Name ________________________________ Date ____________________

Math 10250 Activity 36: Continuous Income Streams (Section 6.2)

Goal: To compute Future Value (FV) and Present Value (PV) of a continuous income stream, which like water it flows into an account earning interest continuously.

•Recall that we calculated the future value (FV) assuming that we made only *one* initial deposit and no other deposits or withdrawals.

Example 1 Jen plans to retire in 30 years and has arranged for money to be continuously deposited into an account at a rate of $10,000 per year.

Q1: What is the FV of the income in 30 years if the interest is compounded continuously at a rate of 7%?

A1: The idea is that the FV is, in fact, an integral and we know that **integral** $\approx$ **Riemann sum**.
Q2: What is the Riemann sum for Jen? Start with partitioning [0, 30] into n (say $n = 3 \cdot 10^{10}$) subintervals of length $\Delta t = 30/n$ (say $\Delta t = 1/10^9$).

A2: Consider the following table.

Interval	$[0, t_1]$	$[t_1, t_2]$	$[t_2, t_3]$	$\dots$	$[t_{n-1}, 30]$
Amt. flowing into account in interval	$10,000 \cdot \Delta t$	$10,000 \cdot \Delta t$	$10,000 \cdot \Delta t$		$10,000 \cdot \Delta t$
FV of amt. at end of 30 years	$10,000 \cdot \Delta t \cdot e^{0.07(30-t_1)}$	$10,000 \cdot \Delta t e^{0.07(30-t_2)}$	$10,000 \Delta t e^{0.07(30-t_3)}$		$10,000 \Delta t e^{0.07(30-30)} = 10,000 \cdot \Delta t$

• Adding up all FVs above gives us

$$\text{Total } FV \approx 10,000 \cdot e^{0.07(30-t_1)} \cdot \Delta t + 10,000 \cdot e^{0.07(30-t_2)} \cdot \Delta t + \cdots + 10,000 \cdot e^{0.07(30-t_n)} \cdot \Delta t.$$

Q3: What is the integral for the FV of Jen's account?

A3: Letting $n \to \infty$ gives us

$$\text{Future Value of income stream} = FV = \int_0^{30} 10,000 e^{0.07(30-t)} dt$$

Then using the FTC we find $FV \approx 1,023,739$

Q4: Now what if Jen plans to retire in T years and deposits at a rate of S dollars per year with interest rate r?

A4: $\text{FV} = \int_0^T S e^{r(T-t)} dt$

Q5: Even better, what if Jen deposits at a rate of $S(t)$ dollars per year? Note that here the rate of deposit is not constant; it changes over time.

A5: The FV of a continuous income stream flowing at a rate of $S(t)$ dollars per year for T years, earning interest at an annual rate r, compounded continuously, is given by

$$\text{FV} = \int_0^T S(t) e^{r(T-t)} dt$$

Q6: What is the **PV** of the same income stream in A4?

A6: *Multiply the last formula for FV by* e^{-rT} *to get*

$$\boxed{\text{PV} = \int_0^T S(t)e^{-rt}dt}$$

Remark: Always keep in mind that $FV = PVe^{rT} \iff PV = FVe^{-rT}$

Note: This PV is the amount of money you would have to invest in an account to have FV at the end of T years (that is compounded continuously with interest rate r).

Example 2 Suppose that money is deposited steadily into a savings account at a constant rate of \$30,000 per year. Find the balance at the end of 5 years if the account pays 10% interest, compounded continuously.

$$FV = \int_0^5 30,000e^{0.1(5-t)}dt = 30,000 \cdot \left(-\frac{1}{0.1}\right) e^{0.1(5-t)}\Big|_0^5 = 300,000e^{0.5} - 300,000 \approx \$194,616.38$$

Example 3 Your company offers you the following two options:

(a) For the next 10 years it deposits money continuously into an account A at a rate of $2000e^{0.1t}$ dollars per year.

(b) At the beginning it deposits \$25,000 into an account B and nothing more during the next 10 years.

If both accounts yield 5% interest, compounded continuously, which option will you choose? Explain your answer. (Hint: Compute PV of the income stream in (a).) (*Compare FV's will give same conclusion*)

- *PV of* $(a) = \int_0^{10} 2,000e^{0.1t}e^{-0.05t}dt = 2,000 \int_0^{10} e^{0.05t}dt = \frac{2,000}{0.05}e^{0.05}\Big|_0^{10} = 40,000(e^{0.5} - 1) \approx \$25,948.85$
- *PV of* $(b) = \$25,000$

PV of (a) is greater than PV of (b) which implies FV of (a) is greater than FV of (b). Thus, option (a) is better.

Ans: PV(a) = \$25,948

Example 4 (a) In 2018, the Social Security program was providing monthly benefits to about 62 million people and total benefit payments for the year were about \$0.94 trillion[1]. Assume that for the next 10 years Social Security continues to pay benefits at this rate (\$0.94 trillion) per year steadily.

(b) On the other hand, in 2018, the Social Security had a surplus (Trust-fund) of about \$2.85 trillion, and an income stream of about \$0.98 trillion per year. Assume, again, that for the next 10 years the Social Security continues to receive this income at the same rate (\$0.98 trillion) per year steadily.

Find the Social Security surplus after 10 years, if during this period the prevailing annual interest rate is 3% compounded continuously. (Hint: Compute the difference of the two FV's.)

$$\text{FV of (a)} = \int_0^{10} 0.94e^{0.03(10-t)}dt = 0.94\left(-\tfrac{1}{0.03}\right)e^{0.03(10-t)}\Big|_0^{10} \approx 10.9622$$

$$\text{FV of (b)} = 2.85e^{0.03\cdot 10} + \int_0^{10} 0.98e^{0.03(10-t)}dt = 2.85e^{0.03\cdot 10} - \tfrac{0.98}{0.03}e^{0.03(10-t)}\Big|_0^{10} \approx 15.2758$$

$$\implies FV \approx 15.2758 - 10.9622 = 4.3136$$

Ans: Surplus = \$4.31 trillion

Question. The depletion of the Social Security Trust-fund is now projected in 2032. How can this happen?

[1] *Source*: 2017 and 2018 Trustees Reports.