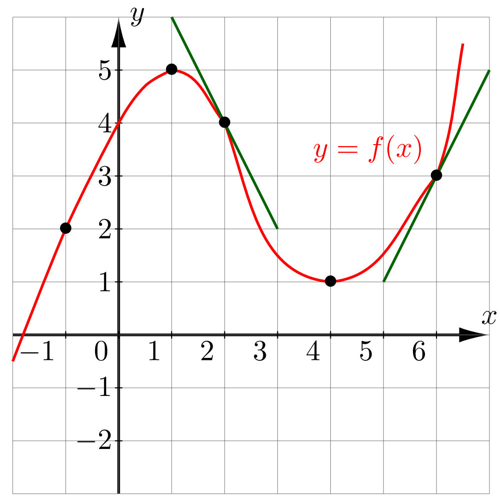
Name ______________________________ Date ____________________

Math 10250 Activity 13: The Derivative of a Function (Section 3.2)

GOAL: To understand that the slope of the graph of a function $f(x)$ is dependent on x. The function that gives the slope of the graph of $f(x)$ is called the derivative of $f(x)$. We will also learn about some basic properties of the derivative, as well as the derivatives of power functions and polynomial.

Example 1 For the function $y = f(x)$ whose graph is shown, compute or estimate the following values:



Slope of the tangent line to the graph of $f(x)$ at $x = 2$.

It is the slope of the tangent line to the graph at $x = 2$, which is $\dfrac{\Delta y}{\Delta x} = \dfrac{-2}{1} = -2.$

Slope of the graph of $f(x)$ at $x = 4$.

It is the slope of the tangent line to the graph at $x = 4$, which is $\dfrac{\Delta y}{\Delta x} = \dfrac{0}{1} = 0.$

Instantaneous rate of change of $f(x)$ at $x = 6$:

It is the slope of the tangent line to the graph at $x = 6$, which is $\dfrac{\Delta y}{\Delta x} = \dfrac{2}{1} = 2.$

Rate of change of $f(x)$ at $x = -1$.

Here, the the tangent line to the graph at $x = -1$ is not given, but we can estimate visually that the slope of the graph is positive and equal to about $\dfrac{\Delta y}{\Delta x} = \dfrac{1}{1/2} = 2.$

Remark: The slope of the graph of $f(x)$ or rate of change of $f(x)$ varies according to x. This gives us a new function called the **derivative** of $f(x)$. We denote the derivative of $f(x)$ by $f'(x)$.

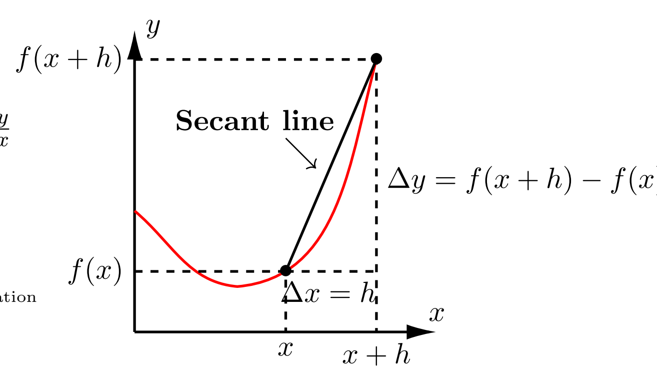
Find the following values for the function in Example 1:

$$f'(1) \stackrel{?}{=} 0 \quad f'(2) \stackrel{?}{=} -2 \quad f'(4) \stackrel{?}{=} 0 \quad f'(6) \stackrel{?}{=} 2 \quad f'(-1) \stackrel{?}{=} 2$$

▶ Difference Quotient and Leibniz's notation

Recalling the limit definition of the rate of change of $f(x)$, we have:

Derivative of $f(x)$ is: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.



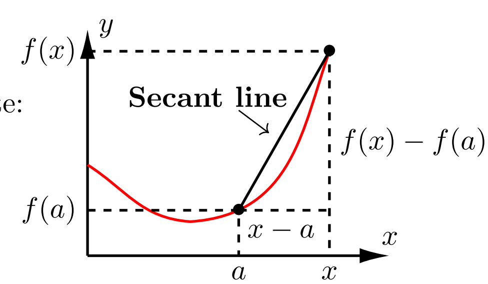
Difference quotient $= \dfrac{f(x+h)-f(x)}{h} = \dfrac{\text{Change in } y}{\text{Change in } x} = \dfrac{\Delta y}{\Delta x}$

So $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{df}{dx}$

↑ Leibniz's notation

We also write: $f'(x) = \dfrac{dy}{dx} = \dfrac{d}{dx}[f(x)]$

▶ Other Notations for the Derivative



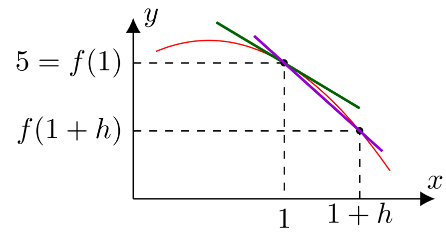
For each fixed value a in the domain of $f(x)$, we can also write:

$f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

Example 2 Suppose that $f(x)$ is a function whose graph goes through the point (1,5) and whose tangent line at that point has the equation $2x + y = 7$. Without computing, find each of the following limits:

(a) $\lim_{h\to 0} \frac{f(1+h)-5}{h} \overset{?}{=}$ (b) $\lim_{\Delta x\to 0} \frac{f(1+\Delta x)-5}{\Delta x} \overset{?}{=}$ (c) $\lim_{x\to 1} \frac{f(x)-5}{x-1} \overset{?}{=}$

All these limits define the derivative of $f(x)$ *at* $x = 1$*, which is the slope of the tangent line* $y = -2x + 7$ *to the graph, that is* $\boxed{-2}$. *Thus, we have*

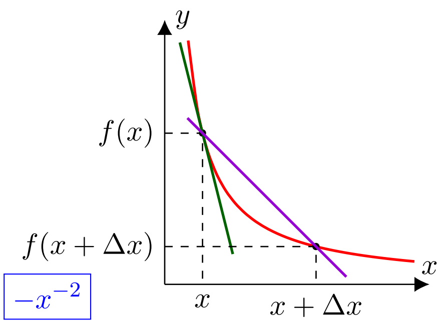


$$\textit{slope} = \boxed{-2} = f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{f(1+h)-5}{h} = (a) = (b) = (c).$$

Example 3 Use the definition of derivative and no other formula to find $f'(x)$ where $f(x) = \frac{1}{x} = x^{-1}$.

Step 1: $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x} \underset{\Delta x\to 0}{\longrightarrow} \frac{\frac{1}{x}-\frac{1}{x}}{0} = \frac{0}{0}$



Step 2: $\frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x} = \frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\frac{\Delta x}{1}} = \frac{-\Delta x}{\Delta x(x+\Delta x)x} = \frac{-1}{(x+\Delta x)x}$

Step 3: $f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{-1}{x(x+\Delta x)} = -\frac{1}{x^2} = \boxed{-x^{-2}}$

▶ **Rules for finding derivatives:** (Notations: $f'(x) = (f(x))' = \frac{d}{dx}[f(x)]$)

0. <u>The derivative of a constant is zero:</u> $\boxed{(c)' \overset{?}{=} 0}$; e.g., $(8)' \overset{?}{=} 0$ or $(\sqrt{2})' \overset{?}{=} 0$ or $(e)' \overset{?}{=} 0$

1. <u>The Power Rule</u> $\boxed{(x^m)' \overset{?}{=} mx^{m-1}}$; e.g., $(x^5)' \overset{?}{=} 5x^{5-1}$ or $(x^{-0.8})' \overset{?}{=} -0.8x^{-0.8-1}$

2. <u>The Constant Multiple Rule</u> $\boxed{\frac{d}{dx}[cf(x)] \overset{?}{=} cf'(x)}$; e.g., $(3x^5)' \overset{?}{=} 3\cdot 5x^4$
   (c ↑ constant)

3. <u>The Sum Rule</u> $\boxed{\frac{d}{dx}[f(x)+g(x)] \overset{?}{=} f'(x)+g'(x)}$; e.g., $(x^2-5x+4)' \overset{?}{=} 2x-5+0$

Q2: Explain why the Sum Rule is true.

A2: *We have* $\frac{d}{dx}[f(x)+g(x)] \doteq \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f'(x)+g'(x)$$

Example 4 $\frac{d}{dx}\left[2x^4 + 3x^{-3} - \frac{\pi}{e}\right] \overset{?}{=} 2\cdot 4x^3 + 3(-3)x^{-4} - 0$

Example 5 Find the equation of the line tangent to the graph $y = x^3 - 2x$ at $x = 2$.

By the point-slope equation of a line, we have $y - y_1 = m(x - x_1)$. *Here,* $x_1 = 2$,

$f(x_1) = 2^3 - 2\cdot 2 = 4$, *and* $m = f'(x) = 3x^2 - 2\Big|_{x=2} = 3\cdot 2^2 - 2 = 10.$

$f'(x) = 3x^2 - 2 \Longrightarrow f'(2) = 3\cdot 2^2 - 2 = 12 - 2 = 10$

Thus we have $\boxed{y - 4 = 10(x-2)}$.