Name ______________________________ Date ____________________

Math 10250 Activity 32: Integration by Parts and Partial Fraction Decomposition (Section 5.3)

GOAL: To find integrals using Integration by Parts and Partial Fraction decomposition.

▶ Integration by parts

IDEA: Recall that Integration by Substitution "reverses" the chain rule. Today we learn another technique, called *integration by parts*, which "reverses" the product rule.

- Let $u(x)$ and $v(x)$ be two differentiable functions. Applying the product rule, we have:

$$\frac{d}{dx}(u(x)v(x)) = u(x)v'(x) + u'(x)v(x)$$

- By the definition of an anti-derivative:

$$u(x)v(x) = \int [u(x)v'(x) + u'(x)v(x)]dx = \int u(x)v'(x)\,dx + \int u'(x)v(x)\,dx$$

- Rearranging terms, we have:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

- Note $\frac{du}{dx} = u'(x) \iff \boxed{du = u'(x)dx}$. Also, $\frac{dv}{dx} = v'(x) \iff \boxed{dv = v'(x)dx}$.

- Suppressing the variable x, we get:

$$\boxed{\int u\ dv = uv - \int vdu.} \rightarrow \textbf{Integration by Parts}$$

Example 1 Use integration by parts to find the following integrals:

(a) $\int xe^{3x}dx = x\frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}dx$

$u = x$
$du = dx$
$dv = e^{3x}dx$
$\frac{dv}{dx} = e^{3x}$
$v = \int e^{3x}dx = \frac{e^{3x}}{3}$

$= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx$

$= \frac{1}{3}xe^{3x} - \frac{1}{3}\cdot\frac{e^{3x}}{3} + c$

$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

(b) $\int x^3 \ln x dx = \int udv = uv - \int vdu$

$u = \ln x$
$du = (\ln x)'dx = \frac{dx}{x}$
$dv = x^3dx$
$\frac{dv}{dx} = x^3$
$v = \int x^3dx = \frac{x^4}{4}$

$= \ln x\frac{1}{4}x^4 - \int \frac{1}{4}x^4\frac{1}{x}dx$

$= \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3dx$

$= \frac{1}{4}x^4\ln x - \frac{1}{4}\frac{1}{4}x^4 + c$

$= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + c$

▶ Partial Fraction Decomposition

Example 2 Find $\int \frac{2}{x^2-3x+2}dx$, by first writing $\frac{2}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$.

Since $x^2-3x+2 = (x-1)(x-2)$, *we have*

$$\frac{2}{x^2-3x+2} = \frac{2}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \Longrightarrow 2 = A(x-2) + B(x-1)$$

- $x = 1 \Longrightarrow 2 = A(1-2) + B(1-1) \Longrightarrow 2 = -A \Longrightarrow A = -2$
- $x = 2 \Longrightarrow 2 = A(2-2) + B(2-1) \Longrightarrow B = 2$

$$\int \frac{2}{x^2-3x+2}dx = \int \left(\frac{-2}{x-1} + \frac{2}{x-2}\right)dx = -2\ln|x-1| + 2\ln|x-2| + c = 2\ln\left|\frac{x-2}{x-1}\right| + c$$

Example 3 Use any integration method to compute the following indefinite integrals:

(a) $\int x\sqrt{2x+9}dx$

$$= x\frac{1}{3}(2x+9)^{\frac{3}{2}} - \int \frac{1}{3}(2x+9)^{\frac{3}{2}}dx$$

$$= x\frac{1}{3}(2x+9)^{\frac{3}{2}} - \frac{1}{3}\cdot\frac{1}{2}\frac{(2x+9)^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$$

$u = x \Longrightarrow du = dx$
$dv = \sqrt{2x+9}dx \Longrightarrow \frac{dv}{dx} = (2x+9)^{\frac{1}{2}}$
$v = \int (2x+9)^{\frac{1}{2}}dx = \frac{1}{2}\frac{(2x+9)^{\frac{3}{2}}}{\frac{1}{2}+1} = \frac{1}{3}(2x+9)^{\frac{3}{2}}$

(b) $\int \frac{x+1}{x^2+2x+8}dx$

$$= \int \frac{du}{u} = \frac{1}{2}\ln|u| + c$$

$$= \frac{1}{2}\ln|x^2+2x+8| + c$$

$u = x^2+2x+8$
$du = (2x+2)dx$
$2(x+1)dx = du$
$(x+1)dx = \frac{1}{2}du$

(c) $\int (\ln x)^2 dx = x(\ln x)^2 - \int x \cdot 2\ln x \cdot \frac{1}{x}dx$

$u = (\ln x)^2$
$du = 2(\ln x)\frac{dx}{x}$
$dv = dx$

$$= x(\ln x)^2 - 2\int \ln x dx$$

$$= x(\ln x)^2 - 2(x\ln x - x) + c$$

$$\int \ln x dx = \ln x \cdot x - \int \frac{1}{x}x dx$$

$$= x\ln x - x$$

$u = \ln x$
$du = \frac{1}{x}dx$
$dv = dx$

(d) $\int \frac{5}{4-x^2}dx = \frac{5}{4}\int \frac{1}{2-x}dx + \frac{5}{4}\int \frac{1}{2+x}dx$

$$= -\frac{5}{4}\ln|2-x| + \frac{5}{4}\ln|2+x| + c$$

$\frac{5}{4-x^2} = \frac{5}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x}$
$\Longrightarrow 5 = A(2+x) + B(2-x)$
• $x = 2 \Longrightarrow 5 = A\cdot 4 \Longrightarrow A = \frac{5}{4}$
• $x = -2 \Longrightarrow 5 = B\cdot 4 \Longrightarrow B = \frac{5}{4}$

Example 4 In a study of students learning a foreign language, the number of new words $w(t)$ (as a function of time) an average student can learn in a day is modeled by the equation $\frac{dw}{dt} = 0.1(1-t)e^{-0.1t}$. If the student begins with 20 new words a day, how many new words a day can he learn after 10 days?

Solving the differential equation, we get

$$w(t) = \int 0.1(1-t)e^{-0.1t}dt = 0.1\int e^{-0.1t}dt - 0.1\int te^{-0.1t}dt = 0.1\frac{e^{-0.1t}}{-0.1} + \int td(e^{-0.1t})$$

$$= -e^{-0.1t} + te^{-0.1t} - \int e^{-0.1t}dt = -e^{-0.1t} + te^{-0.1t} + 10e^{-0.1t} + c = \boxed{(9+t)e^{-0.1t} + c}.$$

Using the initial condition, we obtain: $20 = w(0) = 9e^0 + c \Longrightarrow 20 = 9 + c \Longrightarrow c = 11$.

Thus, $w(t) = (9+t)e^{-0.1t} + 11$ *and* $w(10) = 19e^{-1} + 11 \approx 17.99 \approx \boxed{18 \text{ words}}$