Practice B – Math 10250 Final Exam Solutions

1. F is clearly a linear, not a quadratic function of C, so this answer is false. However, solving for C we can find the inverse relation is $C = \frac{5}{9}(F - 32)$, and we can calculate that when $C = 0$, $F = 32$ and when $C = 100$, $F = 212$.

2. If t is time in the future, than $t = 0$ corresponds to the present. Since the painting increased in value by 10000 over 10 years, it is increasing at a rate of 1000 per year. Moreover, since it's current value is 20000, we deduce that the value in the future is $V(t) = 1000t + 20000$.

3. Equilibrium price/quantity is the price/quantity in which the supply and demand curves are equal. That is $-q + 6 = q + 2 \Longrightarrow 4 = 2q \Longrightarrow q = 2$. Plugging into the demand curve we find the equilibrium price is $p = -2 + 6 = 4$. The supply function is increasing and the demand function is decreasing. The slope of the demand curve is -1.

4. Since the costs are 20 dollars per unit plus 500 dollars, the cost function is $C(x) = 20x + 500$ and the marginal costs are $C'(x) = 20$. The revenue is price time quantity, so $R(x) = (-0.5x + 100)x = -0.5x^2 + 100x$. The profit function is revenue minus costs or $P(x) = -0.5x^2 + 100x - 20x - 500 = -0.5x^2 + 80x - 500$ and therefore the marginal revenue is $P'(x) = -x + 80$.

5. The amount of money invested in stocks in 40 years will be worth $FV = 100e^{0.08(40)} = 100e^{3.2}$. The value of the investment in bonds will be $FV = 100e^{0.007(40)} = 100e^{0.28}$. At any time in the future, these values will be $100e^{0.08t}$ and $100e^{0.007t}$ respectively.

6. We will use $PV = FVe^{-rt}$, we want the $FV = 2000000$, and $t = 40$ and we have $r = 0.08$ therefore, the amount we need today is $PV = 2000000e^{-3.2} \approx 81,524$.

7. To determine when the countries will have the same GDP, we set $G_A = G_B$ and solve for t. This is

$$10e^{0.03t} = 5e^{0.08t} \Longrightarrow 2 = e^{0.05t} \Longrightarrow \ln(2) = 0.05t \Longrightarrow 20\ln(2) = t.$$

   We have $G'_A(t) = 0.3e^{0.03t}$ and $G'_B(t) = 0.4e^{0.08t}$ so at $t = 0$, $G'_A(0) = 0.3$ and $G'_B(0) = 0.4$. Although country A has a larger GDP today, country B's GDP will surpass country A's at time $t = 10\ln(2)$. The GDP of country A is currently 10 trillion dollars and the GDP of country B is currently 5 trillion dollars.

8. This expression is the definition of the derivative of $\ln(x)$, which equals $\frac{1}{x}$.

9. The marginal profit at any value x is the derivative of the profict function at x, which is defined geometrically as the slope of the tangent line. The tangent line passes through the points $(0, 2)$ and $(2, 3)$, and therefore has a slope of $\frac{3-2}{2-0} = \frac{1}{2}$; i.e. the marginal profit at $x = 2$ is $\frac{1}{2}$.

10. The linear approximation is the equation of the line tangent to the graph at $x = 2$. From the previous question, we know the slope of the line is $\frac{1}{2}$, we also know it passes through the point $(2, 3)$. Therefore, the equation of this line is $P(x) \approx 3 + 0.5(x - 2)$.

11. The instantaneous rate of chang eof the function $f(g(x))$ is the derivative, which, by the chain rule equals

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

    Evaluating at 3 we obtain $f'(g(3))g'(3) = f'(3) \cdot 9$. From the graph, we see that the slope of f, i.e. f' at $x = 3$ is $\frac{6-2}{0-3} = -\frac{4}{3}$, and therefore, the instantaneous rate of change is $-\frac{4}{3} \cdot 9 = -12$.

12. This is a related rates problem. We are given that at the current time, $t = 0$, $k(0) = 10,000$ and $k'(0) = 1000$, and we are asked to find $q'(0)$. By the chain rule, we have $\frac{d}{dt}q(t) = 400[k(t)]^{-1/2}k'(t)$, and therefore, $\frac{d}{dt}q(0) = 400[k(0)]^{-1/2}k'(0) = 400[10000]^{-1/2}1000 = \frac{400}{100}1000 = 4000$.

13. To estimate the future revenue, we use linear approximation. $R(t) \approx 155 + 2.5(t-0) = 155 + 2.5t$. The year 2018 corresponds to four years in the future, and therefore, the approximate revenue is $R(4) \approx 155 + 2.5(4) = 165$.

14. We differentiate x^2 and obtain $2x$, which is positive for all x (since consumption must be positive). However, the second derivative is 2, which is also positive. Therefore, x^2 fails the second requirement of a utility function.

15. Acceleration is the second derivative of position, the first derivative is $s'(t) = 6(t-1)^2$, therefore the second derivative is $s''(t) = 12(t-1)$. Acceleration is positive for $t > 1$, and negative for $t < 1$, since the velocity is always positive, the object is accelerating when $t > 1$ and decelerating when $t < 1$.

16. we can either maximize the profits at $x = 0$, or at a critical point inside the interval $0 < x < \infty$. Taking derivatives of the profit function, we obtain $P'(x) = 10xe^{-x} - 5x^2e^{-x} = 5xe^{-x}(2-x)$. Thus, $x = 2$ is the only critical point on the interior of the domain. We compare values at the critical point and endpoints: $P(0) = 0$, $P(2) = 20e^{-2} > 0$, and $\lim_{x\to\infty} P(x) = 0$. Thus, the maximum value this function obtains is at $x = 2$.

17. Revenue is price times quantity or $R(x) = \frac{400x}{x+2}$. We take the derivative to find $R'(x) = \frac{400(x+2)-400x}{(x+2)^2} = \frac{800}{(x+2)^2}$, which never equals zero and is defined for all $x > 0$, thus there are no critical points in the interval $0 \leq x < \infty$. We evaluate $R(0) = 0$, and the limit $\lim_{x\to\infty} R(x) = 400$. Thus, $R(x)$ begins at 0 and increases towards, but never reaches the value of 400; there is not maximum revenue.

18. The profict function is $P(x) = \frac{400x}{x+2} - 8x$. The derivative of the profit function is $P'(x) = \frac{800}{(x+2)^2} - 8$; setting this equal to zero, and solving for x we have

$$\frac{800}{(x+2)^2} = 8 \Longrightarrow (x+2)^2 = 100 \Longrightarrow x + 2 = 10 \Longrightarrow x = 8.$$

This is a global maximum, since $P'(x) > 0$ for $0 < x < 8$, and $P'(x) < 0$ for $x > 8$, and $P(0) = 0$. The profit equals $P(8) = 320 - 64 = 256$.

19. We can estimate the integral of $f(x)$ from 3 to 5 using the midpoint rule, to obtain $\int_3^5 f(x)dx \approx 0.5(25 + 30 + 27 + 18) = 0.5(100) = 50$.

20. The total change in the quantity from 1 to 5 is $\int_1^5 4t\ln t dt$. Using integration by parts, with $u = \ln t \Longrightarrow du = \frac{1}{t}dt$ and $dv = 4tdt \Longrightarrow v = 2t^2$ we obtain

$$\int_1^5 4t\ln t dt = (2t^2\ln t)|_1^5 - \int_1^5 2t^2\frac{1}{t}dt = (2t^2\ln t)|_1^5 - \int_1^5 2tdt = (2t^2\ln t)|_1^5 - t^2|_1^5.$$

Evaluating the above we obtain

$$\int_1^5 4t\ln t dt = (2t^2\ln t)|_1^5 - t^2|_1^5 = 50\ln 5 - 25 + 1 = 50\ln 5 - 24.$$

21. The total amount of gas consumed is the integral $\int_0^{20} r(t)dt$. Since this is a trapezoid, we can calculate the area precisely, and it equals $\frac{1}{2}(10+5)\cdot 20 = 150$ (billions of gallons).

22. The total oil consumption over the next ten years is

$$\int_0^{10}\left(30 + 2te^{-0.01t^2}\right)dt = 30t|_0^{10} + \int_0^{10} 2te^{-0.01t^2}dt = 300 + \int_0^{10} 2te^{-0.01t^2}dt.$$

We make a u substitution; $u = t^2 \Longrightarrow du = 2tdt$ in the last integral to obtain

$$\int_0^{10}\left(30 + 2te^{-0.01t^2}\right)dt = 300 + \int_0^{100} e^{-0.01u}dt = 300 + \frac{1}{-0.01}e^{-0.01u}|_0^{100} = 400 - 100e^{-1}.$$

23. We are asked to solve the initial value problem $y' = 0.02y$, $y(0) = 4$. We know that the function $y(t) = Ae^{0.02t}$ satisfies the differential equation $y' = 0.02y$. Solving for A, we have $y(0) = 4 = Ae^{0.02(0)} = A$. Thus $y(t) = 4e^{0.02t}$.

24. We need to find c such that $\int_0^3 f(x)dx = 1$ i.e. $c\int_0^3 (9-x^2)dx = 1$. Integrating we find $c(9x - \frac{1}{3}x^3)|_0^3 = 1 \Longrightarrow c(27-9) = 1$. Solving for c we find $18c = 1 \Longrightarrow c = \frac{1}{18}$.

25. The total change in profit is equal to the integral of the marginal profit from 0 to 2.5, that is $\int_0^{2.5} MP(x)\,dx$. This integral is equal to the area of the region under the graph of $MP(x)$ from $x = 0$ to $x = 2$ minus the area of the region under the graph of $MP(x)$ from $x = 2$ to $x = 2.5$. Using a Riemann sum with $\Delta x = 0.5$ we estimate the first area by $(7.6 + 6.4 + 4.4 + 1.6)(0.5) = 10$. The second area is equal to $\frac{1}{2} \cdot 2 \cdot 0.5 = 0.5$. Therefore, the total change in profit is equal to 10-0.5=9.5.

26. We have the $MP = MR - MC = -0.1x + 110 - 0.1x - 80 = -0.2x + 30$. We are asked to determine $\int_{50}^{100} MP(x)dx = \int_{50}^{100}(-0.2x+30)dx = -0.1x^2 + 30x|_{50}^{100} = -0.1(10000) + 30(100) + 0.1(2500) - 30(50) = 750$.

27. Recall that exponential decay is of the form $y = y_0e^{-kt}$, where y is the amount at time t and y_0 is the initial amount. If the half life is 10 years, than $\frac{1}{2} = e^{-10k}$ where k is the decay rate. Solving for k we obtain

$$\ln(\frac{1}{2}) = -10k \Longrightarrow k = 0.1\ln 2.$$

We are asked to solve for the time at which one unit decays to 0.2 units, which is to solve for t in the equation

$$0.2 = e^{-0.1\ln(2)t} \Longrightarrow \ln(0.2) = -0.1\ln(2)t \Longrightarrow t = -\frac{\ln(0.2)}{0.1\ln 2}.$$

28. We have $f(0) = \frac{e^{2\cdot 0}-1}{e^{2\cdot 0}+1} = \frac{0}{2} = 0$. Differentiating the function, we obtain $f'(x) = \frac{2e^{2x}(e^{2x}+1)-2e^{2x}(e^{2x}-1)}{(e^{2x}+1)^2} = \frac{4e^{2x}}{(e^{2x}+1)^2} > 0$. Thus, $f'(0) = \frac{4e^{2\cdot 0}}{(e^{2\cdot 0}+1)^2} = \frac{4}{2^2} = 1$, and we see that there is no point x where the derivative of $f(x)$ is zero. To find horizontal asymptotes we take limits as $x \to \pm\infty$. We have

$$\lim_{x\to-\infty} \frac{e^{2x}-1}{e^{2x}+1} = -1, \quad \text{and} \quad \lim_{x\to\infty} \frac{e^{2x}-1}{e^{2x}+1} = \lim_{x\to\infty} \frac{e^{2x}-1}{e^{2x}+1}\frac{e^{-2x}}{e^{-2x}} = \lim_{x\to\infty} \frac{1-e^{-2x}}{1+e^{-2x}} = 1.$$

The linear approximation at $x = 0$ of $f(x)$ is $f(x) \approx f(0) + f'(0)(x - 0) = x$.

29. Letting h be the height and r be the radius, we have the volume of the can is $V = \pi r^2 h = 2000\pi cm^3 \Longrightarrow h = \frac{2000}{r^2}$. The surface area of the can is $S = \pi r^2 + \pi r^2 + \pi 2rh = 2\pi r^2 + 4000\pi\frac{1}{r}$. Taking a derivative and setting it equal to zero, we find $S'(r) = 4\pi r - 4000\pi\frac{1}{r^2} = 0 \Longrightarrow r^3 = 1000 \Longrightarrow r = 10$. The optimal height is then $h = \frac{2000}{100} = 20$. We check that this is a maximum, by evaluating S' for $0 < r < 10$ and $r > 10$.

30. Although fig newtons are delicious, this is not why Isaac Newton is famous!