# Practice B – Math 10250 Exam 2 Solutions

**1.** Assuming the population grows exponentially, we have the population at time $t$ is given by $P(t) = P(0)e^{kt}$. Since at time $t = 0$, the population was 5000, we know $P(0) = 5000$. Furthermore, we know at time $t = 5$, the following equation is satisfied $10000 = 5000e^{k\cdot 5}$. Simplifying, we find $2 = e^{5k}$. Taking natural logarithms of both sides we obtain $\ln(2) = 5k$, or $k = \ln(2)/5$. So after 15 days the population is $P(15) = P(0)e^{(\ln(2)/5)15} = 5000 \cdot 2^3 = \boxed{40000}$.

**2.** By the definition of the derivative, $\frac{d}{dx}\ln(x) = \lim_{h\to 0} \dfrac{\ln(x+h) - \ln x}{h} = \dfrac{1}{x}$ for any $x > 0$. Thus, $\displaystyle\lim_{h\to 0} \frac{\ln(5+h) - \ln 5}{h} = \frac{1}{5}$.

**3.** The equation of a line is $y - y_1 = m(x - x_1)$. If $f(x) = e^{x^2}$, we find the tangent line at $x = -1$ by setting $x_1 = -1$, $y_1 = f(-1) = e$, and using $f'(x) = 2xe^{x^2}$ we find $m = f'(-1) = 2(-1)e^{(-1)^2} = -2e$. Thus, we obtain $y - e = -2e(x+1) \Longrightarrow \boxed{y = -2ex - e}$.

**4.** Currently, we have $q = 200$, and $P(200) = 100$. Furthermore, $MP(200) = MR(200) - MC(200) = 0.15 - 0.05 = 0.1$. Using the linear approximation, $P(q) \approx P(200) + MP(200)(q - 200) \Longrightarrow P(q) \approx 100 + 0.1(q - 200) \Longrightarrow P(q) \approx 80 + 0.1q$ we can estimate $P(120) \approx 80 + 0.1(220) = \boxed{102}$.

**5.** We simplify $y = \ln(x^3) = 3\ln(x)$. Thus, we have $y' = \frac{3}{x} = 3x^{-1}$. Differentiating again, we have $y'' = \boxed{-3x^{-2}}$.

**6.** To solve $3\ln(2x) - \ln(8x) = 2$, we use the product rule for logarithmic functions, to simply the expression into $3\ln 2 + 3\ln(x) - \ln 8 - \ln(x) = 2$. Solving for $\ln(x)$ we obtain $\ln(x) = 1 + \frac{1}{2}\ln 8 - \frac{3}{2}\ln 2 = 1 + \frac{1}{2}\ln 8 - \frac{1}{2}\ln(2^3) = 1$. Exponentiating both sides, we obtain $\boxed{x = e}$.

**7.** Since the graph of $f(x)$ has a corner at $x = 3$, we know it is not differentiable at $x = 3$. $f(x)$ is clearly continuous everywhere, has positive derivative (slope of tangent line) on $(0, 3)$, and appears symmetric around $x = 3$.

**8.** First, we find the cost function $C(x) = 1000 + 5x$. Next, using revenue equals price times quantity, we obtain $R(x) = (20 - 0.01x) \cdot x = 20x - 0.01x^2$. Using profit $P(x) = R(x) - C(x) = 20x - 0.01x^2 - 1000 - 5x = 15x - 0.01x^2$ we find marginal profit $MP(x) = 15 - 0.02x$. Thus $MP(100) = 15 - 2 = \boxed{13}$.

**9.** To find the derivative of $q(x)$, we use the quotient rule .

$$q'(x) = \frac{(x+1) \cdot f'(x) - f(x) \cdot 1}{(x+1)^2} \Longrightarrow g'(1) = \frac{2 \cdot f'(1) - f(1)}{4}.$$

From looking at the graph, we observe $f(1) = 1$, and $f'(1) = 2$, thus $g'(1) = \frac{2\cdot 2 - 1}{4} = \boxed{\frac{3}{4}}$.

**10.** Using the chain rule, we find $g'(x) = f'(2x^2 + 0.5) \cdot (4x + 0)$. Thus, $g'(1) = f'(2.5) \cdot 4$. Observing the graph, we see that $f'(2.5) = 0$, thus $g'(1) = \boxed{0}$.

**11.** (A) If Iodine-131 decays exponentially, then the amount left at time $t$ is $A(t) = A(0)e^{-kt}$. We know that at time $t = 3$, we have 75% of what we started. Therefore, if we started with 100%, we will have

$$0.75 = e^{-k\cdot 3} \Longrightarrow \ln(0.75) = -3k \Longrightarrow k = -\frac{\ln(0.75)}{3}.$$

(i) Thus, the amount at time $t$, given an initial amount $y_0$ is

$$\boxed{y(t) = y_0 e^{\frac{\ln(0.75)}{3}t}}$$

(ii) After two days, the proportion which remains, if we began with 1 unit, is $\boxed{\frac{y(2)}{y_0} = e^{2\frac{\ln(0.75)}{3}}}$.

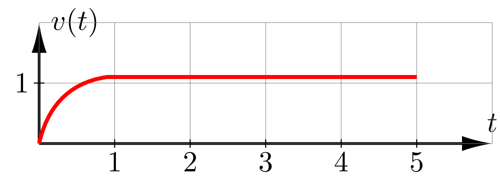
(B) The amount after $t$ days is given by the formula $A(t) = A(0)e^{rt} = 10,000e^{0.05t}$, when the interest is $r = 5\%$ compounded continuously. Thus we will have $30,000 = A(t) = 10,000e^{0.05t}$, when $3 = e^{0.05t}$ or $\ln 3 = 0.05t$ or $t = \ln 3/0.05 = 20\ln 3$.

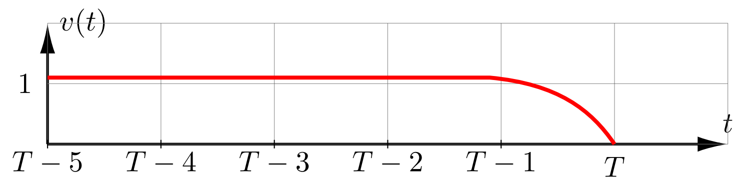
**12.** (A) Since the height at time $t$ is $s(t) = -16t^2 + 48t + 64$, its velocity at time $t$ is $v(t) = s'(t) = -16 \cdot 2t + 48$, and its acceleration at time $t$ is $a(t) = v'(t) = -32$. In particular, the answers to (i) are: $s(0) = 64$ feet above ground, $v(0) = 48$ feet/sec, and since $v(0) > 0$ it was thrown up.

(ii) The acceleration at any time $t$ is $a(t) = -32$ feet/sec$^2$.

(B) Answers may vary, examples are given below.

(ii) The total length of the trip is 85 miles. The first and the last 1 miles combined take 2 minutes. The rest of the trip at constant speed takes $84/70 = 1.2$ hours $= 72$ minutes. So $T = 72 + 2 = 74$ minutes. It follows that the average velocity is $85/74 = 1.14$ miles/min $= 68.92$ mph.




**13.** (A) Since the composition of differentiable functions, is differentiable, and the functions $e^x$, $\sqrt{x}$ and $x^2$ are differentiable for $x > 0$, the function $f(x)$ is differentiable for $x > 0$.
(i) $f(x)$ is differentiable because both $\sqrt{x}$ and $e^{x^2}$ are differentiable for $x > 0$. Thus, applying the product rule we get

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}e^{x^2} + \sqrt{x}e^{x^2} \cdot 2x = \frac{1+4x^2}{2\sqrt{x}}e^{x^2}$$

(ii) A good neighbor of 4.1 is 4 since the linear approximation at $x = 4$ is

$$f(x) \approx f(4) + f'(4)(x-4) = 2e^{16} + \frac{65}{4}e^{16}(x-4)$$

We have

$$f(4.1) \approx 2e^{16} + \frac{65}{4}e^{16}(0.1) = \frac{29}{8}e^{16}$$

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(B) (i) The slope of the secant for $h > 0$ is $\frac{g(h)-g(0)}{h} = \frac{h^3-0}{h} = h^2$, and for $h < 0$ is $\frac{g(h)-g(0)}{h} = \frac{-h^3-0}{h} = -h^2$.

(ii) Since in both cases, we have

$$\lim_{h\to 0}\frac{g(h)-g(0)}{h} = 0,$$

we conclude that this function is differentiable at $x = 0$.

**14.** (A) We have

$$T(x) = \begin{cases} 0.06x, & 0 < x < 115, \\ 6.9, & x > 115 \end{cases}$$

(ii) This function is differentiable for all $x > 0$ except at $x = 115$, where its graph has a corner.

(B) The graph below has no tangent line at $x = 2$ and a vertical tangent line at $x = 3$.