Name ______________________________ Date ____________________

Math 10250 Activity 16: Differentiability and Linear Approximation (Section 3.4)

GOAL: To approximate a differentiable function near a given point $x = a$ with the equation of its tangent line (a simpler linear function) at $x = a$. Discuss continuity versus differentiablity.

▶ **Differentiability** A function $f(x)$ is said to be differentiable if each point of its graph has a non-vertical tangent line. This means that the slope at each point of the graph is a *finite* number.

Graphically, differentiable means that each small segment of the graph of $f(x)$ is almost identical to a straight line. This is illustrated in Figures 1 through 3 below. As you zoom into the point $(a, f(a))$, the segment of the graph of $f(x)$ near point a becomes more and more like its tangent line at $x = a$.

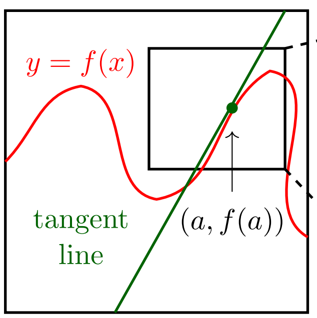


Figure 1.

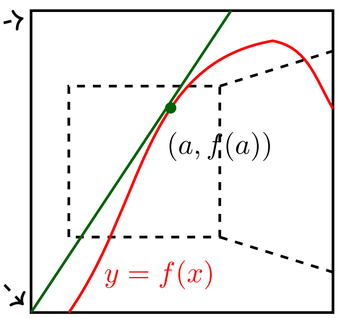


Figure 2.

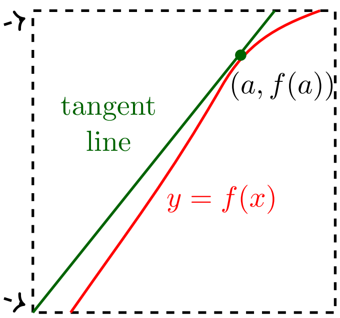


Figure 3.

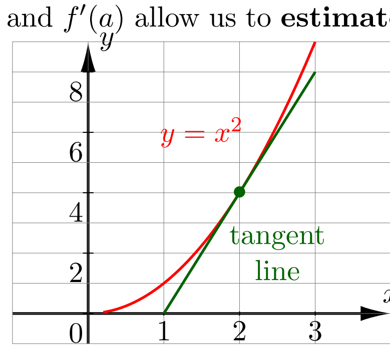
Q1: Referring to Figure 3, what is the equation of the tangent line to the graph of $y = f(x)$ at $(a, f(a))$?

Using the point-slope formula $y - y_1 = m(x - x_1)$ *with* $x_1 = a$, $y_1 = f(a)$ *and* $m = f'(a)$ *we see that the equation of the tangent line is:* $\boxed{y - f(a) = f'(a)(x - a)}$

Since the graph of $f(x)$ near point a is almost the same as its tangent line at a, we have:

$$\boxed{f(x) \approx f(a) + f'(a)(x - a)} \quad \leftarrow \text{Linear approximation of } f(x) \text{ near point a}$$

Remark: If $f(x)$ is a differentiable function at $x = a$, the **two** values $f(a)$ and $f'(a)$ allow us to **estimate** the value of $f(x)$ when x is close to a!



Example 1 (a) Find the tangent line to $f(x) = x^2$ at $x = 2$.

Since $f(2) = 2^2 = 4$, $f'(x) = 2x \implies f'(2) = 4$ the equation of the tangent line is: $\boxed{y = 4 + 4(x - 2)}$

(b) Give the tangent line approximation of $f(x)$ near 2.

The linear approximation is: $\boxed{x^2 \approx 4 + 4(x - 2)}$

(c) Using your answer in (b), estimate the following values and comment on their accuracy:

(i) $f(2.01) \overset{?}{\approx} 4 + 4(2.01 - 2) = 4 + 4(0.01) = \boxed{4.04}$

(ii) $f(1.9) \overset{?}{\approx} 4 + 4(1.9 - 2) = 4 + 4(-0.1) = \boxed{3.6}$

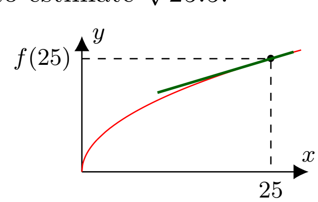
(iii) $f(3) \overset{?}{\approx} 4 + 4(3 - 2) = 4 + 4 = \boxed{8}$

Example 2 Apply linear approximation to the function $f(x) = x^{1/2} = \sqrt{x}$ to estimate $\sqrt{25.5}$.

A good neighbor of $x = 25.5$ *is* $a = 25$ *because it is the closest point at which* $f(a)$ *and* $f'(a)$ *are easy to compute. Thus,* $f(x) = \sqrt{x} = x^{\frac{1}{2}} \implies f(25) = \sqrt{25} = 5$,

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \implies f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{2 \cdot 5} = 0.1$

$\boxed{\sqrt{x} \approx 5 + 0.1(x - 25)} \implies \sqrt{25.5} \approx 5 + 0.1(25.5 - 25) = 5 + 0.1(0.5) = \boxed{5.05}$



Example 3 Use the tangent line approximation to estimate $\sqrt[3]{26.31}$.

A good neighbor of $x = 26.31$ *is* $a = 27$ *since*

$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Longrightarrow f(27) = \sqrt[3]{27} = 3$

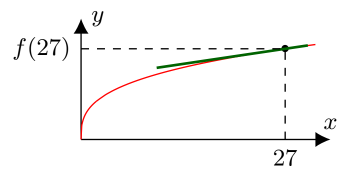
$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}} \Longrightarrow f'(27) = \frac{1}{27}$

$f(x) = \sqrt[3]{x} \approx f(27) + f'(27)(x-27)$

$\Longrightarrow \sqrt[3]{x} \approx 3 + \frac{1}{27}(x-27)$

$\sqrt[3]{26.31} \approx 3 + \frac{1}{27}(26.31-27)$

$\approx 3 + \frac{1}{27}(-0.69) \approx \frac{26.77}{9}$



Example 4 The cost of producing 200 units of a certain item is \$5,000, and the marginal cost of producing 200 units is \$100. Use linear approximation to estimate the cost of producing 202 units.

$C(x) \approx C(200) + C'(200)(x-200) = 5,000 + 100(x-200) \Longrightarrow C(202) \approx 5,000 + 100(202-200) = \boxed{5,200}$

Example 5 Assume that a population grows according to the (exponential) model $\frac{dP}{dt} = 0.02P$. If the population now is 5 millions, use linear approximation to estimate this population 10 years later. (Ans: 6 millions)

Since $P(0) = 5$ and $P'(0) = 0.02P(0) = (0.02)\cdot 5 = 0.1$, we have $P(t) \approx P(0) + P'(0)(t-0) = 5 + 0.1t$. Thus, $P(10) \approx 5 + 0.1 \cdot 10 = \boxed{6}$ millions.

▶ Differentiability and continuity

A **continuous** function is **NOT** differentiable if the graph has a **corner** or a **vertical** tangent line

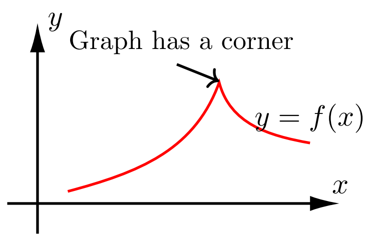


Figure 4

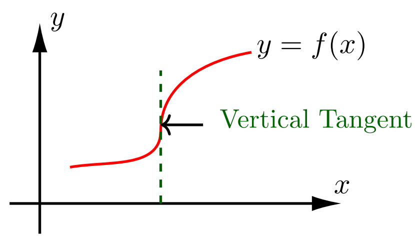


Figure 5

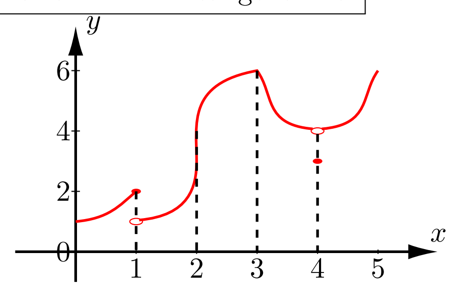


Figure 6

Example 6 According to figure 6, a) $f(x)$ is discontinuous at $x \overset{?}{=} 1, 4$

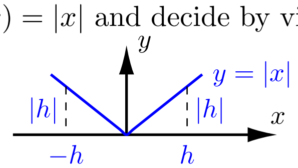
b) $f(x)$ is not differentiable at $x \overset{?}{=} 1, 4, 2, 3$

Definition: A function $f(x)$ is differentiable at point $x = a$ if the graph of $f(x)$ has a non-vertical tangent line at $(a, f(a))$. In terms of slope and limits, this means that

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \textbf{exists} \text{ and is a } \textbf{finite} \text{ number}$$

Example 7 (a) Sketch the graph of $f(x) = |x|$ and decide by visual inspection whether $f(x)$ is differentiable at $x = 0$.

$f(x)$ is not differentiable at $x = 0$



(b) Now, use the limit definition to decide whether $f(x)$ is differentiable at $x = 0$.

Since $\frac{f(0+h)-f(0)}{h} = \frac{|h|}{h}$ *Since* $|h| = h$ *for* $h > 0$, *we get* $\lim_{h\to 0^+} \frac{|h|}{h} = 1$. *And since* $|h| = -h$ *for* $h < 0$, *we also get* $\lim_{h\to 0^-} \frac{|h|}{h} = -1$. *Since the two one-sided limits are not equal, we conclude the limit as* $h \to 0$ *does not exist, which means the function does not have a derivative at* $x = 0$.

Remark: $f(x) = |x|$ is a function that is continuous but NOT differentiable.
Q2: Can a differentiable function not be continuous? **A2:** *No*

Theorem. *If a function f is differentiable at a, then it is continuous at a.*

Quadratic Approximation. If a function $f(x)$ has first and second derivative at a then: $\boxed{f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.}$ This is useful in optimization since it is a complete square when $f'(a) = 0$ (critical point).