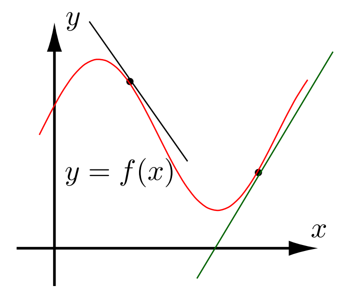
Name ______________________________ Date ___________________

Math 10250 Activity 12: The Slope of a Graph (Section 3.1)

GOAL: Understand the fundamental concept of the slope to a curve using limits and slope of lines. Also realize that slope to a curve is the same as instantaneous rate of change.

The **slope** at the point $(a, f(a))$ on the graph of $y = f(x)$ is the **slope of the tangent line** to the graph at $(a, f(a))$. We need three key concepts to find the slope at each point on the graph of $y = f(x)$:

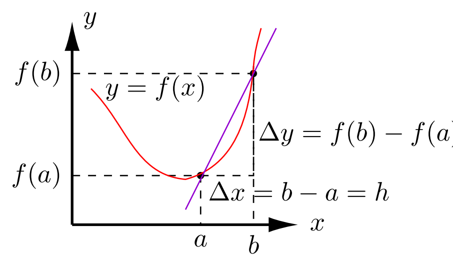


- Slope of line (Already done!)
- Limits (Already done!)
- Average rate of change (To be done!).

▶ Average Rate of Change

Definition: The average rate of change of $f(x)$ over the interval $[a, b] = [a, a+h]$ is:

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} = \frac{f(a+h) - f(a)}{h} = *$$

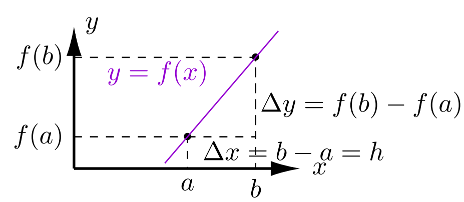


Graphical Interpretation: Use the graph here to explain the graphical meaning of average rate of change of $f(x)$ over an interval [a,b].
Linear Models: $f(x) = mx + b$

Example 1 Find the average rate of change of $f(x) = 2x - 1$ at $x = a$.

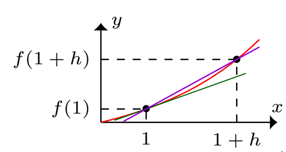
For any a and h, we have: $\dfrac{\Delta y}{\Delta x} = \dfrac{[2(a+h) - 1] - [2a - 1]}{h} = \dfrac{2h}{h} = 2$ *same!*

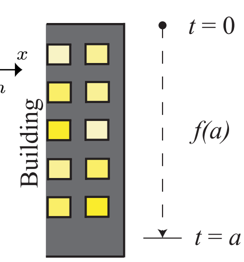
Note: In the general case $f(x) = mx + b$ we have $\dfrac{\Delta y}{\Delta x} = m$, *the same! for any a and h.*



Nonlinear Model of Galileo: It can be shown experimentally that the distance travelled by a stone released at rest from the top of a building is given by $f(t) = 16t^2$.

Q1: Compute the following:





(a) Average speed over $1 \le t \le 3 = \dfrac{\text{Change in distance}}{\text{Change in time}} = \dfrac{f(3) - f(1)}{3 - 1} = 64$

(b) Average speed over $1 \le t \le 1 + h = \dfrac{f(1+h) - f(1)}{h} = \dfrac{16(1+h)^2 - 16 \cdot 1}{h} = ?$

(c) Complete the table:

h	-0.01	-0.001	0	0.001	0.01
$\frac{f(1+h)-f(1)}{h}$	31.984	31.9984	?	32.0016	32.016

Note: The average rate of change is **not** the same!

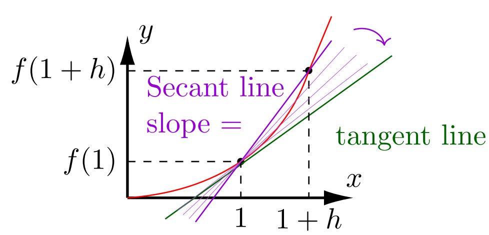
Q2: What is the value of $L = \lim_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$? What physical quantity does L represents?

Since: $* = \dfrac{16(1 + 2h + h^2) - 16}{h} = \dfrac{32h + 16h^2}{h} = \dfrac{h(32 + 16h)}{h} = 32 + 16h$

We have: $L = \lim_{h \to 0}(32 + 16h) = \boxed{32} =$ *velocity = speed.*

Remark: We also call the value L the instantaneous rate of change of $f(t) = 16t^2$ at $t = 1$.

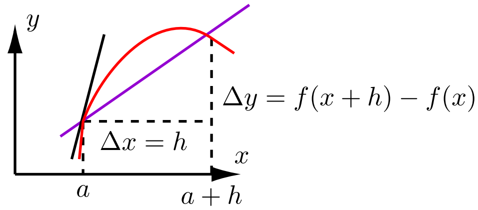
Use the graph here to give a graphical interpretation of the value of $L = \lim_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$.



► Instantaneous Rate of Change

Definition: The instantaneous rate of change of $f(x)$ at $x=a$ is the value of the limit

$$\text{Instantaneous rate of change at } x=a = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$



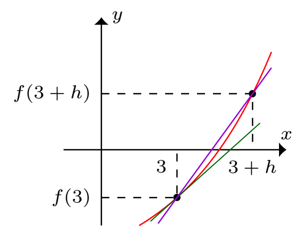
Remark: Graphically, the instantaneous rate of change of $f(x)$ at $x=a$ is the **slope** of the **tangent line** to the curve $y=f(x)$ at the point $(a, f(a))$.

Example 2 Consider the function $f(x)=x^2-5x+4$.

(i) Find the instantaneous rate of change of $f(x)$ at $x=3$ using limits.

Step 1: Find (form) the average rate of change from 3 to $3+h$ (or the slope of the secant line joining $(3, f(3))$ and $(3+h, f(3+h))$.

$$\frac{f(3+h)-f(3)}{h} \overset{①}{=} \frac{[(3+h)^2-5(3+h)+4]-[3^2-5\cdot 3+4]}{h}$$



Step 2: Simplify!

$$\frac{f(3+h)-f(3)}{h} \overset{②}{=} \frac{9+6h+h^2-15-5h+4-9+15-4}{h} = \frac{h+h^2}{h} = 1+h$$

Step 3: Let $h\to 0$ in the average rate of change (or the slope of the secant line). $\lim_{h\to 0}\frac{f(3+h)-f(3)}{h} \overset{③}{=} \lim_{h\to 0}(1+h) = \boxed{1}$

Conclusion: Instantaneous rate of change of $f(x)$ at $x=3$ is $\boxed{1}$

(ii) What is the equation of tangent line to the graph of $y=f(x)$ at $x=3$?

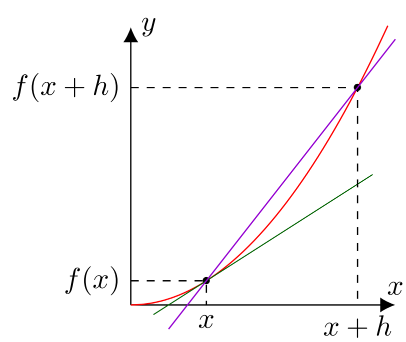
When $x=3$, $y=f(3)=3^2-5\cdot 3+4=9-15+4=-2$. *Also, slope is* 1. *Thus equation of tangent line* $y-(-2)=1\cdot(x-3) \Longrightarrow \boxed{y=x-5}$

(iii) Using the steps in (i), find an expression for the slope of the graph $y=f(x)$ at any given x.

Step 1: $\frac{f(x+h)-f(x)}{h} \overset{①}{=} \frac{(x+h)^2-5(x+h)+4-(x^2-5x+4)}{h} \overset{②}{=} \frac{x^2+2xh+h^2-5x-5h+4-x^2+5x-4}{h} = \frac{h(2x+h-5)}{h} = 2x+h-5$

Step2: $\lim_{h\to 0}(2x+h-5) = \boxed{2x-5}$

Example 3 Using limits, find a formula for the instantaneous rate of change and slope of the following **important functions**:



- $f(x)=x^2$, for any x. Ans. $2x$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \overset{①}{=} \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} \overset{②}{=}$$
$$\lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h} = \lim_{h\to 0}(2x+h) \overset{③}{=} \boxed{2x.}$$

- $f(x)=\sqrt{x}$, for any $x>0$. (In economics it could be a utility function with x being wealth.) Ans. $\frac{1}{2\sqrt{x}}$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \overset{①}{=} \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \overset{②}{=} \lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0}\frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \overset{③}{=} \boxed{\frac{1}{2\sqrt{x}}.}$$