Name ______________________________ Date ____________________

Math 10250 Activity 18: The Product and Quotient Rules (Section 3.6)

GOAL: To learn how to compute the derivatives of a product and a quotient of two functions.

▶ **The Product Rule:** $\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)g(x) + f(x)g'(x)$

Note: $\frac{d}{dx}[f(x)\cdot g(x)] = \frac{d}{dx}[g(x)\cdot f(x)]$.

Example 1 Use the product rule to find the derivatives:

(a) $\frac{d}{dx}[x^2(3x^3-x)] = (x^2)'(3x^3-x) + x^2(3x^3-x)'$
$= (2x)(3x^3-x) + x^2(9x^2-1)$
$= 6x^4 - 2x^2 + 9x^4 - x^2$
$= 15x^4 - 3x^2$

(b) $\frac{d}{dx}[e^{-2x}\ln x] = (e^{-2x})'\ln x + e^{-2x}(\ln x)'$
$= -2e^{-2x}\ln x + e^{-2x}\cdot\frac{1}{x}$

▶ **The Quotient Rule:** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

In general, $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{d}{dx}\left[\frac{g(x)}{f(x)}\right]$.

Example 2 Find the equation of the tangent line to the graph of $y = f(x) = \frac{x}{x^2+1}$ at the point $x = 2$.

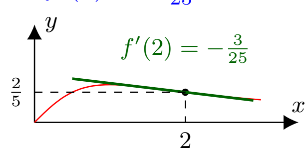
$\frac{dy}{dx} = \frac{(x)'(x^2+1) - x(x^2+1)'}{(x^2+1)^2}$
$= \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

$f(2) = \frac{2}{2^2+1} = \frac{2}{5}$

$f'(2) = \frac{1-2^2}{(2^2+1)^2} = \frac{-3}{25}$

To find the equation of tangent line at $x = 2$ we use the point-slope equation $y - y_1 = m(x - x_1)$ with $x_1 = 2$, $y_1 = f(2) = \frac{2}{5}$ and $m = f'(2) = -\frac{3}{25}$. Thus,

$y - \frac{2}{5} = \frac{-3}{25}(x-2)$



Example 3 Use the appropriate differentiation rules that you have learned so far to find the derivatives below. Some algebra may be helpful.

(a) $\frac{d}{dx}\left(\frac{x^2+x-3}{100}\right)$

$= \frac{1}{100}\frac{d}{dx}(x^2+x-3)$

$= \frac{2x+1}{100}$

(b) $\frac{d}{dx}\left(\frac{x+e^x}{e^x}\right)$

$= \frac{(x+e^x)'e^x - (x+e^x)(e^x)'}{(e^x)^2}$
$= \frac{(1+e^x)e^x - (x+e^x)\cdot(e^x)}{(e^x)^2}$
$= \frac{(1+e^x-x-e^x)\cdot(e^x)}{(e^x)^2} = \frac{1-x}{e^x}$

(c) $\frac{d}{dx}\left(\frac{\ln x}{x^2}\right) = \frac{(\ln x)'(x^2) - (\ln x)(x^2)'}{(x^2)^2}$

$= \frac{\frac{1}{x}\cdot x^2 - \ln x(2x)}{x^4} = \frac{x - 2x\ln x}{x^4}$

$= \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3}$

(d) $\frac{d}{dx}\left(\frac{x^2+x-3}{x^{10}}\right) = \frac{(x^2+x-3)'x^{10} - (x^2+x-3)\cdot(x^{10})'}{(x^{10})^2}$

$= \frac{(2x+1)x^{10} - (x^2+x-3)\cdot(10x^9)}{x^{10\cdot 2}}$

$= \frac{2x^{11}+x^{10}-10x^{11}-10x^{10}+30x^9}{x^{20}} = \frac{-8x^{11}-9x^{10}+30x^9}{x^{20}}$

Faster: (d)$= \frac{d}{dx}[x^{-8}+x^{-9}-3x^{-10}] = -8x^{-9} - 9x^{-10} + 30x^{-11}$

Example 4 Suppose the demand for a certain product is given by $q = f(p)$, where p is the price per unit and q is the number sold. The revenue is given by $R = pq$.

(a) If $f(300) = 20,000$ and $f'(300) = -30$, find dR/dp when $p = 300$.

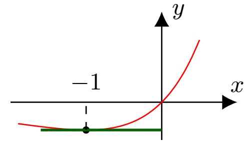
$$R(p) = p \cdot q = p \cdot f(p)$$
$$\frac{dR}{dp} = f(p) + pf'(p)$$

$$\frac{dR}{dp}(300) = f(300) + 300f'(300) = 20,000 + 300 \cdot (-30)$$
$$= 20,000 - 9000 = 11,000 > 0$$

(b) If the product is currently selling for \$300 per unit, should the company increase or decrease the price in order to raise the revenue?

Since $R'(300) > 0$, the company should increase the price.

Example 5 For what x does the graph $y = xe^x$ have slope zero?



Ans: $x = -1$

$$0 = (xe^x)' = e^x + xe^x = e^x(1+x) = 0 \Longrightarrow \boxed{x = -1}$$

Example 6 Find the equation of the tangent line to the graph of $y = \dfrac{1 - \ln x}{1 + \ln x}$ at $x = 1$.

Ans: $y = -2x + 3$

$$\frac{dy}{dx} = \frac{-\frac{1}{x}(1 + \ln x) - (1 - \ln x)\frac{1}{x}}{(1 + \ln x)^2}$$
$$= \frac{-\frac{1}{x} - \frac{1}{x}\ln x - \frac{1}{x} + \frac{1}{x}\ln x}{(1 + \ln x)^2}$$
$$= \frac{-2}{x(1 + \ln x)^2}$$

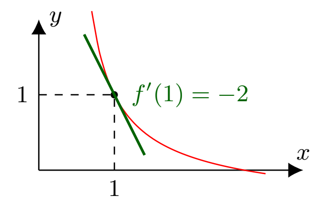
$$y(1) = \frac{1 - 0}{1 + 0} = 1 \qquad y'(1) = \frac{-2}{1 \cdot 1} = -2$$

Using the point-slope equation of a line

$$y - y_1 = m(x - x_1)$$

with $x_1 = 1$, $y_1 = 1$ and $m = -2$, we have

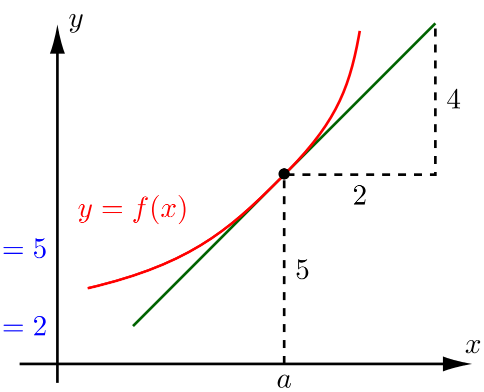
$$y - 1 = -2(x - 1)$$
$$y = -2x + 2 + 1 = \boxed{-2x + 3}$$



Example 7 Let $p(x) = f(x)g(x)$ and $q(x) = \dfrac{f(x)}{g(x)}$. Using the graph of $f(x)$ and $g(x)$ below find

(a) $p'(a)$

$$p'(x) = f'(x)g(x) + f(x)g'(x)$$
$$p'(a) = f'(a)g(a) + f(a)g'(a)$$
$$= 2 \cdot 10 + 5 \cdot (-2) = \boxed{10}$$



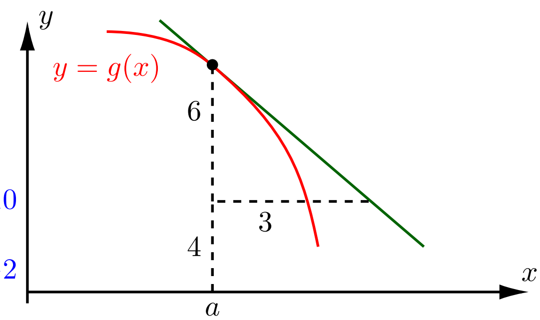
$f(a) = 5$

$f'(a) = \dfrac{4}{2} = 2$

(b) $q'(a)$

Ans: $p'(a) = 10$ and $q'(a) = 0.3$

$$q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$
$$q'(a) = \frac{f'(a)g(a) - f(x)g'(a)}{[g(a)]^2}$$
$$= \frac{2 \cdot 10 - 5 \cdot (-2)}{[10]^2}$$
$$= \frac{20 + 10}{100} = \frac{3}{10} = \boxed{0.3}$$



$g(a) = 10$

$g'(a) = \dfrac{-6}{3} = -2$