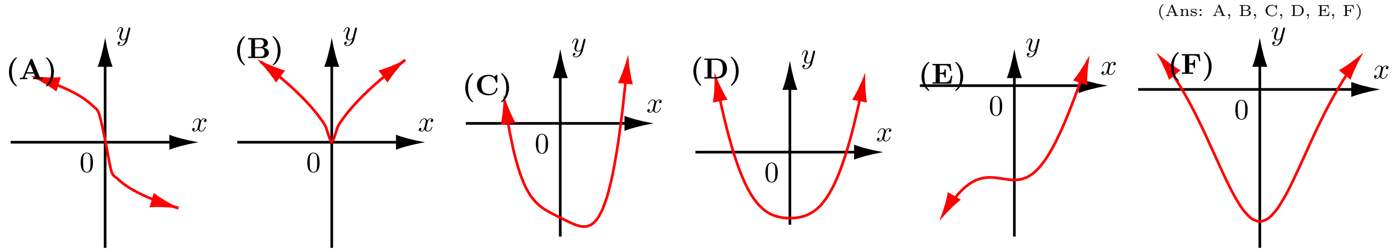
# Math 10250 Final Review

1. Match the following functions with the given graphs without using your calculator:

$f_1(x) = -x^{1/3}$ $\qquad f_2(x) = x^{2/3}$ $\qquad f_3(x) = x^4 - x - 5$

$f_4(x) = \dfrac{5x^4 - 25}{x^2+5}$ $\qquad f_5(x) = \dfrac{5x^3 - 25}{x^2+5}$ $\qquad f_6(x) = \dfrac{5x^2 - 25}{x^2+5}$

(Ans: A, B, C, D, E, F)

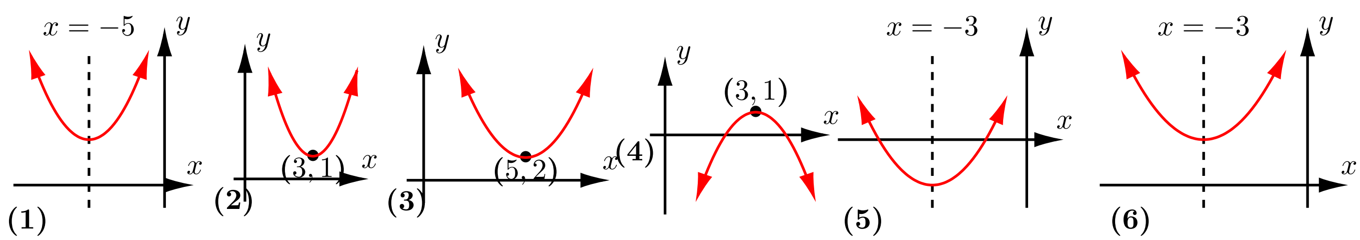



2. Match the graphs to the given quadratic functions. Some graphs are redundant.

$f_1(x) = (x-5)^2 + 2$ $\qquad f_2(x) = a(x-3)^2 + 1 \quad (a < 0)$

$f_3(x) = b(x+3)^2 + 1 \quad (b > 0)$ $\qquad f_4(x) = (x+5)^2 + 2$

(Ans: 3, 4, 6, 1)




3. Find the inverse of the following functions and give its domain.

**a.** $f(x) = e^{-(2x+5)}$ $\qquad$ **b.** $g(x) = 5\ln(2x) + 3$ $\qquad$ **c.** $h(x) = \dfrac{2x-3}{x+1}$

(Ans: $y = -(5 + \ln x)/2$; $x > 0$, $y = 0.5e^{(x-3)/5}$; $-\infty < x < \infty$, $y = (x+3)/(2-x)$ $x \neq 2$)

4. When the price $p$ for a trip to the Bahamas on a cruise ship is \$800, the demand $x$ is 2,000 tickets annually. However, when the price drops to \$400 per ticket then the demand rises to 10,000 tickets. On the cost side, the cruise liner has a \$2,500,000 fixed cost and \$50 expenses per passenger.

   (i) Assuming the demand function $p(x)$ is linear, find a formula for it. (Ans: $p = -0.05x + 900$)

   (ii) Find the cost function $C(x)$. (Ans: $C = 50x + 2,500,000$)

   (iii) Find the revenue function $R(x)$. (Ans: $R = -0.05x^2 + 900x$)

   (iv) Find the profit function. (Ans: $P = -0.05x^2 + 850x - 2,500,000$)

   (v) By complete the square in the profit function, at what level should the company set the price in order to maximize the profit? What is the maximal profit? (Ans: $p$ = \$475; \$1,112,500)

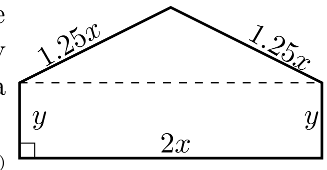
5. The height of a ball thrown upwards from the top of a 1200 ft tall tower with initial velocity 160 feet per second is given by the formula $s(t) = -16t^2 + 160t + 1200$. Assume that the position of the ball is measured from the base of the building.

   (i) Complete the square in order to write $s(t)$ in the form $s(t) = a(t-h)^2 + k$. (Ans: $s(t) = 16(t-5)^2 + 1600$)

   (ii) At what time does the ball reach its maximum height and what is this height? (Ans: $t = 5 \quad h = 1600$)

   (iii) At what time does the ball hit the ground? What is velocity at that moment? (Ans: $t = 15 \quad v = -320$)

6. An landscaper has 30 ft of fencing and wishes to enclosed a five sided figure as shown. Find an expression for the area $A$ enclosed in terms of $x$. By completing the square, find the values of $x$ and $y$ that maximizes the area $A$. What is the maximum area that can be enclosed?

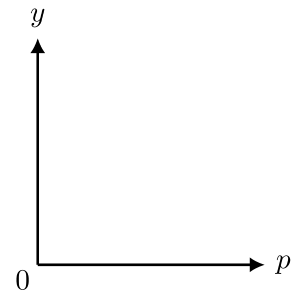



(Ans: $x = 4$ft, $y = 6$ft, max $A = 60\text{ft}^2$)

7. The demand function $D$ and supply function $S$ of a model of jeans for the clothing company Lucky Clover's Wear are given, in term of price p, by

$$D(p) = e^{4-2p} \qquad\qquad S(p) = e^{p+1}$$

**a.** Give a sketch of the demand and supply functions in the axes provided.
**b.** Find the equilibrium price and equilibrium quantity.




(Ans: $p_e = 1$, $q_e = e^2$)

8. Imagine that you are offered two retirement plans.

   (a) When you are 30 years old \$80,000 is deposited in to an IRA, earning an annual interest rate of 8% compounded continuously, till you become 65 years old. (Ans: $80000e^{0.08(35)}$)

   (b) When you are 65 you receive \$1,500,000.

   Which one will you choose?

9. Suppose the half-life of a radioactive substance is 5 years. How long will it take for the substance to be reduced to 20% of its initial amount. $t = 5\ln 5/\ln 2$

10. What should the annual interest rate (compounded continuously) be in order an amount of money earning this rate doubles in 10 years?

11. Consider the rational function $\dfrac{x^2+2x-8}{x-2}$ where $x \neq 2$.

   (i) To make a guess about the value of $\lim\limits_{x\to 2} \dfrac{x^2+2x-8}{x-2}$ fill the following table:

| | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $\dfrac{x^2-2x-8}{x-2}$ | | | | | |

   (ii) Use algebra to compute the exact value of the limit $\lim\limits_{x\to 2} \dfrac{x^2+2x-8}{x-2}$. (Ans: 6)

   (iii) Let $f(x) = \begin{cases} \dfrac{x^2+2x-8}{x-2}, & \text{if } x \neq 2 \\ 0, & \text{if } x = 2 \end{cases}$ Use limits to determine whether $f(x)$ is continuous?

12. Assume that a population grows according to the (exponential) model $\frac{dP}{dt} = 0.02P$. If the population now is 5 millions, use linear approximation to estimate this population 10 years later. (Ans: 6 millions)

13. Assume that a population grows according to the (Logistic) model $\frac{dP}{dt} = 0.02P(1-0.1P)$. If the population now is 5 millions, use linear approximation to estimate this population 10 years later. (Ans: 5.5 millions)

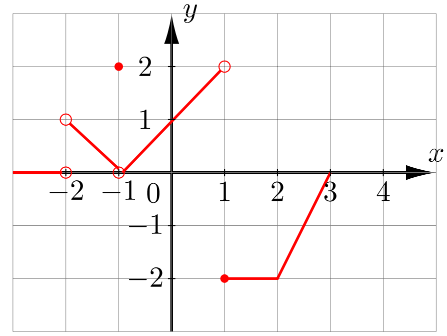
14. The graph of the function $f(x)$ is given in figure below. Find exactly or state that it does not exist each of the following quantities. If it does not exist explain why.

(a) $\lim_{x\to 2} f(x) \overset{?}{=}$ (b) $\lim_{x\to 1^-} f(x) \overset{?}{=}$

(c) $\lim_{x\to 1} f(x) \overset{?}{=}$ (d) $\lim_{x\to -1} f(x) \overset{?}{=}$

(e) $\lim_{h\to 0} \dfrac{f(h)-f(0)}{h} \overset{?}{=}$ (f) $\lim_{h\to 0} \dfrac{f(2+h)+2}{h} \overset{?}{=}$

(Ans: $-2$, 2, does not exist, 0, 1, does not exist)



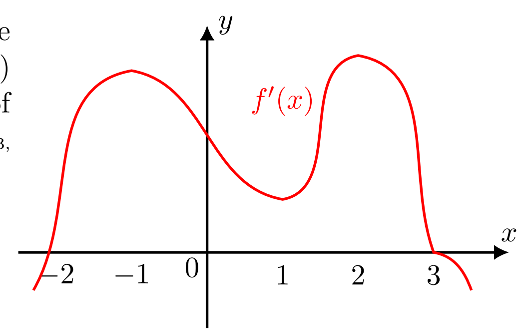

15. For each of the following function $f(x)$ form the difference quotient $\dfrac{f(x+h)-f(x)}{h}$, which is also called the average of $f$ from $x$ to $x+h$, and simplify it so that the quantity $h$ in the denominator is canceled. (i) $f(x) = x^2 - 5x + 1$ (ii) $f(x) = 3/x$ (iii) $f(x) = \sqrt{x}$

16. Let $p(x) = (x^3 - 5x + 1)g(x)$, $q(x) = \dfrac{x^3 - 5x + 1}{g(x)}$, and $r(x) = g(x^3 - 5x + 1)$. Given that $g(-2) = 2$, $g(1) = 3$, $g'(-3) = -1$ and $g'(1) = -4$ find the values of $p'(1)$, $q'(1)$, and $r'(1)$. (Ans: $p'(1) = 6$, $q'(1) = -2$, $r'(1) = 2$)

17. A block of ice with a square base has dimension $x$ inches by $x$ inches by $3x$ inches. If the block of ice is melting so that its surface area $A$ is decreasing at a rate of 2 in$^2$/sec, find the rate at which $x$ is changing when $x = 12$ inches. (Ans: $x' = \frac{-1}{168}$)

18. The carbon dioxide emissions of a small country in January of 2006 is 80% of its emission level in 1996, and is reducing at a rate of 2.5% per year. Let $t$ be the time in years measured from January of 1996. (a) Use linear approximation to find an expression for the percentage $P(t)$ of the emission level of the country in the near future of 2006, (b) use your answer in (a) to estimate the percentage of carbon dioxide emission in June of 2006.

19. The graph of the **derivative** of $f(x)$ is as shown. **(a)** What are the critical points of $f(x)$? **(b)** Find the values of $x$ for which $f(x)$ is (i) increasing, and (ii) decreasing. **(c)** What are the inflection points of $f(x)$. Determine the concavity of $f(x)$? (Ans: CP: $x = -2, x = -3$, Increasing: $-2 < x < 3$, Decreasing: $x < -2$, $x > 3$, Inflection pts: $x = -1, 1, 2$)



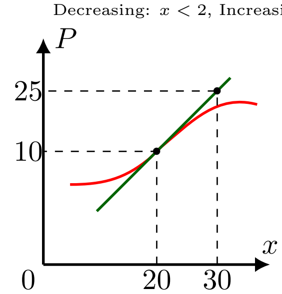

20. (a) Sketch the graph of $f$ given that:

- f(0)=2
- $f'(2) = 0$, $f'(x) < 0$ for $x < 2$ and $f'(x) > 0$ for $x > 2$.
- $f''(4) = 0$, $f''(x) > 0$ for $x < 4$ and $f''(x) < 0$ for $x > 4$.

(b) Determine where $g(x) = e^{f(x)}$ is (1) increasing, and (2) decreasing.

Decreasing: $x < 2$, Increasing $x > 2$

21. The graph of the profit $P(x)$ (in million of dollars) from the sales of a chemical substance ($x$ is the amount sold in million of gallons) is shown in the figure.
**a.** Find the marginal profit at $x = 20$ (Ans: $\frac{3}{2}$)
**b.** Find the linear approximation of $P(x)$ at $x = 20$. (Ans: $L(x) = \frac{3}{2}x - 20$)
**c.** Estimate the profit when $x = 20.5$ (Ans: $L(20.5) = 10.75$)




22. Find the exact value of (i) $\lim_{h\to 0} \dfrac{(3+h)^6 - 3^6}{h}$ (ii) $\lim_{h\to 0} \dfrac{\ln(8+h) - \ln 8}{h}$ (iii) $\lim_{h\to 0} \dfrac{e^{4+2h} - e^4}{h}$

(*Hint:* Think derivatives!)

(Ans: $(i) 6(3)^5$ (ii) $\frac{1}{8}$ (iii) $2e^4$)

23. For each of the following functions: (i) $f(x) = \dfrac{2x}{x+3}$ (ii) $f(x) = \dfrac{x^2-16}{x^2-1}$ (iii) $f(x) = \dfrac{x+1}{x^2-2x-3}$

(Ans: (i) VA:$-3$ HA: 2 (ii) VA: $\pm 1$, HA: 1 (iii) VA: 3, HA: 0)

find vertical asymptote(s), horizontal asymptote(s), $y$-intercept and its zero(s). Then, sketch the graph.

24. Let $g$ be a differentiable function such that its derivative is given by $g'(x) = -(x-5)(x-10)$. Find where the function $g$ has a local maximum and a local minimum, if any. (Ans: Min $x = 5$, Max $x = 10$)

25. Differentiate $1/\sqrt{x}$ with respect to $x$, and use the result to find an approximate value for $1/\sqrt{100.5}$.

(Ans: $L(100.5) = \frac{1}{10} - \frac{1}{4000}$)

26. The following table gives the values of the **continuous** function $f(x)$. For which intervals on the $x$-axis could you be sure that $f(x)$ has a zero.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $-0.5$ | $-2$ | $1$ | $3$ | $0.5$ | $1$ |

27. In economics, a utility function $u$ assigns $u(x)$ units of satisfaction (utiles) to $x$ units of consumption. It is required to satisfy the conditions:

a) $u'(x) > 0$ (the more the consumption the more the satisfaction)

b) $u''(x) < 0$ (each additional unit of consumption gives less satisfaction)

Show that if $\gamma > 0$ and $\gamma \neq 1$, then $u(x) = \dfrac{x^{1-\gamma}}{1-\gamma}$, is a utility function.

28. For the function $f(x) = -x^3 + 3x^2 + 24x - 30$ find the critical points, local maxima/minima, where it is increasing/decreasing and where is concave up/down. Also, find its maximum/minimum in the interval $[1, 5]$.

(Ans: CP: $-2, 4$, Increasing: $-2 < x < 4$, Decreasing: $x > 4$, $x < -2$, Concave up: $x < 6$, Concave down: $x > 6$)

29. For the function $f(x) = x^2 e^{-x^2}$ find its local maxima/minima, where it is increasing/decreasing and where is concave up/down. Also, find its maximum/minimum in the interval $[0, \infty)$. Finally, sketch its graph utilizing its symmetry and its asymptotes. (Ans: $f'(x) = 2xe^{-x^2} - 2x^3e^{-x^2}$, $f''(x) = 2e^{-x^2} - 10x^2e^{-x^2} + 4x^4e^{-x^2}$)

30. A cylindrical can without a top is to be made to contain $288\pi$ ft$^3$ of liquid. Material for the bottom (no top) costs \$16 per ft$^2$ and material for the sides costs \$12 per ft$^2$. Find the dimensions that will minimized the cost of the metal to make the can. (Ans: $r = 6$, $h = 8$)

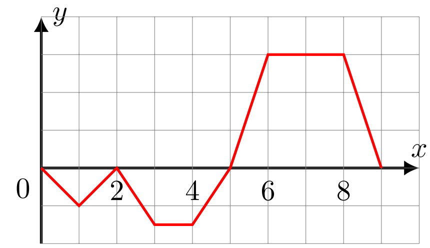
31. A Norman window is one with a semi-circular portion mounted (exactly) on one side of a rectangle. If the **perimeter** of such a window is 30 ft, what are the dimensions that maximizes the amount of light that can enter the window? (Ans: $r = \frac{30}{\pi+4}$, $h = 15 - \frac{60}{\pi+4} - \frac{15\pi}{\pi+4}$)

32. A man launches his boat from point $B$ on a bank of a straight river 4 km wide. He wants to reach point $C$, 12 km downstream on the opposite bank as quickly as possible. Point $D$ is directly across the river from point $B$. He will row his boat to point $P$ between points $C$ and $D$. If he can row at 6 km/hr and run at 10 km/hr, where should he land in order to reach point $B$ as soon as possible? (Ans: 3 km down from D)

33. By using implicit differentiation, find an expression for $\dfrac{dy}{dx}$ where $4x^2 + xe^y = 3y^3 + 4$. Find also the equation of the tangent line at $(1, 0)$. (Ans: $y = -9x + 9$)

34. In an economy, the capital per worker $k$ (in dollars), and its output per worker $q$ are related by the formula $q = 4800k^{1/3}$. Currently $k = 8000$ dollars and it is changing at the rate of 1000 dollars per year. Find the rate at which the output is changing. (Ans: 4,000)

35. The graph of the velocity $v(t)$ of a particle moving on a horizontal straight line is given below. Let $s(t)$ meters be the displacement of the particle after time $t$ minutes. Assume that $s(0) = 1$. Find the exact value of the following quantities and express each of them as a definite integral.



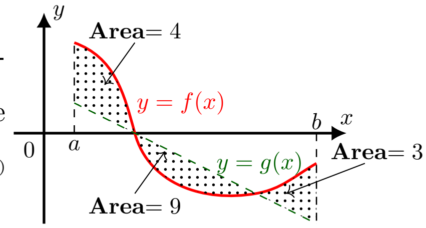

**a.** The position of the particle after 2 minutes.

**b.** Total change in position for the duration $1 \leq t \leq 4$.

**c.** Total distance covered by the particle.

(Ans: (a) 0 ft, (b) $-11/4$ ft, (c) 13 ft)

36. Let $f(x)$ and $g(x)$ be functions as shown below. Find the definite integral $\int_a^b [f(x) - g(x)]dx$. Compare this answer to the area between the graph of $f(x)$ and $g(x)$ over $[a, b]$. (Ans: $-2$, Area $= 16$)




37. The marginal cost of a product is given by $MC(x) = 4x^3 - \dfrac{1}{x^2}$ where $x$ is the number of items produced in units of millions. Find the total change in the cost (in dollars) if production level changes from 2 million to 4 million. (Ans: \$239.75)

38. The demand function $D$ (in units of thousand) of a line of jeans $D(p) = \dfrac{100}{2p+1}$ where $p$ is the price in dollars. Sketch a graph showing the function and the average demand for the jeans for the price level between \$15 and \$30.

39. Find the area enclosed between the graph of $f(x) = (1-x)^{10}x$ and the $x$-axis. (Ans: 1/132)

40. Mid point rule and trapezoidal rule with $n = 5$ to compute the value of $\int_1^2 \ln x \, dx$. Find the exact value of the definite integral and check your answers. (Ans: $2\ln 2 - 1$)

41. Evaluate the following indefinite integrals:

**a.** $\displaystyle\int (e^{2x} - 2e^{-x} + e + x^{3/2})\, dx$

**b.** $\displaystyle\int x(x^2+1)^{3/2}\, dx$

**c.** $\displaystyle\int \frac{x^2}{\sqrt{1-x}}\, dx$

**d.** $\displaystyle\int \frac{(\ln x)^2}{x}\, dx$

**e.** $\displaystyle\int_0^2 (x-1)^{25}\, dx$

**f.** $\displaystyle\int_1^4 \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\, dx$

**g.** $\displaystyle\int_0^1 xe^{3x}\, dx$

**h.** $\displaystyle\int_1^3 x^2 \ln x\, dx$

**i.** $\displaystyle\int_0^5 \frac{1}{t^2+7t+12}\, dt$

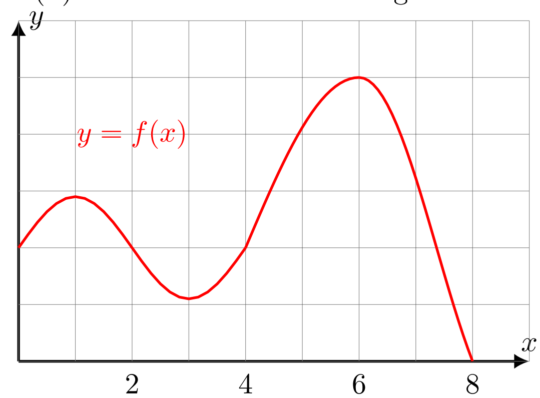
(Ans: a. $\frac{1}{2}e^{2x} + 2e^x + ex + \frac{2}{5}x^{5/2} + C$ b. $\frac{1}{5}(x^2+1)^{5/2} + C$ c. $-\frac{2}{15}\sqrt{1-x}(3x^2+4x+8) + C$ d. $(\ln x)^3/x + C$ e. 0 f. $4sqrt2/3 - 5\sqrt{5}/12$ g. $\frac{2}{9}e^3 + \frac{1}{9}$ h. $9\ln 3 - 26/9$ i. $\ln(32/27)$ )

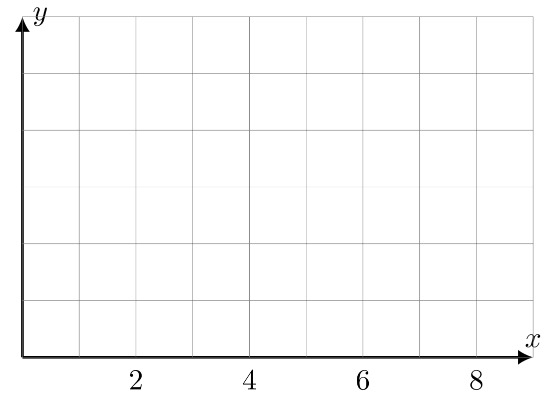
42. The rate of change $r(x)$ of the intensity of pollution on a 5 mile straight road between two factories $A$ and $B$ is given by $r(x) = \dfrac{20}{25-x^2}$ where $0 \leq x < 5$ is the distance from $A$ in miles. What is the total change in intensity of pollution experienced by a man walking on the road from a point two miles from $A$ to a point two miles from $B$. (Ans: $\ln 144 - \ln 49$)

43. Compute $\dfrac{d}{dt}\displaystyle\int_0^t x^9 e^{-0.5x}\, dx$. (Ans: $t^9e^{-0.5t}$)

44. The instantaneous rate of change of a quantity $Q(t)$ is given by $\dfrac{dQ}{dt} = 10te^{-t^2}$. Compute the total change of this quantity from $t = 0$ to $t = 1$. (Ans: $5 - 5/e$)

45. Sketch the graph of $F(x) = \int_0^x f(t)\,dt$ if graph of $f(t)$ is as show below. Indicate clearly the concavity of $F(x)$ and where it is increasing and decreasing.




46. Solve the initial value problem $\frac{dy}{dt} = te^{0.1t}$, $y(0) = 2$. (Ans: $10te^{0.1t} - 100e^{0.1t} + 102$)

47. A function $f(x)$ on an interval $[a, b]$ is a probability density function (pdf) if $f(x) \geq 0$ and $\int_a^b f(x)dx = 1$. Find $c$ such that $f(x) = cx^4$ is a pdf on $[-1, 1]$. Do the same for $f(x) = ce^{-0.1x}$ on $[0, 10]$, and for $f(x) = cxe^{-0.5x^2}$ on $[0, 10]$. (Ans: $5/2$, $1/(10 - 10e^{-1})$, $1/(1 - e^{-50})$)

48. Suppose you deposit \$15,000 in an account paying 3% annual interest compounded continuously, and do not make any further deposit or withdrawals. Find the average amount of money in the account during the first 5 years. (Ans: 16183.4)

49. Find the equilibrium quantity and price, consumer surplus and producer surplus for the demand curve $D(q) = (q - 10)^2 + 1$ and the supply curve $S(q) = q^2 + 1$ for $0 \leq q \leq 10$. (Ans: $q_e = 5, p_e = 26, CS = \frac{500}{3}, PS = \frac{250}{3}$)

50. Which of the following savings accounts yields the most after **ten** years:

(a) For the next 10 years, money is deposited continuously into Account A which pays 6% interest compounded continuously, at a rate of $2000 + 200t$ dollars per year. (Ans: $\text{FV}_a$=\$39,743.89)

(b) An account B with principal \$25,000 paying interest at 5% compounded **monthly.** (Ans: $\text{FV}_b$=\$41,175.24)

(c) On the first of January this year and for the next four years (no more after), \$6000 is deposited into Account C earning 4% annual interest compounded continuously. (Ans: $\text{FV}_c$=\$41,379.97)

51. If a continuous income stream flows into a saving account at the rate of \$10,000 per year and earns 7% interest, compounded continuously, find the time required for the balance to become \$1,000,000. (Ans: $\frac{\ln 8}{0.07}$)

52. A continuous income stream flows into a saving account at the constant rate of $S$ dollars per year and earns 8% interest, compounded continuously. Find $S$ so that the balance becomes \$2,000,000 in 40 years.

53. A retired person has \$2 million in an IRA paying an annual interest rate of 5%, compounded continuously. Over the next 20 years she plans to withdraw money continuously at the constant rate of $S$ dollars per year. Find the value of $S$ so that there is \$1 million left in her account at the end of 20 years. ( Ans: $S = \frac{0.05(2 - e^{-1})}{1 - e^{-1}}$ or $\frac{0.1e - 0.05}{e - 1}$ million)

54. Define the derivative of a function $f(x)$ at $x$ and provide the different names that people call it.

55. (a) Define the definite integral of a function $f(x)$ over an interval $[a, b]$ and provide its relation to the area under the graph of $f(x)$.

(b) Find $\int_{-1}^1 x^7 dx$. (Ans: 0)

(c) Find the area between $f(x) = x^7, -1 \leq x \leq 1$, and the $x$-axis. (Ans: $\frac{1}{4}$)

56. Describe the theorem that connects integration and differentiation.

57. How do we compute the present and future value of an income stream?

58. What is calculus good for?