Name ______________________________ Date __________________

Math 10250 Activity 28: Applied Optimization Problems (Section 4.5)

GOAL: To use what we learned about optimizing $f(x)$ on an interval to solve special word problems called optimization problems.

Recall the steps that we want to follow to solve a word problem:

Steps to solving word problems

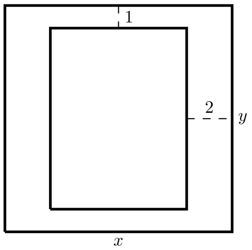
1. Understand the problem.
    - Read the problem carefully.
    - What is given?
    - What do you want to find? (What do you want to maximize or minimize?)
    - What is a reasonable estimate for the answer?
2. Construct a mathematical model.
    - Make a sketch, if possible.
    - Write equations, with domains, relating the unknown quantities.
3. Simplify the model (using substitution, etc.).
4. Solve simplified problem (using familiar techniques or by approximation methods).
5. Check your answer.

Example 1 A book publisher is designing a book whose pages have 2 inches margins on each side and 1 inch margins on the top and bottom. The publisher wants the total page area to be 200 square inches. Find the page dimensions that will maximize the **printed** area.

a) What information do you know? Introduce any variables that help to simplify this situation.

width $= x$, *length* $= y$

b) Sketch and label a picture of the page described by the problem.



c) The total page area is 200 in^2. Write an equation using this information.

$x \cdot y = 200$

d) What do you want to maximize? What function will you use? On what interval do you optimize the function? (This is important for determining the method of optimization.)

Maximize the printed area, which is $\boxed{(x-4)(y-2) = A}$. *This is the mathematical model.*

$$\left.\begin{array}{l}\textit{Margin on each side is } 2in \implies x \geq 4 \\ \textit{Margin on the top and bottom is } 1in \implies y \geq 2 \\ \textit{Using } xy = 200 \textit{ this reads } \frac{200}{x} \geq 2\end{array}\right\} \implies x \leq 100$$

e) Now finish the problem and find the variable that will maximize the area.

Solving $x \cdot y = 200$ *for* y *gives* $y = \dfrac{200}{x}$. *Thus* $A(x) = (x-4)\left(\dfrac{200}{x} - 2\right) = 200 - 2x - \dfrac{800}{x} + 8$

Simplified model: $\boxed{\textit{Maximize } A(x) = 208 - 2x - \dfrac{800}{x} \textit{ over } [4, 100]}$.

Step 1. *Find the critical point:* $A'(x) = -2 + \dfrac{800}{x^2} \overset{cp}{=} 0 \iff 2x^2 = 800 \implies x^2 = 400 \implies \boxed{cp\!: \ x = 20}$

Step 2.

x	4	20	100
$A(x)$	0	128	0

↑
max

Method II: Using 2nd derivative test

$A''(x) = -\dfrac{1600}{x^3} \implies A''(20) = \dfrac{-1600}{20^3} < 0$

Thus x is max since it is the only c. p.

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Example 2 The amount of production P of a company is given by the function $P = \sqrt{KL}$ where K is the capital and L is the labor provided by the company. Suppose each unit of capital is \$10,000, and each unit of labor costs \$20,000 and the company's budget is \$180,000. (a) Find the possible values of L and (b) find the K and L that maximize P. (Ans: $0 \le L \le 9$, $L = 9/2$, $K = 9$)

Budget: $10,000K + 20,000L = 180,000 \Longrightarrow K + 2L = 18 \iff K = 18 - 2L$

If $K = 0$ *then* $L = 9$. *Thus* $0 \le L \le 9$.

$P = P(L) = \sqrt{L(18 - 2L)} = (18L - 2L^2)^{\frac{1}{2}}$

$P' = \frac{1}{2}(18L - 2L^2)^{-\frac{1}{2}}(18 - 4L) \overset{cp}{=\!=} 0$

$\Longrightarrow 4L = 18 \Longrightarrow \boxed{cp\text{: } L = \frac{9}{2}} \Longrightarrow \boxed{K = 9}$

L	0	$\frac{9}{2}$	9
P	0	$\frac{9}{\sqrt{2}}$	0

$\uparrow$

max

Example 3 Find the point on the curve $y = x^2$ $(x \ge 0)$ that is the closest to the point $(0, 2)$. (Ans: $(\sqrt{1.5}, 1.5)$)

The distance from (x, x^2) *to* $(0, 2)$ *is*

$d = \sqrt{(x-0)^2 + (x^2 - 2)^2}$

It suffices to minimize d^2, *that is*

$f(x) = x^2 + (x^2 - 2)^2, \ x \in [0, \infty).$

$f'(x) = 2x + 2(x^2 - 2) \cdot 2x \overset{cp}{=\!=} 0$

$\Longrightarrow 2x(2x^2 - 3) = 0 \Longrightarrow \boxed{cp\text{: } x = 0, \sqrt{\frac{3}{2}}}$

$\lim_{x \to \infty} f(x) \approx big + big = \infty$

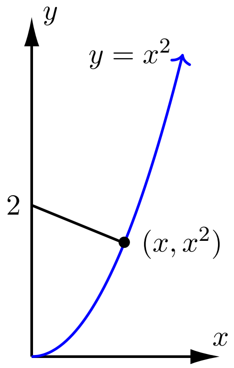
x	0	$\sqrt{3}/2$	∞
$f(x)$	2	$25/16$	$\lim_{x\to\infty} f(x) = \infty$

$\uparrow$

min

x		$\sqrt{1.5}$	
sign of $f'(x)$	$-$	0	$+$
info about $f(x)$	$\searrow$		$\nearrow$

min



Example 4 A house is located at a point H in the woods, 3 miles from the nearest point A on a road. A telephone switching station is located at point a S on the road, 6 miles from A. The homeowner wants to run a telephone cable through the woods from H to P (where P is a point between A and S) and then along the road from P to S. The cost of laying the cable through the woods is three times as expensive per mile as it is along the road. Where should the point P be chosen to minimize the cost? (Ans: $3/(2\sqrt{2})$ miles from A)

Using the picture we derive the model:

$$\boxed{\begin{array}{l}\textit{Cost: } C(x) = 1 \cdot (6 - x) + 3\sqrt{x^2 + 9} \\ \textit{The domain of } C(x) \textit{ is } [0, 6].\end{array}}$$

To minimize $C(x)$ *we have:*

$C'(x) = -1 + \frac{3}{2}(x^2 + 9)^{-\frac{1}{2}} \cdot 2x$

$C'(x) \overset{cp}{=\!=} 0 \iff \dfrac{3x}{\sqrt{x^2 + 9}} = 1$

$\iff 3x = \sqrt{x^2 + 9} \iff 9x^2 = x^2 + 9$

$\iff 8x^2 = 9 \iff x^2 = \dfrac{9}{8}$

$\iff \boxed{cp\text{: } x = \dfrac{3}{2\sqrt{2}}}$

$C\left(\frac{3}{2\sqrt{2}}\right) = 6 - \frac{3}{2\sqrt{2}} + 3\sqrt{\frac{9}{8} + 9} \approx 14.49$

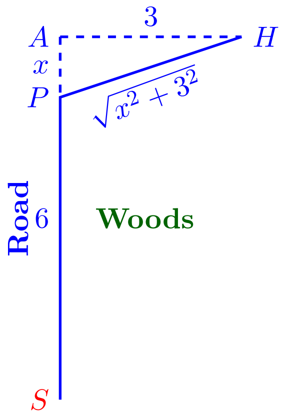
x	0	$3/(2\sqrt{2})$	6
$C(x)$	15	14.49	$9\sqrt{5} \approx 20.12$

$\uparrow$

min

x		$\frac{3}{2\sqrt{2}}$	
sign of $C'(x)$	$-$	0	$+$
info about $C(x)$	$\searrow$		$\nearrow$

min



Remark. The path of light from the **Sun** to a point at the bottom of the **sea** follows a path similar to SPH above, since light travels faster through air than water. With calculus, you can explain this physical phenomenon. Yes! you can!