Name ________________________________ Date ____________________

Math 10250 Activity 10: Logarithmic Functions (Section 2.3)

GOAL: Learn logarithmic functions as inverses of exponential functions and use them to model various interesting situations, like intensity of earthquakes, noise level, and acidity of beer.

Q1: What "undoes" the exponential function $f(x) = b^x$? (For example, if $f(x) = 2^x$ then $3 \overset{f}{\mapsto} 2^3 = 8$.)

A1: The **logarithmic function with base *b***, denoted $\log_b$. (If $g(x) = \log_2 x$ then $8 \overset{g}{\mapsto} \log_2 8 = 3$.)

Definition: $\log_b$ (for $b > 0, b \neq 1$) is defined by

$$\boxed{\log_b x = y \quad \Leftrightarrow b^y = x, x > 0.}$$

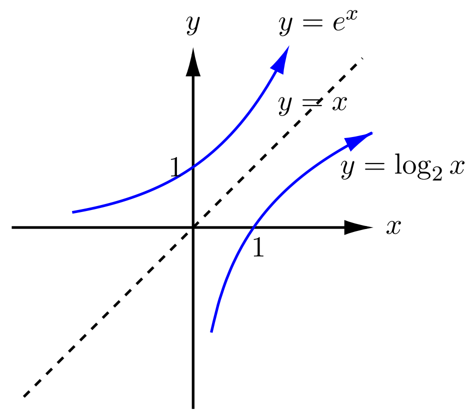
Example 1 Express the following logarithms as an integer or fraction without using a calculator.

(a) $\log_3 9 = y \iff 3^y = 9 = 3^2 \iff y = 2$

(b) $\log_{(0.1)} 1000 = y \iff (0.1)^y = 1000 \iff (10)^{-y} = 10^3 \iff y = -3$

▶ The graph of $\log_b x$ for $b > 1$

As an example, first graph $y = 2^x$ and obtain the graph of $y = \log_2 x$.



Properties of logarithmic functions

- $\log_b 1 \overset{?}{=} 0$
- domain $\overset{?}{=} (0, \infty)$ and range $\overset{?}{=} (-\infty, \infty)$
- It's continuous and increasing.
- $\lim_{x \to \infty} \log_b x \overset{?}{=} \infty$ and $\lim_{x \to 0^+} \log_b x \overset{?}{=} -\infty$

Note: The most common choices for b are $\boxed{10,\ e \text{ and } 2}$.

▶ The laws of logarithms. (*Reversing the laws of exponents*) Let $s, t > 0$. Then

(1) $\log_b(st) \overset{?}{=} \log_b s + \log_b t$;e.g., $\log_2(3 \cdot 8) \overset{?}{=} \log_2 3 + \log_2 8$

(2) $\log_b \left(\frac{s}{t}\right) \overset{?}{=} \log_b s - \log_b t$;e.g., $\log_2 \left(\frac{3}{8}\right) \overset{?}{=} \log_2 3 - \log_2 8$

(3) $\log_b(t^r) \overset{?}{=} r \log_b t$ for any number r ;e.g., $\log_2(3^7) \overset{?}{=} 7 \log_2 3$

(4) $\log_b 1 \overset{?}{=} 0$

(5) $\log_b \left(\frac{1}{t}\right) \overset{?}{=} -\log_b t$;e.g., $\log_2 \left(\frac{1}{8}\right) \overset{?}{=} -\log_2 8 = -3$

Q2: Can you explain property (1)? *It follows from the corresponding law of exponent.*

A2: *Letting $u = \log_b s$ and $v = \log_b t$, we have $s = b^u$ and $t = b^v$. So $s \cdot t = b^u \cdot b^v \underset{exp.}{\overset{(1)}{=}} b^{u+v} \iff \log_b s \cdot t \overset{def}{=} u + v = \log_b s + \log_b t$.*

Example 2 Use the approximation $\log_{10} 0.5 \approx -0.301$ to estimate $\log_{10} 20$.

$$\log_{10} 20 = \log_{10}\left(\frac{10}{0.5}\right) = \log_{10} 10 - \log_{10} 0.5 \approx 1 - (-0.301) = 1 + 0.301 = 1.301$$

Example 3 Use the approximation $\log_2 3 \approx 1.585$ and $\log_2 5 \approx 2.322$ to estimate $\log_2 45$.

$$\log_2 45 = \log_2(5 \cdot 9) = \log_2 5 + \log_2 3^2 = \log_2 5 + 2\log_2 3 \approx 2.322 + 2 \times 1.585 = 5.492$$

Example 4 Suppose A and b are positive numbers with $\log_3 A = b$. Write $\log_3\left(\frac{3}{\sqrt[3]{A}}\right)$ in terms of b.

$$\log_3\left(\frac{3}{A^{\frac{1}{3}}}\right) = \log_3 3 - \log_3 A^{\frac{1}{3}} = 1 - \frac{1}{3}\log A = 1 - \frac{b}{3}$$

Example 5 A bank teller claims that a saving account with principal of \$1000 earning interest at a annual rate of 1.3%, compounded weekly, after T years would at least double. What is the smallest possible T in whole years? Ans. 54 years.

$$A = 1000\left(1 + \frac{0.013}{52}\right)^{52\cdot t} \overset{\text{must}}{=} 2000 \Longrightarrow \left(1 + \frac{0.013}{52}\right)^{52\cdot t} = 2$$

$$\Longrightarrow 52 \cdot t \log_2\left(1 + \frac{0.013}{52}\right) = \log_2 2 = 1 \Longrightarrow t = \frac{1}{52\log_2\left(1 + \frac{0.013}{52}\right)}$$

▶ Logarithms with base 10

Logarithms with base 10, called **common logarithms**, are used in many well-known applications.

1 The Richter scale

$$\text{Richter value } = \log_{10}\left(\frac{x}{A}\right),$$

where A is the amplitude of the seismic wave of a reference earthquake and x is the amplitude of the seismic wave of the earthquake in question.

Example 6 One of the worst earthquakes in history occurred in Tokyo and registered 8.3 on the Richter scale. A more recent earthquake in California in 1989 registered 7.2. How much more severe was the earthquake in Tokyo in terms of the amplitude of its seismic wave? Ans. $10^{1.1}$ larger.

$$\left.\begin{array}{l} 8.3 = \log_{10}\left(\frac{X_T}{A}\right) \Longrightarrow \frac{X_T}{A} = 10^{8.3} \\ 7.2 = \log_{10}\left(\frac{X_C}{A}\right) \Longrightarrow \frac{X_C}{A} = 10^{7.2} \end{array}\right\} \Longrightarrow \frac{X_T}{X_C} = 10^{8.3-7.2} = 10^{1.1} \Longrightarrow X_T = 10^{1.1}X_C \Longrightarrow \boxed{X_T = 12.59X_C}$$

2 The decibel scale

$$\text{Noise level in decibels } = 10\log_{10}\left(\frac{x}{I}\right),$$

where I is the amplitude of a minimal audible sound wave and x is the amplitude of another sound wave. **Read Text Example 2.3.3 (p. 141).**

3 The pH scale

$$\text{pH value } = -\log_{10}[H^+],$$

where $[H^+]$ is the concentration of hydrogen ions in a solution. **Read Text Example 2.3.4 (p. 142).**