; \forall j > k, \; w^k_+ = \boldsymbol{e}_k.$$

Denote by $\mathcal{S}$ the composition

$$\mathcal{S} = \mathcal{P}_{k+1} \circ \cdots \circ \mathcal{P}_m.$$

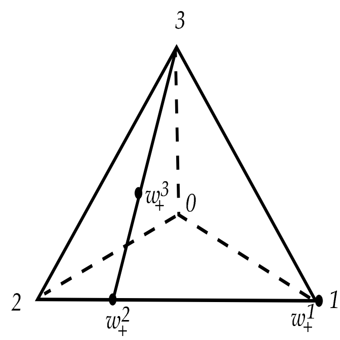


FIGURE 17. *The sequence of shadows $w^m_+, \ldots, w^k_+$, when $m = 3$ and $k = 1$.*

From the conservation of parallelism and the iterated cone description of Φ we deduce

$$\mathcal{S}\big(\mathcal{N}^-_k(w_+, \varepsilon_-)\big) = W^-_k(c\varepsilon_-),$$

for some $c > 1$. We set $w_+(t) = \Phi_t w_+$. Note that

$$\Phi_t \circ \mathcal{S} = \mathcal{S} \circ \Phi_t$$

Using the conservation of parallelism we deduce that the map

$$\mathcal{S} : \Phi_t\big(\,\mathcal{N}_k^-(w_+, \varepsilon_-)_{reg}\,\big) \to \mathcal{S}\Phi_t\big(\,\mathcal{N}_k^-(w_+, \varepsilon_-)_{reg}\,\big)$$

is a homothety. Now observe that

$$\mathcal{S}\Phi_t\big(\,\mathcal{N}_k^-(w_+, \varepsilon_-)_{reg}\,\big) = \Phi_t\mathcal{S}\big(\,\mathcal{N}_k^-(w_+, \varepsilon_-)_{reg}\,\big) = \Phi_t W_k^-(c\varepsilon_-)_{reg}.$$

We conclude that

$$\Phi_t\big(\,\mathcal{N}_k^+(w_+, \varepsilon)\,\big) = \mathcal{N}_k^+(w_+(t), \varepsilon_-(t)),$$

and $\varepsilon_-(t) \to 1$ as $t \to \infty$ (see Figure 18).

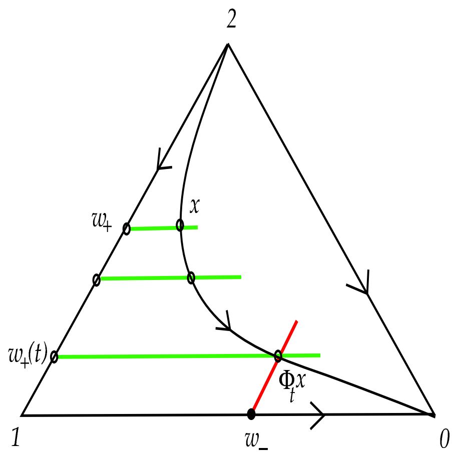


FIGURE 18. $\Phi_t(\mathcal{N}_k^-(w_+, \varepsilon_-)\,)$ *is depicted as the moving horizontal segment that is increasing in length.*

Denote by $T_k = T_k(w_+, w_-, \varepsilon_-, \varepsilon_+)$ the smallest real number $T > -1$ with the property that

$$w_+(t) \in W_k^+(\varepsilon_+) \;\text{ and }\; \varepsilon_-(t) > 1 - t_k(w_-), \;\; \forall t > T.$$

If $t > T_k(w_+, w_-, \varepsilon_-, \varepsilon_+)$, then affine k-dimensional piece $\Phi_t\big(\,\mathcal{N}_k^-(w_+, \varepsilon_-)_{reg}\,\big)$ intersects the affine $(m-k)$-dimensional piece $\mathcal{N}_k^+(w_-, \varepsilon_+)$ at a unique point (see Figure 18)

$$y_t = (w_+(t) - \boldsymbol{e}_k) + (w_- - \boldsymbol{e}_k) + \boldsymbol{e}_k.$$

If we think of $\boldsymbol{e}_k$ as the origin of our affine space then we can rewrite the above equality in the simpler form

$$y_t = w_+(t) + w_-.$$

Hence, the normal slice $\mathcal{N}_k(w_+, w_-, \varepsilon) := \mathcal{N}_k^-(w_+, \varepsilon_-)_{reg} \times \mathcal{N}_k^+(w_-, \varepsilon_+)_{reg}$ intersects Γ_{reg}^t at a single point

$$(x, y) = \big(\,x(w_+, w_-, t),\, y(w_+, w_-, t)\,\big) = \big(\,\Phi_{-t}(\,w_+(t) + w_-),\, w_+(t) + w_-\,\big), \tag{12.8}$$

and the intersection is transversal. □

Observe that the map

$$(w_+, w_-, \varepsilon_-, \varepsilon_+) \mapsto T_k(w_+, w_-, \varepsilon_-, \varepsilon_+)$$

is upper semicontinuous in the variables (w_-, w_+), i.e. if

$$T_k(w_+, w_-, \varepsilon_-, \varepsilon_+) < T,$$

then there exist open neighborhoods $U_\pm$ of $w_\pm$ in $\boldsymbol{Int}\, W_k^\pm$ such that, for all

$$T_k(u_+, u_-, \varepsilon_-, \varepsilon_+) < T, \;\; \forall (u_+, u_-) \in U_+ \times U_-.$$

Given T and $U_\pm$ as above we obtain for every $t > T$ a tame continuous map given by (12.8),

$$U_+ \times U_- \stackrel{\psi_t}{\longmapsto} \big(x(u_+, u_-, t), y(u_+, u_-, t)\,\big) \in \Gamma^t_{reg},$$

which is a homeomorphism onto its image. Γ^t_{reg} admits a natural orientation induced by the homeomorphism

$$\Gamma^t_{reg}(x, \Phi_t x) \mapsto x \in \boldsymbol{Int}\, \Delta_m.$$

We conclude that the homeomorphism ψ_t induces an orientation $\boldsymbol{or} = \boldsymbol{or}_t$ on $U_+ \times U_-$ which is independent of $t > T$. For a different pair of points (w'_+, w'_-), and corresponding neighborhood $U'_+ \times U'_-$, the orientation $\boldsymbol{or'}$ on $U'_+ \times U'_-$ obtained by the above procedure coincides with $\boldsymbol{or}$ on the overlap. We obtain in this fashion an orientation $\boldsymbol{or}_k$ on $\boldsymbol{Int}\, W_k^+ \times \boldsymbol{Int}\, W_k^-$. We would like to give a more explicit description of $\boldsymbol{or}_k$.

To achieve this, we place $w_\pm$ very close to $\boldsymbol{e}_k \in W_k^\pm$, and we choose $\varepsilon_\pm$ relatively large, say $\varepsilon_\pm = \frac{1}{2}$. Then $T_k(w_+, w_-, \varepsilon_-, \varepsilon_+) < 0$, and it suffices to understand the homeomorphism ψ_t, $t = 0$. In this case the equation (12.8) takes the simple form

$$(u_-, u_+) \mapsto (u_- + u_+, u_- + u_+).$$

Thus $\boldsymbol{or}_k$ is the orientation with the property that the map

$$T_{\boldsymbol{e}_k} W_k^+ \times T_{\boldsymbol{e}_k} W_k^- \to T\Delta_m, \;\; (u_+, u_-) \mapsto u_+ + u_-$$

is orientation preserving, where we recall that we have fixed an orientation $\boldsymbol{or}_m$ on $\boldsymbol{Int}\, \Delta_m$.

Let us observe that we have a natural tube $\mathfrak{T}$ around $\{1\} \times \boldsymbol{Int}\, W_k^+ \times \boldsymbol{Int}\, W_k^-$ inside $\mathbb{R} \times \boldsymbol{Int}\, \Delta_m \times \Delta_m$ defined as follows.

- Fix continuous definable functions $\varepsilon_\pm : \boldsymbol{Int}\, W_k^\mp \to (0, 1)$ such that

$$\mathcal{N}_k^\mp(w_\pm, \varepsilon_\mp) \subset \Delta_m.$$

- Set $d : W_k^+ \times \boldsymbol{Int}\, W_k^- \to (0, 1)$,

$$\begin{aligned} d(w_+, w_-) &= \frac{2T\big(\, w_+, w_-, \varepsilon_-(w_+), \varepsilon_+(w_-)\,\big)}{\sqrt{1 + 4T\big(\, w_+, w_-, \varepsilon_-(w_+), \varepsilon_+(w_-)\,\big)^2}} \\ &= \tau\Big(\, 2T\big(\, w_+, w_-, \varepsilon_-(w_+), \varepsilon_+(w_-)\,\big)\,\Big). \end{aligned}$$

- Define

$$\mathfrak{T} = \bigcup_{(w_-, w_+) \in \boldsymbol{Int}\, W_k^+ \times \boldsymbol{Int}\, W_k^-} [d(w_-, w_+), 2] \times \mathcal{N}_k^-(w_+, \varepsilon_-(w_+)\,) \times \mathcal{N}^+(w_-, \varepsilon_+(w_-)\,).$$

• Define $\pi : \mathcal{T} \to \boldsymbol{Int}\, W_k^+ \times \boldsymbol{Int}\, W_k^-$ by $\pi(t,x) = \big(\pi_k^+(x), \pi_k^-(x)\big)$ where $\pi_k^\pm$ is the projection onto the affine plane spanned by $W_k^\pm$ and parallel with the plane spanned by $W_k^\mp$. The fiber of π over (w_+, w_-) is the PL ball

$$\mathcal{B}(w_+,w_-) := [d(w_-,w_+),2] \times \mathcal{N}_k^-(w_+,\varepsilon_-(w_+)\,) \times \mathcal{N}^+(w_-,\varepsilon_+(w_-)\,)$$

Then Γ_{reg}^t intersects the boundary of the ball $\mathcal{B}(w_+,w_-)$ exactly once, in the region

$$\{d(w_+,w_-)\} \times \mathcal{N}_k^-(w_+,\varepsilon_-(w_+)\,)_{reg} \times \mathcal{N}^+(w_-,\varepsilon_+(w_-)\,)_{reg}.$$

That intersection is transversal. Using the generalized Stokes formula, Remark 12.5(a), and the equality (12.6) we obtain the following result.

THEOREM 12.10. *Consider and affine m-simplex $\Delta_m = [\boldsymbol{e}_0, \dots, \boldsymbol{e}_m]$, and an orientation $\boldsymbol{or}_m$ on its relative interior. Denote by Φ the simplicial flow determined by the above ordering of the vertices of Δ_m. Equip the cartesian product $W_k^+ \times W_k^-$ with the orientation $\boldsymbol{or}_k^+$ defined by the property that the map*

$$W_k^+ \times W_k^- \ni (w_+,w_-) \mapsto w_+ + w_- - \boldsymbol{e}_k$$

is an orientation preserving map from $W_k^+ \times W_k^-$ to the affine plane spanned by Δ_m and equipped with the orientation $\boldsymbol{or}_m$. Then

$$\sum_k [W_k^+ \times W_k^-, \boldsymbol{or}_k^+] - [\Gamma^0] = \partial[\Gamma^{[0,\infty)}\,] + [\Gamma_\partial^{[0,\infty)}].$$

Similarly, we define an orientation $\boldsymbol{or}_k^-$ on $W_k^- \times W_k^+$ with the property that the switch map

$$(W_k^- \times W_k^+, \boldsymbol{or}_k^-) \to (W_k^+ \times W_k^-, \boldsymbol{or}_k^+)$$

is orientation preserving. Then

$$[\Gamma^0] - \sum_k [W_k^- \times W_k^+, \boldsymbol{or}_k^-] = \partial[\Gamma^{(-\infty,0]}\,] + [\Gamma_\partial^{(-\infty,0]}]. \qquad \square$$

We would like to use the above result, and the technique of Harvey-Lawson [**23**] to construct a canonical chain homotopy between the DeRham complex of a compact, real analytic manifold, and the simplicial chain complex associated to a tame triangulation of the manifold. Before we do this we would like to clarify a few issues.

Suppose M is a compact, orientable, real analytic manifold without boundary. We assume M is embedded in some Euclidean space E. Let $m := \dim M$. We fix an orientation $\boldsymbol{or}_M$ on M and a tame triangulation of M, which is a pair $(\mathcal{K}, \Delta)$, where $\mathcal{K}$ is a CSC and Δ is a tame homeomorphism

$$\Delta : |\mathcal{K}| \to M.$$

We assume that the restriction of Δ on the relative interiors of the faces of $\mathcal{K}$ is C^2.

For every (combinatorial) face $S \in \mathcal{K}$ we denote by Δ_S the image of the closed face $|S| \subset |\mathcal{K}|$ via the homeomorphism Δ, and by Δ_S° the image via Δ of the relative interior of $|S|$. We fix orientations $\boldsymbol{or}_S$ on Δ_S° so that the orientations on the top dimensional faces coincide with the orientations induced by the orientation of M.

We denote by $C_j(\mathcal{K}, M)$ the subgroup of tame integral currents $\mathcal{T}_j(M)$ spanned by $[\Delta_S, \boldsymbol{or}_S]$, $\#S = j+1$. The chain complex $(C_\bullet(\mathcal{K}, M), \partial)$ is isomorphic to the simplicial chain complex associated to $\mathcal{K}$. We form a cochain complex $(C^k_{\mathbb{R}}(\mathcal{K}), \delta)$ by setting

$$C^k_{\mathbb{R}}(\mathcal{K}) := C_{m-k}(\mathcal{K}) \otimes \mathbb{R}, \quad \delta = \partial.$$

We see that this cochain complex is naturally isomorphic to the simplicial chain complex with real coefficients determined by $\mathcal{K}$.

Consider the barycentric subdivision $D\mathcal{K}$ of $\mathcal{K}$. We denote by b_S the vertex of $D\mathcal{K}$ corresponding to the (open) face S of $\mathcal{K}$. We have a canonical homeomorphism $|D\mathcal{K}| \to |\mathcal{K}|$, and we thus a tame homeomorphism

$$\Delta' : |D\mathcal{K}| \to M.$$

We set

$$x_S := \Delta'(b_S) \in \Delta_S \subset M.$$

The simplicial complex $D\mathcal{K}$ is the nerve of the poset $(\mathcal{K}, \subset)$. We have a natural admissible function on the poset $\mathcal{K}$,

$$f : \mathcal{K} \to \mathbb{Z}, \quad f(S) = \dim S.$$

This defines a dynamical ordering of $D\mathcal{K}$, and thus a tame flow Ψ on $|D\mathcal{K}|$ and, via Δ', a conjugate tame flow Φ on M. We will refer to these flows as the *Stieffel flows* determined by a triangulation of M. The simplices Δ_S are invariant subsets of the Stieffel flow on M. The phase portrait of the Stieffel flow on a 2-simplex Δ_S is depicted in Figure 19.

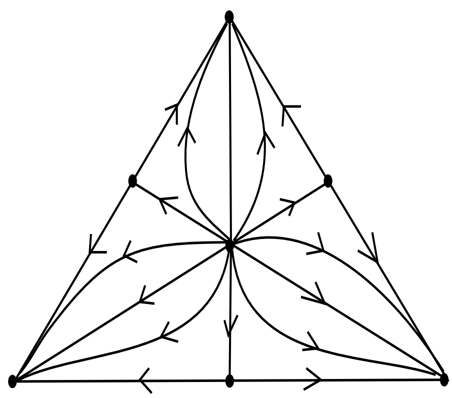


FIGURE 19. *The Stieffel flow on a triangle.*

From the definition, it follows immediately that the only stationary points of the flow Φ are the barycenters x_S, and the unstable variety of x_S is the open face Δ°_S. It is equipped with the orientation $\boldsymbol{or}_S$.

If Δ_S is a face of dimension k with barycenter x_S, then we define the *normal star* of x_S to be the union of all $(m-k)$ simplices of the barycentric subdivision whose vertices are barycenters of faces $T \supseteq S$. We denote by $\mathbf{St}^\perp(x_S)$ the normal star. It is a tame $(m-k)$-manifold with boundary. Its boundary is called the normal link of x_S, and it is denoted by $\mathbf{Lk}^\perp(x_S)$.

For a barycenter S, we denote by $\mathcal{M}_S$ the collection of maximal faces of $\mathcal{K}$ which contain S. Each $\Sigma \in \mathcal{M}_S$ determines a top dimensional face $\Delta_\Sigma \supset \Delta_S$. Δ_Σ is a Φ invariant set and we denote by Φ^Σ the restriction of Φ to Δ_Σ. The barycenter

x_S is a stationary point of Φ^Σ. We denote by $W^+_{S,\Sigma}$ the stable variety of x_S in Δ_Σ with respect to Φ^Σ. We have the equality

$$W^+_{S,\Sigma} = \big(\mathbf{St}^+(x_S) \setminus \mathbf{Lk}^\perp(x_S)\big) \cap \Delta_\Sigma.$$

We deduce that the stable variety of x_S in M with respect to the flow Φ is

$$W^\perp_S = \bigcup_{\Sigma\in\mathcal{M}_S} W^+_{S,\Sigma} = \mathbf{St}^\perp(x_S) \setminus \mathbf{Lk}^\perp(x_S).$$

We have a natural homeomorphism

$$h_S : \mathbf{St}^\perp(x_S) \times \Delta^\circ_S \to \mathcal{T}_S,$$

where $\mathcal{T}_S$ is a tubular neighborhood of Δ°_S. Using the orientation $\boldsymbol{or}_S$ on Δ°_S, and the orientation $\boldsymbol{or}_M$ on $\mathcal{T}_S$, we obtain an orientation $\boldsymbol{or}^\perp_S$ on $\mathbf{St}^\perp(x_s)$ such that

$$\boldsymbol{or}^\perp_S \times \boldsymbol{or}_S \overset{h_S}{\longmapsto} \boldsymbol{or}_M.$$

This defines an orientation $\boldsymbol{or}^\perp_S$ on W^+_S.

Define again $\tau(t) = \frac{t}{\sqrt{1+t^2}}$.

$$\Gamma^\pm_M = \big\{\, (\tau(t), x, \Phi_t x\,) \in [-1,1]\times M\times M;\ \ \pm t\geq 0,\ \big\},$$
$$\Gamma^t_M = \big\{\, (x, \Phi_t x) \in M\times M \,\big\}.$$

Denote by $\mathcal{M}$ the set of maximal simplices of $\mathcal{K}$. For $\Sigma\in\mathcal{M}$ we define

$$\Gamma^\pm_\Sigma = \big\{\, (\tau(t), x, \Phi_t x\,) \in [-1,1]\times \Delta_\Sigma\times \Delta_\Sigma;\ \ \pm t\geq 0,\ \big\},$$
$$\Gamma^t_\Sigma = \big\{\, (x, \Phi_t x) \in M\times M \,\big\}.$$

As before, these tame sets are equipped with natural orientations and define currents $[\Gamma^\pm_M]$, $[\Gamma^\pm_\Sigma]$. Moreover

$$[\Gamma^\pm_M] = \sum_{\Sigma\in\mathcal{M}} [\Gamma^\pm_\Sigma].$$

Using Theorem 12.10 and the fact that

$$\sum_{\Sigma\in\mathcal{M}} \partial[\Delta_\Sigma, \boldsymbol{or}_M] = \partial[M, \boldsymbol{or}_M] = 0,$$

we deduce

$$\partial[\Gamma^+_M] = [\Gamma^\infty_M] - [\Gamma^0_M] = \sum_{S\in\mathcal{K}} [W^+_S, \boldsymbol{or}^\perp_S]\times[W^-_S, \boldsymbol{or}_S] - [\Gamma_0]$$
$$= \sum_{S\in\mathcal{K}} [\mathbf{St}^\perp(x_S), \boldsymbol{or}^\perp_S]\times[\Delta_S, \boldsymbol{or}_S] - \Gamma^0,$$

and similarly,

$$\partial[\Gamma^-] = [\Gamma^0] - [\Gamma^{-\infty} = [\Gamma^0_M] - \sum_{S\in\mathcal{K}} (-1)^{\dim S(m-\dim S)}[\Delta_S, \boldsymbol{or}_S]\times[\mathbf{St}^\perp(x_S), \boldsymbol{or}^\perp_S].$$

Now we can start using the formalism of kernels developed by Harvey-Lawson in [**23**]. For the reader's convenience we briefly recall it here.

Suppose that we are given a roof, i.e., a diagram of the form

$$X_0 \xleftarrow{\ f_0\ } Y \xrightarrow{\ f_1\ } X_1$$

where X_0, X_1, Y are oriented smooth manifolds, and f_0, f_1 are smooth maps. Assume K is a k-dimensional *kernel* for this roof, i.e., a k-dimensional current in Y such that f_0 is proper over $\operatorname{supp} K$. Then we obtain a linear map

$$K_\# : \Omega^m(X_1) \to \Omega_{k-m}(X_0), \;\; K_\# \alpha = (f_0)_*\big((f_1^*\alpha) \cap K \big).$$

The operator $K_\#$ is called the *linear operator associated to the kernel* K. We have the following homotopy formula

$$(\partial K)_\# \alpha = K_\#(d\alpha) + (-1)^m \partial K_\sharp \alpha, \;\; \forall \alpha \in \Omega^m(X_1). \tag{12.9}$$

Indeed,

$$K_\#(d\alpha) = (f_0)_*\big(d(f_1^*\alpha) \cap K \big) \overset{(12.1)}{=} (f_0)_*\big(f_1^*\alpha \cap \partial K - (-1)^m \partial(f_1^*\alpha \cap K) \big)$$
$$= (\partial K)_\# \alpha - (-1)^m \partial K_\# \alpha.$$

We can rewrite this in operator form

$$(\partial K)_\# = K_\# \circ d + (-1)^m \partial \circ K_\#. \tag{12.10}$$

Let us point out that if X_0, X_1 are compact, oriented smooth manifolds, $m_i = \dim X_i$, $F : X_0 \to X_1$ is a smooth map, and $K = \Gamma_F \subset X_0 \times X_1$ is the graph of F, then the map

$$[\Gamma_F]_\# : \Omega^m(X_1) \to \Omega_{m-m_0}(X_0),$$

is essentially the pullback by F. More precisely, for every $\alpha \in \Omega^m(X_1)$ we have

$$[\Gamma_F]_\# \alpha = (-1)^{m(m_0-m)}\big((F^*\alpha) \cap [X_0, \boldsymbol{or}_{X_0}] \big).$$

We apply this formalism to the roof

$$M \xleftarrow{\;\pi_0\;} \mathbb{R} \times M \times M \xrightarrow{\;\pi_1\;} M, \qquad \pi_i(t, x_0, x_1) = x_i, \;\; i = 0, 1,$$

and the currents

$$[\Gamma_M^-] \in \Omega_{m+1}(\mathbb{R} \times M \times M), \;\; [\Gamma_M^0], [\Gamma_M^{-\infty}] \in \Omega_m(\mathbb{R} \times M \times M).$$

Clearly $[\Gamma_M^-]$, $[\Gamma_M^0]$ and $[\Gamma_M^{-\infty}]$ are kernels for this roof, and

$$\partial[\Gamma_M^-] = [\Gamma_M^0] - [\Gamma_M^{-\infty}].$$

Since M does not have boundary we deduce $\partial[\Gamma_M^{-\infty}] = 0$. We obtain operators

$$[\Gamma_M^0]_\#, \;\; [\Gamma_M^{-\infty}]_\# : \Omega^j(M) \to \Omega_{m-j}(M),$$

and

$$[\Gamma_M^-]_\# : \Omega^j(M) \to \Omega_{m+1-j}(M),$$

satisfying for every $\alpha \in \Omega^j(M)$ the equalities

$$[\Gamma_M^0]_\# \alpha - [\Gamma_M^{-\infty}]_\# \alpha = [\Gamma_M^-]_\# d\alpha + (-1)^j \partial [\Gamma_M^-]_\# \alpha, \tag{12.11}$$

and

$$[\Gamma_M^0]_\sharp \alpha = (-1)^{j+1} [\Gamma_M^{-\infty}]_\# d\alpha = (-1)^{j+1} \partial [\Gamma_M^{-\infty}]_\# \alpha. \tag{12.12}$$

Observe that

$$[\Gamma_M^0]_\# \alpha = \alpha \cap [M, \boldsymbol{or}_M],$$

and

$$[\Gamma_M^{-\infty}]_\# \alpha = \sum_{S\in\mathcal{K},\,\dim S = m-k} (-1)^{k(m-k)} \big\langle \alpha,\, [\mathbf{St}^\perp(x_S), \boldsymbol{or}_S^\perp]\,\big\rangle [\Delta_S, \boldsymbol{or}_S].$$

The equality (12.12) shows that the maps

$$[\Gamma_M^0]_\#,\ \ [\Gamma_M^{-\infty}]_\# : (\Omega^\bullet(M), d) \to (\Omega_{m-\bullet}(M), \partial)$$

are morphisms of chain complexes, while the equality (12.11) shows that they are chain homotopic. The morphism $[\Gamma_M^0]_\sharp$ is one-to-one, while the image of the morphism $[\Gamma_M^{-\infty}]_\#$ is the simplicial complex $(C^\bullet_{\mathbb{R}}(\mathcal{K}), \delta)$. We have obtained the following result.

THEOREM 12.11. *The Stieffel flow associated to a tame triangulation of a compact, real analytic manifold without boundary determines a chain homotopy between the DeRham complex and the simplicial chain complex with real coefficients associated to that triangulation.* □

REMARK 12.12. In the above proof, the tameness assumption is needed only to guarantee that the flow Φ is a finite volume flow on M. We can reach this conclusion under weaker assumptions. We know that the flow Ψ on the geometric realization $|\mathcal{K}|$ is tame, and thus has finite volume. If the homeomorphism $\Delta : |\mathcal{K}| \to M$ happens to be bi-Lipschitz then the flow Φ will also have finite volume.

If M is only a smooth, then the triangulation procedure employed by H. Whitney in [**49**, Chap. IV.B] produces triangulations with this property. In this case, for every $t \in \mathbb{R}$ the map $\Phi_t : M \to M$ is bi-Lipschitz because the conjugate map $\Psi_t : |\mathcal{K}| \to |\mathcal{K}|$ is such. Then, for every smooth form $\alpha \in \Omega^k(M)$ the pullback $\Phi_t^*\alpha$ is defined almost everywhere and it is a form with L^∞-coefficients. Moreover

$$[\Gamma_M^t]_\#\alpha = \Phi_t^*\alpha \cap [M, \boldsymbol{or}_M].$$

The current $[\Gamma_M^t]$ converges in the flat norm to $[\Gamma^{-\infty}]$ as $t \to -\infty$, and we deduce that $\Phi_t^*\alpha \cap [M, \boldsymbol{or}_M]$ converges in the sense of currents to

$$[\Gamma_M^{-\infty}]_\# \alpha = \sum_{S\in\mathcal{K},\,\dim S = m-k} (-1)^{k(m-k)} \big\langle \alpha,\, [\mathbf{St}^\perp(x_S), \boldsymbol{or}_S^\perp]\,\big\rangle [\Delta_S, \boldsymbol{or}_S].$$

Intuitively, this means that as $t \to -\infty$ the form $\Phi_t^*\alpha$ begins to concentrate near the barycenters x_S, and along the normal planes to the face Δ_S°. □

REMARK 12.13. Suppose M is a compact, orientable real analytic manifold, $m = \dim M$. Fix an orientation $\boldsymbol{or}_M$ on M, and a tame Morse pair (ξ, f). We denote by Φ the flow generated by ξ. For every $p \in \mathbf{Cr}_\Phi$ we denote by $\lambda(p)$ the Morse index of p, and by $W^\pm(p)$ the stable/unstable manifold of p with respect to Φ.

Suppose that the flow Φ is tame and satisfies the *dimension condition*

$$q, p \in \mathbf{Cr}_\Phi \ \text{ and } \ W^-(q) \cap W^+(p) \neq \emptyset \Longrightarrow \lambda(q) > \lambda(p).$$

We fix orientations $\boldsymbol{or}_p^\pm$ on $W^\pm(p)$ such that the natural map

$$T_pW^-(p)M \oplus T_pW^+(p) \to T_pM$$

is an isomorphism of *oriented* vector spaces. Arguing as in the proof of Theorem 12.10 we deduce that

$$\partial[\Gamma_\Phi^{[-\infty,0]}] = [\Gamma_\Phi^0] - [\Gamma_\Phi^{-\infty}] = [\Gamma_\Phi^0] - \sum_{p\in\mathbf{Cr}_\Phi} [W^-(p)\times W^+(p), \boldsymbol{or}_p^- \times \boldsymbol{or}_p^+]$$

Then for every $\alpha \in \Omega^k(M)$ we have

$$[\Gamma^{-\infty}]_\# \alpha = (-1)^{k(m-k)} \sum_{\lambda(p)=m=k} \big\langle\, \alpha, [W^+(p), \boldsymbol{or}_p^+]\,\big\rangle \cdot [W^-(p), \boldsymbol{or}_p^-].$$

Denote by W the subspace of $\Omega_\bullet(M)$ spanned by the set $[W^-(p), \boldsymbol{or}_p^-]$, $p \in \mathbf{Cr}_\Phi$. If the flow Φ satisfies the Morse-Whitney condition[1] then the subspace W is a *subcomplex* of the complex of currents $(\Omega_\bullet(M), \partial)$.

This subcomplex is known as the Morse-Floer complex. We deduce that the Morse-Floer complex of a tame Morse pair (ξ, f) whose flow satisfies the Morse-Smale condtion is homotopic to the DeRham complex, and with the chain complex determined by a tame triangulation of M. □

[1]As explained in [**23**], the Morse-Smale condition suffices for W to be a subcomplex of $(\Omega_\bullet(M), \partial)$.

APPENDIX A

An "elementary" proof of the generalized Stokes formula

In this appendix we want to present a proof of the Stokes formula (12.2) which does not use the advanced results of geometric measure theory in [**20, 21**]. We continue to use the notations in the proof of Theorem 12.4.

We begin by constructing a system of tubes $(T_I, \pi_I, \rho_I \varepsilon_I)$ around the open faces $D_f(I)$ of D. As in the proof of [**4**, Prop. 7.1], for every $\theta_0 \in (0, \frac{\pi}{2})$ we can choose the tube system so that the following additional conditions are satisfied

$$T_I \cap T_J \neq \emptyset \Longleftrightarrow I \subset J \text{ or } J \subset I,$$

$$\forall y \in D_f \cap T_I \cap T_J : \left| \measuredangle\big(\nabla\rho_I(y), \nabla\rho_J(y) \big) - \frac{\pi}{2} \right| < \theta_0.$$

Define

$$\bar{\varepsilon}_I : T_I \to (0, \infty), \;\; \bar{\varepsilon}_I(y) := \varepsilon_I\big(\pi_I(y) \big), \;\; \mathfrak{T} := \bigcup_{\#I \leq m} T_I,$$

so that $\mathfrak{T}$ is an open neighborhood of $\boldsymbol{bd}(D)$.

As in the proof of [**4**, Prop. 7.1], we fix a C^3 definable function

$$h : [0, \infty] \to [0, 1], \;\; h(t) = \begin{cases} t & t \leq 1/3 \\ 1 & t > 2/3 \end{cases}$$

and define $\ell_I, \ell : \mathbb{R}^n \to [0, \infty)$ by

$$\ell(x) = \begin{cases} h\big(\frac{\rho_I}{\bar{\varepsilon}_I} \big) & y \in T_I \\ \\ 1 & y \notin T_I \end{cases}, \;\; \ell = \prod_{\#I \leq m} \ell_I.$$

We will say that ℓ is the *boundary profile* associated to the isolating system of tubes. As explained in the proof of [**4**, Prop. 7.1] the profile ℓ satisfying the following properties.

($\mathbf{P}_1$) $\ell^{-1}(0) = \boldsymbol{bd}(D)$.
($\mathbf{P}_2$) ℓ is C^3 on $\mathbb{R}^n \setminus \boldsymbol{bd}(D)$.
($\mathbf{P}_3$) For every open neighborhood U of $\mathbb{D}$ there exists $\varepsilon > 0$ such that $f^{-1}([0, \varepsilon]) \subset U$.
($\mathbf{P}_4$) There exists $\delta > 0$ such that any $t \in (0, \delta)$ is a regular value of ℓ.
($\mathbf{P}_5$) If $(x_k) \in \mathbb{R}^n \setminus \boldsymbol{bd}(D)$ is a sequence which converges to a point $x \in D_f(I)$, and if the line spanned by $\nabla\ell(x_k)$ converges to a line L_∞, then the limit line L_∞ is perpendicular to the tangent space $T_x D_f(I)$.

We have depicted in Figure 20 a tame 2-simplex, with a tube system and the associated boundary profile.

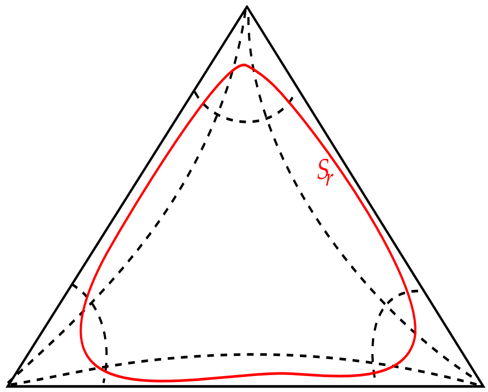


FIGURE 20. *A tube system around the boundary of a tame simplex (dotted lines) and its associated profile.*

For every $r \in (0,1)$ and $I \subsetneq \{0,\dots,m\}$ we denote by $T_I(r)$ the closed tube

$$T_I(r) := \{\, y \in T_I;\ \ \rho_I(y) \le r\bar{\varepsilon}_I(y)\,\}.$$

We set

$$\mathfrak{T}(r) := \bigcup_{\#I \le m} T_I(r),\ \ \mathfrak{T}^0(r) := \bigcup_{\#I < m} T_I(r).$$

Note that $\mathfrak{T}^0(r)$ is a neighborhood of the $(m-2)$-dimensional skeleton of D_f. Moreover, if $r \le \frac{1}{3}$ then

$$T_I(r) = \{\ell_I \le r\,\}.$$

We can find a tame function $\tau : (0,1) \to (0,1)$, $r \mapsto \tau(r)$ satisfying the following conditions.

- $\tau(r)$ is a regular value of $\ell|_{D^f}$.
- $D^f \cap \{\ell = \tau(r)\} \subset \mathfrak{T}(r)$.

We set $S_r := D^f \cap \{\ell = \tau(r)\}$. The set S_r is a compact, tame oriented C^3-submanifold of D_f of dimension $(m-1)$ which approaches $\boldsymbol{bd}(D^f)$ as $r \to 0$ (see Figure 20 for a 2-dimensional rendition of S_r). The manifold S_r has a natural orientation as boundary of

$$D_r := D \cap \{\ell \ge \tau(r)\,\}.$$

We will prove that

$$\forall \eta \in \Omega^{k-1}_{cpt}(\mathbb{R}^n):\ \lim_{r \searrow 0} \int_{S_r} \eta = \sum_{k=0}^{m} (-1)^k \langle \eta, [D^k_f, \boldsymbol{or}_k]\rangle. \tag{A.1}$$

Clearly, (A.1) implies (12.2).

For every $x \in D^k_f$, we denote by C_x the fiber of the projection

$$\pi_{I_k} : D \cap T_{I_k} \to D^k_f.$$

This fiber is a C^3-curve, and the map

$$C_x \to (0,1),\ \ y \mapsto s_x(y) := \frac{1}{\varepsilon_i(x)}\rho(y)$$

is a C^3-diffeomorphism. We think of s_x as a parameter along C_x so that the restriction of ℓ to C_x can be regarded as a function of one variable $s = s_x$.

LEMMA A.1. *There exists a definable function* $\delta \mapsto r_1 = r_1(\delta)$*, such that, for all* $x \in D_f^k \setminus \mathcal{T}^0(\delta)$*, and all* $r < r_0$*, the equation* $\ell(y) = \tau(r)$ *has at* exactly one solution $y(r,\delta) \in C_x$. *In other words, for any* $x \in D_f^k$*, and any* $r < r_1(\delta)$ *the manifold* S_r *intersects the fiber* C_x *at a single point* $y(r,\delta)$.

PROOF. Let $x \in D_f^k \setminus \mathcal{T}^0(\delta)$. Then, along C_x we can use the parameter $s = s_x$, and we can think of the restriction of ℓ to C_x as a C^3 function of a single variable s. Then

$$\ell|_{C_x} = \ell_{I_k}|_{C_x} \prod_{I \neq I_k} \ell_{I_k}|_{C_x}.$$

Observe that

$$\ell_{I_k}|_{C_x} = s,\ \ \ell(s) = \ell|_{C_x}(s) = su(s),\ \ u(s) := \prod_{I \neq I_k} \ell_{I_k}|_{C_x}(s).$$

There exists a constant $\nu = \nu(\delta) > 1$, *independent* of $x \in D_f^k \setminus \mathcal{T}^0(\delta)$, such that

$$u(s) > \frac{1}{\nu}, \tag{A.2a}$$

$$|u'(s)| < \nu,\ \ \forall s \in (0,1). \tag{A.2b}$$

To find one solution of the equation $su(s) = \tau(r)$ we regard it as a fixed point problem

$$s = g(s) = \frac{\tau(r)}{u(s)}.$$

Observe that $u(0) > 0$ so it suffices to have $g(s) \leq 1$, i.e.,

$$\tau(r) \leq u(s),\ \ \forall s \in [0,1].$$

Using (A.2a) we deduce that if

$$\tau(r) \leq \frac{1}{\nu(\delta)},$$

then we have at least one solution. In fact any, solution σ of this equation must satisfy the inequality

$$0 < \sigma < \nu(\delta)\tau(r)$$

To prove the uniqueness, it suffices show that the derivative of $s \mapsto su(s)$ is positive in the interval

$$0 < s < \min\big\{\, 1, \nu(\delta)\tau(r) \,\big\}.$$

We have

$$\ell'(s) = u(s) + su'(s) \overset{(A.2b)}{\geq} u(s) - \nu s \overset{(A.2a)}{\geq} \frac{1}{\nu} - \nu^2\tau(r).$$

Hence, if $\tau(r) < \frac{\delta}{\nu(\delta)^3}$, we also have uniqueness. Moreover, the unique solution satisfies

$$s < \min\{\delta, \nu(\delta)\tau(r)\},$$

i.e.,

$$y(r,\delta) \in T_{I_k}(\delta). \tag{A.3}$$

By definable selection, we can find a definable function $r_1(\delta)$ such that for all $r < r_1(\delta)$ we have $\tau(r) < \frac{\delta}{\nu^3}$. □

Set

$$D_f^k(\delta) = D_f^k \setminus \mathcal{T}^0(\delta),\ \ S_r^k(\delta) = \big\{\, y \in T_{I_k} \cap D;\ \ \pi_{i_k}(y) \in D_f^k(\delta)\,\big\}.$$

$$S_r(\delta) = S_r \setminus \bigcup_k S_r^k(\delta).$$

Let $y \in S_r$. For every *oriented, orthonormal* frame $\boldsymbol{f} = (f_1, \dots, f_{m-1})$ of $T_y S_r$ we get a scalar

$$\eta(\boldsymbol{f}, y) = \eta_y(f_1, \dots, f_{m-1}).$$

This scalar is *independent* of the frame $\boldsymbol{f}$, and thus defines a C^3-function η_r on S_r. Moreover, there exists $C_1 > 0$ such that

$$|\eta_r(y)| \leq C_1,\ \ \forall r,\ \ \forall y \in S_r^k.$$

Denote by $\mathcal{H}^{m-1}$ the $(m-1)$-dimensional Hausdorff measure. We have

$$\int_{S_r} \eta = \int_{S_r} \eta_k(y) d\mathcal{H}^{m-1}(y).$$

In particular

$$\Big|\int_{S_r(\delta)} \eta\Big| \leq C_1 \mathcal{H}^{m-1}(S_r(\delta)).$$

For $r < r_1(\delta)$ we have

$$\int_{S_r} \eta = \int_{S_r(\delta)} \eta + \sum_k \int_{S_r^k(\delta)} \eta.$$

Hence

$$\Big|\int_{S_r} \eta - \sum_{k=0}^m (-1)^k \int_{D_f^k} \eta\Big|$$

$$\leq \underbrace{\Big|\int_{S_r(\delta)} \eta\Big|}_{T_1(r,\delta)} + \underbrace{\sum_{k=0}^m \Big|\int_{S_r^k(\delta)} \eta - (-1)^k \int_{D_f^k(\delta)} \eta\Big|}_{T_2(r,\delta)} + \underbrace{\sum_{k=0}^m \Big|\int_{D_f^k} \eta - \int_{D_f^k(\delta)} \eta\Big|}_{T_3(\delta)}$$

We will prove the following things.

LEMMA A.2. *There exists a function $\varepsilon \mapsto \delta_1(\varepsilon)$ such that if $\delta < \delta_1(\varepsilon)$ and $r < r_1(\delta)$ then*

$$T_1(r, \delta) < \frac{\varepsilon}{3}.$$

LEMMA A.3. *There exists a function $\varepsilon \mapsto \delta_3(\varepsilon)$ then*

$$T_3(\delta) < \frac{\varepsilon}{3}.$$

LEMMA A.4. *There exists a function $(\varepsilon, \delta) \mapsto r_2(\delta, \varepsilon)$ such that if $r < r_2(\delta, \varepsilon)$ we have*

$$T_2(r, \delta) < \frac{\varepsilon}{3}.$$

Assuming the above results, the equality (A.1) is proved as follows. Fix $\varepsilon > 0$. Choose $\delta < \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$. Then, if $r < \min\{r_1(\delta), r_2(\delta, \varepsilon)\}$ we have

$$T_1(r,\delta) + T_2(r,\delta) + T_3(\delta) < \varepsilon.$$

Using (A.3) we deduce that if $r < r_1(\delta)$ then

$$S_r(\delta) \subset S_r \cap \mathcal{T}^0(2\delta).$$

Lemma A.2 and Lemma A.3 are both consequences of the following result.

LEMMA A.5. *Suppose X is a tame C^3-manifold of dimension $(m-1)$. Then*

$$\lim_{\delta\searrow 0} \mathcal{H}^{m-1}\big(X \cap \mathcal{T}^0(\hbar)\big) = 0.$$

PROOF. Denote by $\mathbf{Graff}^{m-1}$ the Grassmannian of affine planes in $\mathbb{R}^n$ of codimension $(m-1)$. Denote by μ_{m-1} and invariant measure on $\mathbf{Graff}^{m-1}$, and set

$$X_\hbar := \boldsymbol{cl}\big(X \cap \mathcal{T}^0(\hbar)\big).$$

Then, from Crofton formula (see [**4, 13**]) we deduce

$$\mathcal{H}^{m-1}\big(X \cap \mathcal{T}^0(\hbar)\big) = \mathcal{H}^{m-1}(X_\hbar) = \int_{\mathbf{Graff}^{m-1}} \chi(L \cap X_\hbar) d\mu_{m-1}(L).$$

The function

$$(0,1) \times \mathbf{Graff}^{m-1} \ni (\hbar, L) \mapsto \chi(L \cap X_\hbar)$$

is definable and thus its range is finite. From dominated convergence theorem we deduce

$$\lim_{\hbar\searrow 0} \mathcal{H}^{m-1}(X_\hbar) = \int_{\mathbf{Graff}^{m-1}} \lim_{\hbar\searrow 0} \chi(L \cap X_\hbar).$$

Suppose $L \in \mathbf{Graff}^{m-1}$ is such that

$$\chi_0(L) := \lim_{\hbar\searrow 0} \chi(L \cap X_\hbar) \neq 0.$$

Then the definable set $L \cap X_\hbar$ is nonempty for all $\hbar$ sufficiently small. In particular, we can find a definable function

$$\hbar \mapsto x_\hbar \in L \cap X_\hbar$$

defined in a neighborhood of 0. Then the limit $x_0 = \lim_{\hbar\searrow 0} x_\hbar$ exists and it is a point in the intersection of L with the $(m-2)$-skeleton of D. We denote this skeleton by $D^{(m-2)}$. Thus

$$\chi_0(L) \neq 0 \Longrightarrow L \cap D^{(m-2)} \neq \emptyset.$$

The function

$$\mathbf{Graff}^{m-1} \ni L \mapsto \chi_0(L) \in \mathbb{Z}$$

is definable and thus bounded. Hence

$$\int_{\mathbf{Graff}^{m-1}} |\chi_0(L)| d\mu_{m-1}(L) \leq C\mu_{m-1}\big(\{L \in \mathbf{Graff}^{m-1};\ \ L \cap D^{(m-2)} \neq \emptyset\}\big)$$

By Sard's theorem, the definable set

$$\{L \in \mathbf{Graff}^{m-1};\ \ L \cap D^{(m-2)} = \emptyset\}$$

is *dense*[1] in $\mathbf{Graff}^{m-1}$.

[1]A typical codimension $(m-1)$ affine plane will not intersect a manifold of dimension $\leq (m-2)$, and $D^{(m-2)}$ is a finite union of such manifolds, $D_f(I)$, $\#I \leq m-1$.

Hence, if $d = d(m,n)$ denotes the dimension of $\mathbf{Graff}^{m-1}$, then

$$\dim\{L \in \mathbf{Graff}^{m-1};\ \ L\cap D^{(m-2)} \neq \emptyset\} < d.$$

Up to a multiplicative constant $c > 0$ we have

$$\mu_{m-1} = c\mathcal{H}^d,$$

from which we deduce

$$\begin{aligned}&\mu_{m-1}\big(\{L \in \mathbf{Graff}^{m-1};\ \ L\cap D^{(m-2)} \neq \emptyset\}\big)\\ &\quad= \mathcal{H}^d\big(\{L \in \mathbf{Graff}^{m-1};\ \ L\cap D^{(m-2)} \neq \emptyset\}\big) = 0.\end{aligned}$$

This completes the proof of Lemma A.5. □

Lemma A.3 is clearly a special case of Lemma A.5. To see that Lemma A.2 is also a special case of Lemma A.5 observe that for every $\hbar > 0$ there exists $\delta_1(\hbar)$ such that if $\delta < \delta_1(\hbar)$ and $r < r_1(\delta)$ then $S_r(\delta) \subset \mathcal{T}^0(\hbar)$.

Proof of Lemma A.4. Fix $\delta < \frac{1}{2}$. We have to prove that for every $k = 0,\dots,m\}$ we have

$$\lim_{r\searrow 0}\int_{S_r^k(\delta)}\eta = (-1)^k\int_{D_f^k(\delta)}\eta.$$

By Lemma A.1, for $r < r_1(\delta)$, the projection π_{I_k} induces a homeomorphism

$$S_r^k(\delta) \to D_f^k(\delta).$$

For simplicity we write $\bar{y} = \pi_{I_k}(y)$. We want to prove that

$$\lim_{r\to 0}\sup_{y\in S_r^k(\delta)}\operatorname{dist}\big(T_yS_r^k(\delta), T_{\bar{y}}D_f^k(\delta)\big) = 0. \tag{A.4}$$

We argue by contradiction. We can therefore find a constant $c > 0$ and definable map $r \mapsto y_r \in S_r(\delta)$ such that

$$\operatorname{dist}\big(T_{y_r}S_r^k(\delta), T_{\bar{y}_r}D_f^k(\delta)\big) > c,\ \ \forall r < r_(\delta). \tag{A.5}$$

Both limits $\lim_{r\to 0} y_r$ and $\lim_{r\to 0}\bar{y}r$ exist and they coincide with a point $y_0 \in \boldsymbol{cl}(D_f^k(\delta)) \subset D_f^k$. From the Whitney regularity condition (a) and the property $(\mathbf{P}_5)$ of the boundary profile ℓ we deduce

$$\lim_{r\to 0} T_{y_r}S_r^k(f) = T_{y_0}D_f^k.$$

Clearly

$$\lim_{r\to 0} T_{\bar{y}_r}D_f^k(\delta) = T_{y_0}D_f^k.$$

This contradicts (A.5) and completes the proof of (A.4).

The equality (A.4) show that the map

$$S_r^k(\delta) \ni y \mapsto \bar{y} \in D_f^k(\delta)$$

is a C^3-diffeomorphism for r sufficiently small, and changes the orientation by a factor of $(-1)^k$. For every $y \in D_f^k(\delta)$ we denote by $\eta(y)$ the pairing between η and oriented orthonormal frame of $T_yD_f^k$.

Using the change in variables formula we can write

$$\int_{S_r^k(\delta)}\eta = (-1)^k\int_{D_f^k(\delta)} J_r(\bar{y})\eta_r(\bar{y})d\mathcal{H}^{m-1}(\bar{y}),$$

where $\eta_r(\bar{y})$ is the pullback of the function $\eta_r|_{S_r^k(\delta)}$ to $D_f^k(\delta)$, and $J_r(\bar{y})$ is the Jacobian of the change in variables. The equality (A.4) and the continuity of the form η imply that

$$\lim_{r \to 0} J_r(\bar{y}) = 1 \quad \text{and} \quad \lim_{r \to 0} \eta_r(\bar{y}) = \eta(\bar{y})$$

uniformly on $D_f^k(\delta)$. This completes the proof of Lemma A.4 and of Theorem 12.4. □

APPENDIX B

On the topology of tame sets

We would like to include a few topological facts concerning tame sets. These are not needed in the main body of the paper, yet they may shed some light on the subtleties of tame topology.

As we mentioned in Section 1, any compact tame set S can be triangulated, i.e., there exists an affine finite simplicial complex, and a tame homeomorphism $\varphi : K \to S$.

Clearly, if $\varphi_i : K_i \to S$, $i = 0, 1$, are two tame triangulations, then the map

$$\varphi_1 \circ \varphi_0^{-1} : K_0 \to K_1$$

is a tame homeomorphism. It turns out that the existence of a tame homeomorphism between two compact PL spaces imposes a severe restriction on these spaces. More precisely, M. Shiota has proved (see [**41**, Chap. IV]) the *tame Haupvermutung*, namely that two compact PL spaces are PL-homeomorphic if and only if they are tamely homeomorphic. Given this result, we can define the link of a point in a compact tame space to be its PL link as defined e.g. in [**39**, Chap. 2].

To appreciate the strength of Shiota's result, consider the classical example of Cannon-Edwards [**6, 31**], the double suspension of a non-simply connected homology 3-sphere, say the Poincaré sphere $\Sigma(2,3,5)$. This is a simplicial complex K which is homeomorphic, *but not PL-equivalent* to the 5-sphere, equipped with the triangulation as boundary of a 6-simplex. The tame *Hauptvermutung* implies that X and S^3 are not tamely homeomorphic. In particular, there cannot exist a semi-algebraic homeomorphism from the round 5-sphere to X.

In the above paragraphs, we have defined the link of a point in a compact tame space indirectly, via triangulations and the tame Hauptvermutung. We can attempt a more intrinsic approach, namely given a point p_0 in a compact tame set X, and a tame continuous function $w : X \to [0, \infty)$, such that $w^{-1}(0) = \{p_0\}$, we can define the link of p_0 as the level set $\{w = \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small. The homeomorphism type of this set is independent of $\varepsilon > 0$, but at this point, we do not know how to eliminate the dependence on w.

To understand the subtleties of this question consider a closely related problem.

Suppose $w : \mathbb{R}^n \to [0, \infty)$ is a tame continuous function such that $w^{-1}(0) = \{0\}$. Then there exists $r_0 > 0$ such that for every $x \in \mathbb{R}^n$, $|x| = 1$, the function

$$[0, r_0] \ni t \mapsto w(tx)$$

is strictly decreasing, i.e., in a neighborhood of 0 the function w is a Lyapunov function for the radial flow.

If w is a C^1-function, then this fact follows from the non-depravedness arguments in [**17**, Sec. 2.4]. When w is merely continuous (and tame), this seems to be a rather slippery problem.

Let us observe that if Φ is a gradient-like tame flow on the compact tame space, x_0 is an isolated stationary point of Φ and $u, v : X \to \mathbb{R}$ are two Lyapunov functions such that $u(0) = v(0) = 0$,, then unstable links

$$\mathcal{L}_u^-(x_0) = W^-(x_0) \cap \{u = -\varepsilon\},\ \ \mathcal{L}_v^-(x_0) = W^-(x_0) \cap \{v = -\varepsilon\},$$

are *tamely* homeomorphic for ε sufficiently small. In other words, the tame homeomorphism type of the unstable link, is a *dynamical* invariant of the stationary point.

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Index