Name ______________________________ Date ____________________

Math 10250 Activity 11: Natural Logarithm and Applications (Section 2.4)

GOAL: Define the **natural** logarithmic function $\ln x$ as the inverse of the **natural** exponential function, $f(x) = e^x$ and use it to solve equations when the unknown is an exponent as is the case when we need to determine doubling time or half-life time.

Last time: We met the logarithmic function with base b. Recall, $\boxed{\log_b x = y \Leftrightarrow x = b^y, \quad x > 0}$.

Q1: What do we get when we let $\boldsymbol{b = e}$?

A1: The **natural logarithm**, $\ln x = \log_e x$, $x > 0$. Therefore $\boxed{\ln x = y \quad \Leftrightarrow \quad x = e^y, \quad x > 0.}$

• Since $\boldsymbol{\ln x}$ is the **inverse** of $\boldsymbol{e^x}$, we have the following two useful formulas:

$$\boxed{\ln(e^x) = x, \quad \text{any } x} \quad \text{and} \quad \boxed{e^{\ln x} = x, \quad x > 0.}$$

Sketch the graph of $\boldsymbol{\ln x}$:

Q2: What are the **basic properties of $\boldsymbol{\ln x}$**?

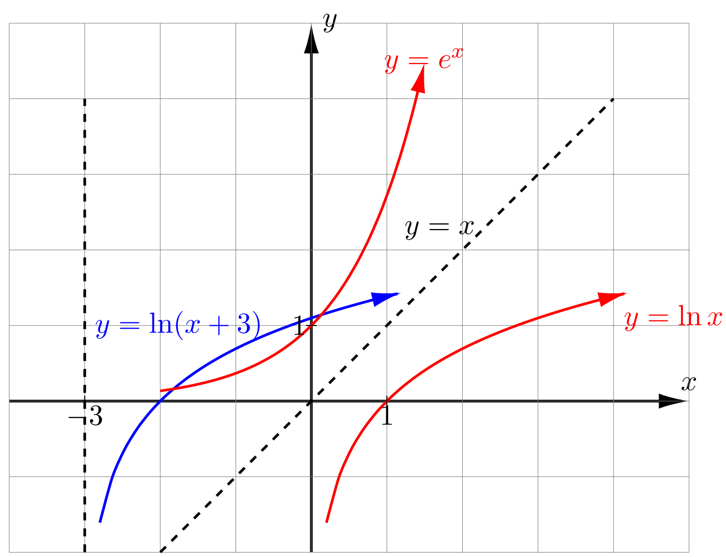
A2: • domain $\stackrel{?}{=}$ $(0, \infty)$ and range $\stackrel{?}{=}$ $(-\infty, \infty)$

• It's continuous and increasing.

• $\lim\limits_{x\to\infty} \ln x \stackrel{?}{=} \infty$ and $\lim\limits_{x\to 0^+} \ln x \stackrel{?}{=} -\infty$.

• $\ln 1 \stackrel{?}{=} 0$, $\ln e \stackrel{?}{=} 1$, and $\ln(1/e) \stackrel{?}{=} -1$

• Laws of logarithms with $b = e$.



Example 1 Sketch the graph of $y = \ln(3 + x)$.

Its graph is horizontal translation by 3 unit to the left of the graph $y = \ln x$

Example 2 Solve $e^{3-2x} = 8$ for x.

$$\ln(e^{3-2x}) = \ln 8 \Longrightarrow 3 - 2x = \ln 8 \Longrightarrow 3 - \ln 8 = 2x \Longrightarrow x = \frac{3 - \ln 8}{2}$$

▶ Converting exponentials from base b to base e

Q3: How do we convert b^x to $e^{(\text{something})}$?

A3: Using $\boldsymbol{b = e^{\ln b}}$ we have the **conversion formula**: $\boxed{b^x = \left(e^{\ln b}\right)^x = e^{(\ln b)\cdot x}\ .}$

Example 3 Rewrite $\sqrt[3]{7}$ as an exponential with base e.

$$\sqrt[3]{7} = 7^{\frac{1}{3}} = \left(e^{\ln 7}\right)^{\frac{1}{3}} = e^{\frac{1}{3}\ln 7}$$

Example 4 Evaluate the given expression as a number in decimal form without using a calculator.

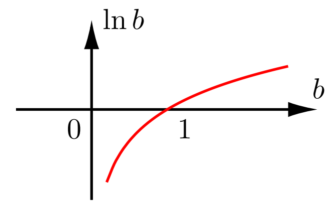
(a) $\ln\left(\dfrac{1}{\sqrt[4]{e}}\right) = \ln 1 - \ln \sqrt[4]{e} = 0 - \ln e^{\frac{1}{4}} = -\dfrac{1}{4}$ | (b) $e^{2\ln 3} = \left(e^{\ln 3}\right)^2 = 3^2 = 9$

Example 5 Simplify $e^{\ln(5x)+\ln(2/x)} = e^{\ln\left(5x\cdot\frac{2}{x}\right)} = e^{\ln 10} = 10.$

▶ Exponential growth and decay

Recall: In Section 2.1 we saw that the equation for exponential growth and decay is:

$$\boxed{y = y_0 b^t = y_0 e^{(\ln b)t}},$$



since $b^x = e^{(\ln b)x}$.

- If $b > 1$ then $\ln b$ = growth constant. ← exponential growth
- If $0 < b < 1$ then $\ln b < 0$. $|\ln b|$ = decay constant. ← exponential decay

Example 6 If \$10,000 is deposited in an account paying 5% interest per year, compounded continuously, how long will it take for the balance to reach \$20,000?

$$20,000 = 10,000e^{0.05t} \Longrightarrow \ln 2 = 0.05t \Longrightarrow t = \frac{\ln 2}{0.05} \Longrightarrow \boxed{t_{double} = \frac{\ln 2}{r}}$$

Example 7 Polonium-210 has a decay constant of 0.004951, with time measured in days. How long does it take a given quantity of polonium-210 to decay to half the initial amount? In other words, what is the half-life of polonium-210?

$y = y_0 \cdot e^{-kt}$, $k = 0.004951$
want $\frac{y_0}{2} = y_0 e^{-kt}$

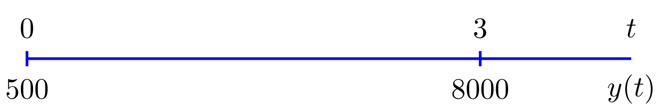
$$\Longrightarrow \tfrac{1}{2} = e^{-kt} \Longrightarrow \ln \tfrac{1}{2} = \ln e^{-kt}$$
$$\Longrightarrow -\ln 2 = -kt \Longrightarrow t = \tfrac{\ln 2}{k}$$
$$\Longrightarrow t = \tfrac{\ln 2}{0.004951} \approx 140$$

Fact: For any radioactive substance: $\boxed{\text{Half-life} = \frac{\ln 2}{k}}$

Example 8 A bacteria culture starts with 500 bacteria and is growing exponentially. After 3 hours there are 8000 bacteria.

(a) Find a formula of the form $y = Ae^{kt}$ for the number of bacteria after t hours.

$$y(t) = 500e^{kt} \Longrightarrow 8000 = 500e^{k\cdot 3} \Longrightarrow e^{k3} = \frac{80}{5} = 16 \Longrightarrow \ln e^{k3} = \ln 16$$
$$\Longrightarrow 3k = \ln 16 \Longrightarrow k = \frac{\ln 16}{3} \Longrightarrow \boxed{y(t) = 500e^{\frac{1}{3}\ln 16 \cdot t}}$$



(b) Find the number of bacteria after 4 hours.

$$y(4) = 500e^{\frac{1}{3}\ln 16 \cdot 4} = 500e^{\frac{4}{3}\ln 16} = 500(16)^{\frac{4}{3}}$$

(c) When will the population reach 30,000?

$$\left.\begin{array}{l} 30,000 = 500e^{kt},\ \ k = \frac{1}{3}\ln 16 \\ e^{kt} = \frac{300}{5} = 60 \Longrightarrow kt = \ln 60 \end{array}\right\} \Longrightarrow t = \frac{\ln 60}{k} = \frac{\ln 60}{\frac{1}{3}\ln 16} = 3\frac{\ln 60}{\ln 16} = 4.43$$

Application (Log-Normal Model) In Finance and Economics a theoretical model for the value of the stock market $S(t)$ is given by the formula

$$\boxed{S(t) = S_0 e^{(r - \frac{1}{2}\sigma^2)t} e^{\sigma\sqrt{t}\,Z},}$$

where Z is a standard normal random variable, r is the risk free interest rate, σ is the volatility, and S_0 is the value of the stock market at time $t = 0$. Take the natural logarithm of this formula and see if you can understand it better.