Name ______________________________ Date ____________________

Math 10250 Activity 31: Integration by Substitution (Section 5.2 continued)

GOAL: To compute integrals using Integration by Substitution.

Example 1 Find the following indefinite integrals. Check your answer by differentiating.

(a) $\int (3x+5)^5 dx = \int u^5 \frac{1}{3} du = \frac{1}{3}\cdot\frac{1}{6}u^6 + c = \frac{1}{18}(3x+5)^6 + c$

$u = 3x+5$
$du = 3dx$

Check: $\left[\frac{1}{18}(3x+5)^6\right]' = \frac{1}{18}\cdot 6(3x+5)^5\cdot 3 + 0 = (3x+5)^5$

(b) $\int 4t^3\sqrt{t^4+1}dt = \frac{1}{2}\int \sqrt{u}du = \frac{u^{1/2+1}}{1/2+1} + c = \frac{2}{3}(t^4+1)^{\frac{3}{2}} + c$

$u = t^4+1$
$du = 4t^3 dt$

Check: $\left[\frac{2}{3}(t^4+1)^{\frac{3}{2}}\right]' = \frac{2}{3}\cdot\frac{3}{2}(t^4+1)^{\frac{1}{2}}\cdot 4t^3$

(c) $\int \frac{2x+1}{x^2+x+5}dx = \int \frac{du}{u}du = \ln|u| + c = \ln|x^2+x+5| + c$

$u = x^2+x+5$
$du = (2x+1)dx$

Check: $[\ln|x^2+x+5| + c]' = \frac{2x+1}{x^2+x+5}$

(d) $\int \frac{x}{(2x^2+1)^3}dx = \int \frac{1}{u^3}\cdot\frac{1}{4}du = \frac{1}{4}\cdot\frac{u^{-3+1}}{-3+1} + c = -\frac{1}{8}\cdot\frac{1}{u^2} + c = -\frac{1}{8}\cdot\frac{1}{(2x^2+1)^2} + c$

$u = 2x^2+1$
$du = 4xdx$
$xdx = \frac{1}{4}du$

Check: $\left[-\frac{1}{8}\cdot\frac{1}{(2x^2+1)^2} + c\right]' = -\frac{1}{8}\cdot\frac{-2(2x^2+1)(4x)}{(2x+1)^4}$

(e) $\int \frac{dx}{x\ln x}dx = \int \frac{1}{u}du = \ln|u| + c = \ln|\ln x| + c$

$u = \ln x$
$du = \frac{1}{x}dx$

Check: $[\ln|\ln x| + c]' = \frac{1}{x\ln x}$

(f) $\int \frac{t}{3-t}dt = \int \frac{3-u}{u}(-du) = \int\left(1-\frac{3}{u}\right)du = u - 3\ln|u| + c = (3-t) - 3\ln|3-t| + c$

$u = 3-t$
$t = 3-u$
$du = -dt$

Check: $[(3-t) - 3\ln|3-t| + c]' = -1 - \frac{-3}{3-t}$

(g) $\int \frac{x\ln(x^2+1)}{x^2+1}dx = \frac{1}{2}\int udu = \frac{1}{4}u^2 + c = \frac{1}{4}[\ln(x^2+1)]^2 + c$

$u = \ln(x^2+1)$
$du = \frac{2x}{x^2+1}dx$

Check: $\left[\frac{1}{4}[\ln(x^2+1)]^2\right]' = \frac{1}{4}\cdot 2\ln(x^2+1)\frac{2x}{x^2+1}$

(h) $\int \frac{dx}{1+\sqrt{x}} = \int \frac{1}{u}2(u-1)du = 2\int\left(1-\frac{1}{u}\right)du = 2(u-\ln|u|) + c = 2[1+\sqrt{x} - \ln(1+\sqrt{x})] + c$

$u = 1+\sqrt{x}$
$du = \frac{dx}{2\sqrt{x}} = \frac{dx}{2(u-1)}$
$dx = 2(u-1)du$

Example 2 A consulting firm, hired by a city to help to develop its downtown renewal project, predicted that the number of jobs would increase at a rate of

$$\frac{dy}{dx} = 1,600\left(1 - \frac{x}{\sqrt{1+x^2}}\right),$$

where y is the number of jobs and x is the amount of money invested, in million of dollars. If no money is invested, no new jobs will be created. If the prediction is correct, how many new jobs will be created by an investment of \$6 million?

The initial condition is $y(0) = 0$. *Solving the DE, we have*

$$y = 1600\int\left(1 - \frac{x}{\sqrt{1+x^2}}\right)dx = 1600\left(x - \int \frac{1}{\sqrt{u}}\cdot\frac{1}{2}du\right)$$

$u = 1 + x^2$
$du = 2xdx$
$xdx = \frac{1}{2}du$

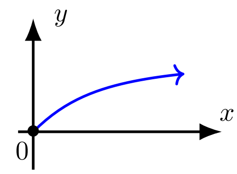
$$= 1600\left(x - \frac{1}{2}\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + c$$

$$= 1600\left(x - u^{\frac{1}{2}}\right) + c = 1600\left(x - (1+x^2)^{\frac{1}{2}}\right) + c$$

Using the initial condition, we have:

$$0 = y(0) = 1600(0 - \sqrt{1}) + c = -1600 + c \Longrightarrow c = 1600$$

$$y(x) = 1600(x - \sqrt{1+x^2}) + 1600 = \boxed{1600(x - \sqrt{1+x^2} + 1)}$$



To find the new jobs created by an investment of \$6 million, we compute:

$$y(6) = 1600(6 - \sqrt{1+36} + 1) = 1600(7 - \sqrt{37}) \approx \boxed{1,468}$$

Example 3 The marginal revenue MR for a product is given by $\dfrac{60}{(q+3)^2}$ where q denotes the quantity of the product sold.

(a) Write down an initial value problem involving the revenue R. Recall that an initial value problem consists of a differential equation and an initial condition.

$$\frac{dR}{dq} = \frac{60}{(q+3)^2}, \quad R(0) = 0.$$

(b) Solve the initial value problem in Part (a), and determine the behavior of the revenue if more and more of the product is sold.

$$R(q) = \int \frac{60}{(q+3)^2}dq = 60\frac{(q+3)^{-2+1}}{-2+1} + c = -\frac{60}{q+3} + c$$

$$0 = R(0) = -\frac{60}{0+3} + c = -20 + c \Longrightarrow c = 20$$

$$\boxed{R(q) = 20 - \frac{60}{q+3}}$$

$$\lim_{q\to\infty} R(q) = \boxed{20}.$$