Name ____________________________ Date ____________________

Math 10250 Activity 24: Second Derivative Tests (Section 4.2 continued)

GOAL: To use information given by $f''(x)$ to locate local maxima and minima.

▶ The second derivative test for local maxima and minima

Q1: If $f'(c) = 0$ and $f''(c) > 0$, what can we conclude about c?
A1:
$f'(c) = 0 \quad \Rightarrow \quad$ Slope is zero at c.
$f''(c) > 0 \quad \Rightarrow \quad$ Graph is *concave up* near c.
Therefore, c is a local *minimum*.

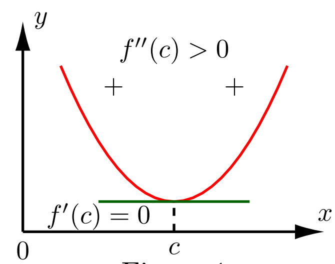


Figure 1

Explanation: By quadratic approximation at c, we have

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2$$
$$= f(c) + \underbrace{\frac{1}{2}f''(c)}_{\text{positive}}(x-c)^2 \Longrightarrow f(x) \geq f(c)$$

Q2: If $f'(c) = 0$ and $f''(c) < 0$, what can we conclude about c?

A2:
$f'(c) = 0 \quad \Rightarrow \quad$ Slope is zero at c.
$f''(c) < 0 \quad \Rightarrow \quad$ Graph is *concave down* near c.
Therefore, c is a local *maximum*.

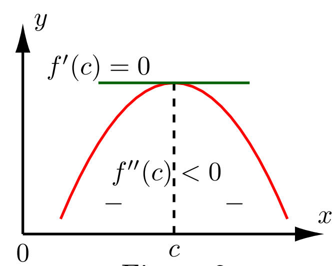


Figure 2

Explanation: By quadratic approximation at c, we have

$$f(x) \approx f(c) + \underbrace{\frac{1}{2}f''(c)}_{\text{negative}}(x-c)^2 \Longrightarrow f(x) \geq f(c)$$

This gives us another test for local extrema.

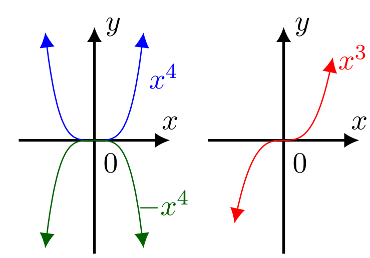


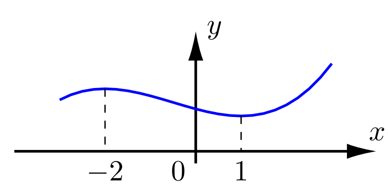
Figure 3

Second derivative test
Suppose $f'(x)$ and $f''(x)$ exist around c and $f'(c) = 0$.
- If $\boldsymbol{f''(c) > 0}$ then there is a *local minimum* at $x = c$. (Like Fig. 1)
- If $\boldsymbol{f''(c) < 0}$ then there is a *local maxmum* at $x = c$. (Like Fig. 2)
- If $\boldsymbol{f''(c) = 0}$ then the test is *inconclusive*. (Like Fig. 3)

Method for finding extrema of f using the second derivative test
1. Compute f' and find all critical points of f. (That is, find all points in domain where $f'(x) = 0$ or $f'(x)$ does not exist.)
2. Compute f'' and use second derivative test (on solutions to $f'(x) = 0$) to find local minima and maxima.

Example 1 Use the second derivative test to determine all local maximum and minumum points of $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 7$.

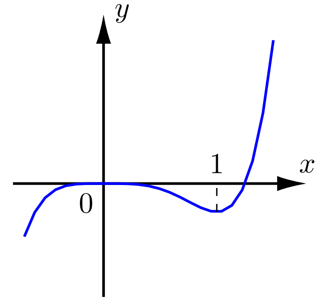
$f'(x) = x^2 + x - 2 = (x+2)(x-1) = 0 \Longrightarrow x = -2, 1$ *critical points*
$f''(x) = 2x + 1$
$f''(-2) = 2(-2) + 1 = -4 + 1 = -3 < 0 \Longrightarrow$ *local max.*
$f''(1) = 2(1) + 1 = 2 + 1 = 3 > 0 \Longrightarrow$ *local min.*



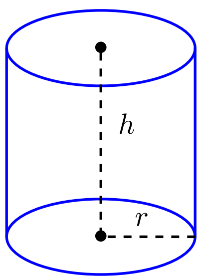
Example 2 Find the critical points of $f(x) = x^5 - \frac{5}{4}x^4$ and determine whether each is a local minimum or maximum. Use the second derivative test wherever possible.

$f'(x) = 5x^4 - 5x^3 = 5x^3(x-1) = 0 \Longrightarrow$
$x = 0, 1$ *critical points*
$f''(x) = 20x^3 - 15x^2 = 5x^2(4x-3)$
At $x = 1$ *we have* $f''(1) = 5 > 0$
$\Longrightarrow$ *local min.*
At $x = 0$ *we have* $f''(x) = 0$
$\Longrightarrow$ *2nd derivative test inconclusive*

x		0		1	
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$
		max		*min*	



Example 3 A paint manufacturer needs to construct a cylindrical can that holds 250π cm^3 (≈ 785 cm^3) of its product. To reduce its cost for the can, the manufacturer needs to construct one with minimal surface area. Can you help the manufacturer to decide on the dimensions of the can? Note that the can is a cylinder closed on both ends. ($r = 5$ cm, $h = 10$ cm)



Volume $= 250\pi$
$\pi r^2 \cdot h = 250\pi$
$\Longrightarrow h = \frac{250}{r^2}$

Surface area
$\pi r^2 + \pi r^2 + 2\pi r \cdot h = F(r, h)$
$\uparrow$ *bottom* $\uparrow$ *top* $\uparrow$ *arround*

$f(r) = F\left(r, \frac{250}{r^2}\right) = 2\pi r^2 + 2\pi r \cdot \frac{250}{r^2}$
or $f(r) = 2\pi r^2 + 500\pi r^{-1}$
$f'(r) = 4\pi r - 500\pi r^{-2} = 0 \Longrightarrow 4r = \frac{500}{r^2} \Longrightarrow$
$r^3 = \frac{500}{4} = 125 \Longrightarrow \boxed{r = 5}$
$f''(r) = 4\pi + 1000\pi r^{-3} \Big|_{r=5} > 0 \Longrightarrow$ *min*
$h = \frac{250}{5^2} = \boxed{10} \Longrightarrow$ *height=diameter*

Note: Not true for coke can (beauty reasons).

Example 4 Using the information for the **continuous** function $f(x)$ below, find all critical points of $f(x)$ and classify them using (i) the first derivative test and (ii) the second derivative test, if possible. State also all inflection points.

- $f'(x) < 0$ for $(-\infty, -2) \cup (-1, 0) \cup (0, 1)$.
- $f'(x) > 0$ for $(-2, -1) \cup (1, +\infty)$.
- $f'(-2) = 0 = f'(0)$; $f'(-1)$ and $f'(1)$ do not exist.
- $f''(0) = 0 = f''(2)$; $f''(-1)$ and $f''(1)$ do not exist.
- $f(x)$ is concave up for $(-\infty, -1) \cup (-1, 0) \cup (1, 2)$.
- $f(x)$ is concave down for $(0, 1) \cup (2, +\infty)$.

Critical points: $x = -2, -1, 0, 1$

$x = 0$ *2nd derivative test inconclusive*

$x = \pm 1$ *2nd derivative test inconclusive*

$x = -2$, $f''(-2) \geq 0$ *So the 2nd derivative test may not apply if* $f''(-2) = 0$.

x		-2		-1		0		1		2	
$f'(x)$	$-$	0	$+$	*DNE*	$-$	0	$-$	*DNE*	$+$		$+$
$f''(x)$	$+$		$+$	*DNE*	$+$	0	$-$	*DNE*	$+$	0	$-$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$		$\nearrow$
		min		*max*				*min*