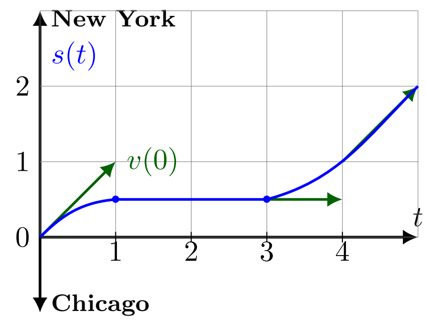
Name ______________________________ Date __________________

Math 10250 Activity 15: The Derivative as a Rate (Section 3.3 continued)

GOAL: To learn that when the function is cost, revenue and profit then its derivative is called **marginal** and realize that the derivative has different names in different situations.

Example 1 Suppose you are driving due east on a straight road towards New York. At 1:00 pm (set $t = 0$) you drive by a gas station (set $s = 0$) at the speed of 60 miles per hour (1 mile per minute) and begin to decelerate for 1 minute, reaching a full stop. You pause for 2 minutes to check your map then start moving again eastward, accelerating for 1 minute reaching the speed of 60 miles per hour again. Draw a graph illustrating your position $s(t)$ beginning at 1:00pm. Measure position in miles and time in minutes.



▶ **The second derivative of** $f(x)$ $\longmapsto$ $\boxed{f''(x) = (f'(x))'}$

Example 2 For the functions $f(t) = 3t^3 + 6t^2 + 7t + 11$ and $h(t) = t^4 - \dfrac{1}{t^2}$ below, find the indicated higher order derivative and evaluate it at $t = 1$.

(a) $f''(t) = (9t^2 + 12t + 7)'$
$= 18t + 12$

$f''(1) \stackrel{?}{=} 18 \cdot 1 + 12 = \boxed{30}$

(b) $h'''(t) = (4t^3 + 2t^{-3})''$
$= (12t^2 - 6t^{-4})'$
$= 24t + 24t^{-5}$

$h'''(1) \stackrel{?}{=} 24 + 24 = \boxed{48}$

▶ **Marginal functions** are the derivatives (rate of change) of:

- Cost function $= C(x) =$ cost of producing x items.
- Revenue function $= R(x) =$ income earned from the sale of x items.
- Profit function $= P(x) \stackrel{?}{=} R(x) - C(x)$.

marginal cost $= MC(x) = C'(x)$ ← rate at which cost is changing

marginal revenue $= MR(x) = R'(x)$ ← rate at which revenue is changing

marginal profit $= MP(x) = P'(x) = R'(x) - C'(x)$ ← rate at which profit is changing

Example 3 The revenue function of a certain company is $R(x) = 40x - 0.2x^2$ and its cost function is $C(x) = 20x + 100$. Find the marginal revenue, marginal cost and marginal profit functions.

$$MR(x) = R'(x) = 40 - 0.4x, \quad MC(x) = C'(x) = 20,$$
$$MP(x) = P'(x) = R'(x) - C'(x) = 40 - 0.4x - 20 = \boxed{20 - 0.4x}$$

Example 4 The marginal cost and revenue for a certain footwear manufacturer are given by $MC(x) = 500$ and $MR(x) = 1000 - 3x$, where x is the number of pairs of shoes produced and sold per week.

4(a) What is the marginal profit? $P'(x) = R'(x) - C'(x) = 1000 - 3x - 500 = \boxed{500 - 3x}$

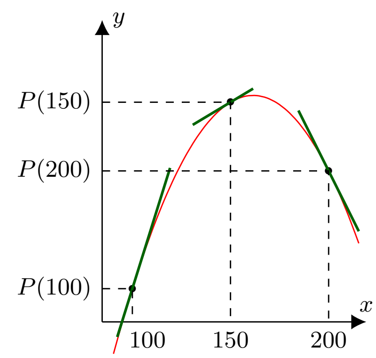
4(b) If the company is currently producing 100 pairs per week, should it increase or decrease production in order to raise its profit? Explain your answer.

$P'(100) = 500 - 3 \cdot 100 = 200 > 0 \Longrightarrow$ *Increasing production is good! since then profit is rising! (having positive slope).*

4(c) What if the company is currently producing 150 pairs per week? 200 pairs per week?

$P'(150) = 500 - 3 \cdot 150 = 500 - 450 = 50 > 0 \Longrightarrow$ *Increase production is good! again!*

$P'(200) = 500 - 3 \cdot 200 = 500 - 600 = -100 < 0 \Longrightarrow$ *Decreasing production is good! since then profit is falling! (having negative slope).*



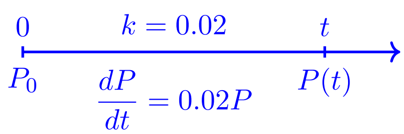
Marginal Profit Rule:

- If $MP(x) > 0$ then increasing production is beneficial.
- If $MP(x) < 0$ then decreasing production is beneficial.

▶ Modeling population growth

For very simple populations, the growth rate of the population $P(t)$ is proportional to its size at any given time t. This is expressed by the following **differential equation**: $\boxed{\dfrac{dP}{dt} = kP}$ k = proportionality constant

Example 5 Assume that the U.S. population P increases at a rate equal to 0.02 of its size at any given time t. Write a differential equation modeling this situtation.



Later we will solve this equation and will find the familiar formula

$$P(t) = P_0 e^{0.02t},$$

where P_0 is the initial population.

Example 6 Suppose $P_A(t)$ and $P_B(t)$ are the population sizes of two colonies of insects, both of which are functions of time (in days). Translate each of the following statements into a formula involving the functions and/or their derivatives:

(a) Initially, Colony B is three times as large as colony A. $P_B(0) = 3P_A(0)$

(b) Colony B is three times as large as colony A at all times. $P_B(t) = 3P_A(t)$

(c) Colony A is growing at three times the rate of colony B at all times. $P'_A(t) = 3P'_B(t)$

(d) Colony A is growing at a rate of 1000 insects per day after three weeks. $P'_A(21) = 1000$

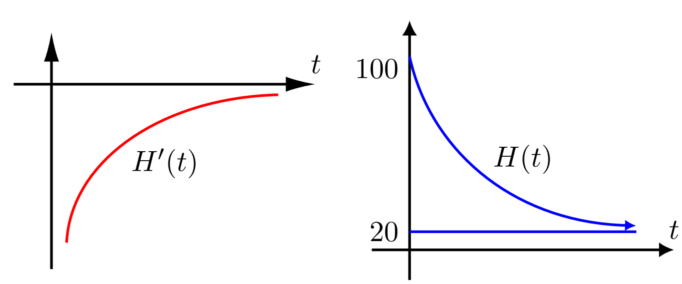
(e) The rate at which colony B grows is proportional to its size. $\boxed{\dfrac{dP_B}{dt} = rP_B}$

▶ Another example of rates

Example 7 A kettle filled with water at temperature 100°C is set in a kitchen. The temperature of the kitchen is 20°C. Let $H(t)$ be the temperature (in °C) of the water at time t (in minutes). If the graph of the **derivative** of $H(t)$ is given in the figure, answer the following questions:

(a) Is the water cooling off or heating up? Explain your answer.

Since H' is negative, which means the slope of the graph of the temperature $H(t)$ is negative, we know that $H(t)$ is sloping down, i.e. the water is cooling.



(b) Sketch a graph for the temperature $H(t)$ in the axes provided. Explain your answer.

Water temperature decreases and approaches room temperature.