Practice B – Math 10250 Exam 3 Solutions

1.) The functions $P(t) = Ae^{0.02t}$ satisfy the differential equation $P'(t) = 0.02P(t)$, where A is an arbitrary constant. Next, we need to impose the initial condition $P(0) = 100$. Thus $P(0) = Ae^0 = A = 100$. We therefore conclude that the solution to our initial value problem is $P(t) = 100e^{0.02t}$.

2.) Recall that the marginal profit function is by definition the first derivative of the profit function, i.e. $MP(x) = P'(x)$. In our problem $MP(x) = -x + 100$, so that

$$\int (-x + 100)dx = -\frac{x^2}{2} + 100x + C$$

where C is an arbitrary constant. We therefore conclude that $P(x) = -\frac{x^2}{2} + 100x + C$ for a well defined value of the constant C which has to be determined by the condition $P(0) = 0$. We then have $C = 0$ and $P(x) = -\frac{x^2}{2} + 1000x$. Finally, we compute that that $P(10) = -\frac{10^2}{2} + 10000 = -50 + 10000 = 9950$.

3.) The oil production is assumed to be $P(t) = 30 + te^{-\frac{1}{50}t}$. By applying the product rule, we have $P'(t) = e^{-\frac{1}{50}t} - \frac{t}{50}e^{-\frac{1}{50}t} = e^{-\frac{1}{50}t}(1 - \frac{t}{50})$. We therefore conclude that the only critical point is at $t = 50$. Since the sign of P' changes from positive to negative at the critical point $t = 50$, we have that this is the maximum point. Thus, the production will peak in $2006 + 50 = 2056$.

4.) Observe that the function $y''(x) = (x^2 - 1)^2(2x - 3)^2$ is always ≥ 0. Therefore the second derivative of $y(x)$ never changes sign and then there are no inflection points.

5.) Given $f(x) = x^3 - 4x^2 + 5x - 2$, we have that $f'(x) = 3x^2 - 8x + 5$. Thus, we can find the critical points of $f(x)$ by solving the quadratic equation $3x^2 - 8x + 5 = 0$. By applying the celebrated quadratic formula we then obtain:

$$x = \frac{8 + \sqrt{64 - 60}}{6} = \frac{5}{3}, \quad x = \frac{8 - \sqrt{64 - 60}}{6} = 1.$$

The critical points are $x = 1$ and $x = \frac{5}{3}$.

6.) Since $f''(x) = e^{-x}(x^2 - 4x + 2)$ we have that the sign of f'' is the same as the sign of the polynomial $g(x) = x^2 - 4x + 2$. The zeros of the polynomial $g(x)$ are given by the points $x = 2 - \sqrt{2}$ and $x = 2 + \sqrt{2}$, in fact if we apply the quadratic formula we obtain

$$x = \frac{4 + \sqrt{16 - 8}}{2} = 2 + \sqrt{2}, \quad x = \frac{4 - \sqrt{16 - 8}}{2} = 2 - \sqrt{2}.$$

We then have that $g(x) > 0$ for $x > 2 + \sqrt{2}$ and $x < 2 - \sqrt{2}$. We therefore conclude that for $x < 2 - \sqrt{2}$ the concavity of $f(x)$ is up since $f''(x) > 0$. Thus part (a) must be false.

7.) Since $R'(x) = 50e^{-x}$ is always positive, we conclude that $R(x) = 50 - 50e^{-x}$ is increasing on the infinite interval $[0, \infty)$. Thus, the minimum point is at $x = 0$. Next, we observe that $\lim_{x\to\infty}(50 - 50e^{-x}) = 50$ so that the line $y = 50$ is a horizontal asymptote. Since the function is approaching the asymptote from below we do not have a maximum.

8.) By definition $y(x) = \frac{x}{x-1}$, so that by applying the quotient rule we can compute both $y'(x)$ and $y''(x)$. More precisely, we have

$$y'(x) = \frac{(x-1) - x}{(x-1)^2} = -\frac{1}{(x-1)^2}, \quad y''(x) = -\frac{0 - 2(x-1)}{(x-1)^4} = \frac{2}{(x-1)^3}$$

Thus, for $x < 1$ the concavity of $y(x)$ is down and therefore (c) must be false.

9.) By implicit differentiation, if we take the derivative with respect to x of the equation $x^2 + y^2 = 4$ we obtain the identity

$$2x + 2yy' = 0$$

which then implies $y' = -\frac{x}{y}$. Next, let us substitute into this equation the point $(x, y) = (\sqrt{2}, \sqrt{2})$. Thus, we obtain

$$y' = -\frac{\sqrt{2}}{\sqrt{2}} = -1.$$

10.) From looking at the graph of $f'(x)$ we know that such a function has four zeros at $x = -1$, $x = 1$, $x = 2$ and $x = 3$. By definition of critical points, we conclude that $f(x)$ has four critical points at $x = -1$, $x = 1$, $x = 2$ and $x = 3$. Thus, (a) must be false.

11 i.) We have that $\displaystyle\int (x^5 - e^{-3x} + x + 1)dx = \frac{x^6}{6} + \frac{e^{-3x}}{3} + \frac{x^2}{2} + x + C$. Thus, the revenue function has to be of the form

$$R(x) = \frac{x^6}{6} + \frac{e^{-3x}}{3} + \frac{x^2}{2} + x + C$$

where C is a constant which has to be defined by the condition $R(0) = 0$. But then

$$R(0) = 0 + \frac{1}{3} + 0 + 0 + C = 0$$

which implies $C = -\frac{1}{3}$. Concluding, the revenue function is given by

$$R(x) = \frac{x^6}{6} + \frac{e^{-3x}}{3} + \frac{x^2}{2} + x - \frac{1}{3}.$$

11 ii.) By substituting $u = x^6 + x^2 + 12$, we have that $du = (6x^5 + 2x)dx$. We therefore compute:

$$\int \frac{3x^5+x}{x^6+x^2+12}dx = \frac{1}{2}\int \frac{6x^5+2x}{x^6+x^2+12}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln(u)+C = \frac{1}{2}\ln(x^6+x^2+12)+C$$

12 i.) Since $y(x) = 2x^3 - 9x^2 + 12x + 6$, we have that $y'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$. Thus, the only critical points of $y(x)$ are give by the zeros of the quadratic polynomial $x^2 - 3x + 2$. By applying the quadratic formula we obtain:

$$x = \frac{3+\sqrt{9-8}}{2} = 2, \quad x = \frac{3-\sqrt{9-8}}{2} = 1.$$

In conclusion, $x = 1$ and $x = 2$ are the only critical points of $y(x)$.

12 ii.) Since the critical points are $x = 0$ and $x = 1$ which are both inside the closed interval $[-1, 2]$, we have to evaluate $p(x) = 2x^3 - 3x^2 + 10$ at those critical points and at the end points of the interval and then pick the maximum and minimum values. We have

$$p(0) = 0 + 0 + 10 = 10, \quad p(1) = 2 - 3 + 10 = 9,$$

and

$$p(-1) = -2 - 3 + 10 = 5, \quad p(2) = 16 - 12 + 10 = 14.$$

We therefore conclude that the Max is at $x = 2$ and the Min is at $x = -1$.

13) Since $V = 20\pi = \pi h r^2$, we have that $h(r) = \frac{20}{r^2}$. We can then express the cost function as a function of the radius only, more precisely we have

$$C(r) = 2(\pi r^2 + \pi r^2) + 3(2\pi r h(r)) = 4\pi r^2 + 6\pi r\frac{20}{r^2} = 4\pi r^2 + \frac{120\pi}{r}.$$

Next, we compute $C'(r) = 8\pi r - \frac{120\pi}{r^2}$ so that the critical point is at

$$C'(r) = 0 \quad \Rightarrow \quad 8\pi r = \frac{120\pi}{r^2} \Rightarrow \quad r^3 = \frac{120}{8} = 15$$

which therefore implies $r = 15^{\frac{1}{3}}$. Since the sign of C' passes from being negative to positive at the critical point $r = 15^{\frac{1}{3}}$ this is the minimum. Finally, the optimal height is $h(15^{\frac{1}{3}}) = \frac{20}{15^{\frac{2}{3}}}$.

14) The x-intercepts are the solution of the equation

$$f(x) = \frac{e^x}{e^x - 1} = 0 \quad \Rightarrow \quad e^x = 0$$

which we know has no solutions since the exponential function is always strictly positive. Next, we observe that for $x = 0$ we have $e^0 - 1 = 1 - 1 = 0$, so that the y-axis is a vertical asymptote for $f(x)$. In particular, we do not have a y-intercept as the function $f(x)$ is not defined for $x = 0$. In conclusion: no x-intercepts or y-intercept.

Regarding the horizontal asymptotes we need to compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. First, since $\lim_{x\to-\infty} e^x = 0$ we conclude that

$$\lim_{x\to-\infty} \frac{e^x}{e^x-1} = \frac{0}{0-1} = 0$$

so that $y = 0$ is a horizontal asymptote. Second, let us observe that

$$\lim_{x\to\infty} \frac{e^x}{e^x-1} = \lim_{x\to\infty} \frac{\frac{e^x}{e^x}}{\frac{e^x-1}{e^x}} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{e^x}} = 1$$

since $\lim_{x\to\infty} e^x = \infty$. Thus, $y = 1$ is a horizontal asymptote. In conclusion: $y = 0$ and $y = 1$ are the horizontal asymptotes.

Next, we want to compute the critical points. As a first step, we compute $f'(x)$ by applying the quotient rule

$$f'(x) = \frac{e^x(e^x-1) - e^x \cdot e^x}{(e^x-1)^2} = -\frac{e^x}{(e^x-1)^2}.$$

No critical points since $f'(x)$ is always strictly negative. This fact tells you that $f(x)$ is always a decreasing function.

Regarding the concavity we need to compute $f''(x)$. By applying the quotient rule we obtain:

$$f''(x) = \frac{-e^x(e^x-1)^2 - (-e^x)2(e^x-1)e^x}{(e^x-1)^4} = \frac{-e^{2x} + e^x + 2e^{2x}}{(e^x-1)^3} = \frac{e^{2x}+e^x}{(e^x-1)^3}$$

so that $f''(x) > 0$ for $x > 0$ and $f''(x) < 0$ for $x < 0$. In conclusion, the concavity of $f(x)$ is up for $x > 0$ and down for $x < 0$. Summarizing, the graph of $f(x)$ is as follows: