Name ________________________________ Date ____________________

Math 10250 Activity 22: First Derivative Tests (Section 4.1 continued)

GOAL: To use information given by $f'(x)$ to find where $f(x)$ is increasing and decreasing, and to locate maxima and minima.

First, let's review what we learned last time.

▶ The derivative test for increasing and decreasing functions

Method for finding where a function f is increasing/decreasing

1. Find all **critical points** of f. (That is, find all points in the domain where $f'(x) = 0$ or $f'(x)$ does not exist.)
2. Find points where f has a **vertical asymptote** or is undefined.
3. Plot points in 1 and 2 on the x-axis (making intervals).
4. Take one point a in each interval and compute $f'(a)$. The sign of $f'(a)$ is the sign of f' throughout that interval.
5. f is **increasing** on intervals where f' is *positive*.
   f is **decreasing** on intervals where f' is *negative*.

Example 1 Find all values of x for which $f(x) = \dfrac{1}{x^2 - x}$ is increasing or decreasing with the steps outlined below.

Step 1: Find all **critical points** of f. (That is, all points c in the domain where $f'(c) = 0$ or $f'(c)$ does not exist.)

$$f'(x) = \frac{-(2x-1)}{(x^2-x)^2} = 0 \Longrightarrow 2x - 1 = 0 \Longrightarrow \boxed{x = \frac{1}{2}} \text{ critical points}$$

Step 2: Find points where f has a **vertical asymptote** (v.a.) or is undefined. Answer: *$x = 0$ and $x = 1$*

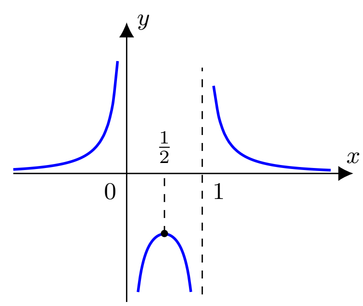
$x^2 - x = 0 \Longrightarrow x(x-1) = 0 \Longrightarrow x = 0$ *and* $x = 1$ *are v.a.*

Step 3: Draw a number line, mark all the points found in Steps 1 and 2, and find the sign of $f'(x)$ in each interval between marked points.

Step 4: Write down the values of x for which f is increasing ($f'(x) > 0$) and those for which f is decreasing ($f'(x) < 0$).

Sign table:

x		0		1/2		1	
sign of $f'(x)$	+	⋮	+	0	−	⋮	−
info about $f(x)$	↗	v.a.	↗	max	↘	v.a.	↘



The first derivative test for maxima and minima

If $f(x)$ has a critical point at c, then

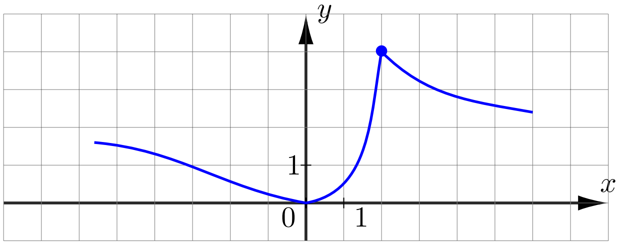
- there is a local maximum at $x = c$ if $f'(x)$ changes its sign from $+$ to $-$, and
- there is a local minimum at $x = c$ if $f'(x)$ changes its sign from $-$ to $+$.

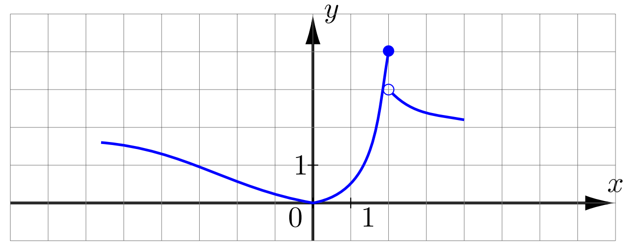
Example 2 In Example 1, where does $f(x)$ have a local maximum or local minimum, if any?

At $x = \dfrac{1}{2}$, $f(x)$ has a local maximum

Example 3 Sketch the graphs of **two different** functions sharing the same properties below. The graphs should have at least one feature that is markedly different.

- $f'(x) < 0$ on $(-\infty, 0)$ or $(2, \infty)$.
- $f'(0) = 0$ but $f'(2)$ does not exist.
- $f'(x) > 0$ on $(0, 2)$.
- $\lim_{x \to +\infty} f(x) = 2 = \lim_{x \to -\infty} f(x)$.
- $f(0) = 0$ and $f(2) = 4$.





▶ **Global Maximum and Global Minimum**

Q1: How can we determine the global maximum or global minimum of a given function?

A1: One way is to study how the function increases and decreases.

Example 4 Find the local and global extrema, if any, of $f(x) = x^2 e^{-x}$ for $-\infty < x < \infty$.

Step 1: Find all **critical points** of f.

$f'(x) = 2x \cdot e^{-x} + x^2 e^{-x}(-1) = x \cdot e^{-x}(2 - x) = 0.$ *There are critical points at* $\boxed{x = 0 \text{ and } x = 2}$.

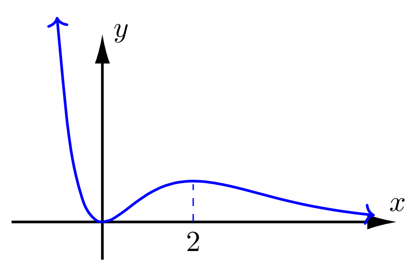
Step 2: Find points where f has a **vertical asymptote** or is undefined. Answer: *None*

Step 3: Find the values of $f(x)$ at all critical points and the behavior of $f(x)$ at $\pm\infty$.

$f(0) = 0, \quad f(2) = 4e^{-2}, \quad \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x^2}{e^x} \approx \frac{\text{big}}{\text{Huge}} = 0 \quad \textit{and} \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2}{e^x} \approx \frac{\text{big}}{\text{small}} = \infty$

Step 4: Give a rough sketch of the graph of $f(x)$ indicating clearly where f is increasing and decreasing.

x	$-\infty$		0		2		$+\infty$
sign of $f'(x)$		$-$	0	$+$	0	$-$	
info about $f(x)$		↘	*global min*	↗	*local max*	↘	



Step 5: Read off the global maximum and the global minimum from the sketch above. If there are none state so.

Global min at $x = 0$. *Since for all* x *we have* $f(x) \geq f(0) = 0$.