Math 10360 – Example Set 08A
Section 15.4 Double Integral in Polar Coordinates

Polar Coordinates. For circular regions centered at the origin, it is often simpler to parametrize (label) the region in terms of polar coordinates (r distance from the origin, and θ angle measured from the positive half of the x-axis) instead of the cartesian (or rectangular) coordinates (x, y).

The relations between the polar coordinates (r, θ) of a given on the plane and its cartesian coordinates (x, y) are given by

$$x = r\cos(\theta); \qquad y = r\sin(\theta)$$

As a convention, we may restrict $r > 0$ and $0 \le \theta < 2\pi$. Can you see the following the relations?

$$x^2 + y^2 = r^2; \qquad \tan\theta = \frac{y}{x}$$

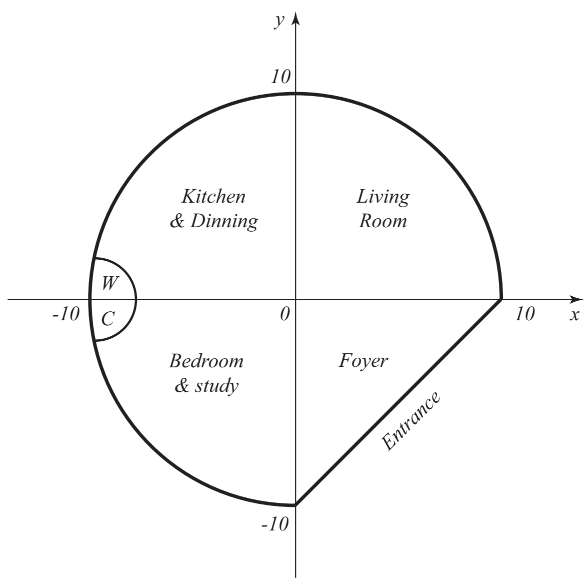
1. Convert the following rectangular coordinates to polar coordinates: **(a)** $(3, -3)$ and **(b)** $(-2, -\sqrt{3})$.

2. Convert the following polar coordinates to rectangular coordinates: **(a)** $(\sqrt{2}, \pi/4)$ and **(b)** $(2, 7\pi/6)$.

3. Recall the hobbit house where the height of the house over each point on the floor (under the roof) is given by the function

$$f(x, y) = 40 - 2x + 2y \quad \text{meter}$$

Find the volume of the house enclosed by the roof over the kitchen, bedroom and living room area.



Math 10360 – Example Set 08B
Section 15.4 Double Integrals in Polar Coordinates

1. (15.4 Appln) The density of a quarter of a disc of radius 2m centered at the origin sitting in the first quadrant is given by the function

$$f(x, y) = 2x^2 + y^2 \text{ kg/m}^2.$$

Find the total mass of the quarter disc in kg.

2. (15.4 Appln) A variety of deep sea worm is distributed about a hydrothermal vent according to the population density

$$\rho(r, \theta) = \frac{8000 \sin^2(\theta/2)}{9 + r^2}$$

thousand per sq. miles where $1 \leq r \leq 3$ is the distance (in miles) from the vent. Find the total population of the sea worm.

Math 10360 – Example Set 08C
Separable Equations (9.1)

Separable Equations (9.1). Let $p(x)$ be a smooth function in x and $q(y)$ be a smooth function in y. Then a differential equation $\dfrac{dy}{dx} = F(x,y)$ is said to be separable if $F(x,y)$ can be written as a product of two smooth functions $p(x)$ and $q(y)$. That is $\dfrac{dy}{dx} = p(x)\cdot q(y)$.

Question: Is $\dfrac{dy}{dx} = \dfrac{p(x)}{q(y)}$ separable? What about $\dfrac{dy}{dx} = \dfrac{q(y)}{p(x)}$?

Separable Equations are solved by Method of Separation:

$$\frac{dy}{dx} = p(x)\cdot q(y) \iff y'(x) = p(x)\cdot q(y) \iff \frac{1}{q(y)}\cdot y'(x) = p(x).$$

Integrating both sides with respect to x gives: $\displaystyle\int \frac{1}{q(y)}\cdot y'(x)\,dx = \int p(x)\,dx.$

Note that $\dfrac{dy}{dx} = y'(x)$ ie $dy = y'(x)dx$. Thus we have: $\displaystyle\int \frac{1}{q(y)}\,dy = \int p(x)\,dx.$

Performing the integrals on the left and right hand sides give the an equation where we can usually find y in terms of x or, in general, an implicit relation between x and y.

Remark: One can skip some steps in the Method of Separation above by writing:

$$\frac{dy}{dx} = p(x)\cdot q(y) \iff \frac{1}{q(y)}\,dy = p(x)\,dx \iff \int \frac{1}{q(y)}\,dy = \int p(x)\,dx.$$

1. Solve for the general solution of $y'(x) = 3x^2y$. Find the particular solution such that $y(0) = -2$.

Application to Autonomous Equations: $\dfrac{dy}{dt} = F(y)$

2. (9.1/9.2) Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. A roast turkey is taken from an oven when its temperature is 175 ^0F and is placed on a table in a room where the temperature is 70 ^0F. Temperature of the turkey falls to 160 ^0F after half an hour. Apply Newton's Law of Cooling to find the temperature of the turkey after 45 minutes.

3. (9.1) Some scientists found that the fraction $y(t)$ of species of fish in a lake is changing according to the model

$$64\frac{dy}{dt} = -1 - (y-1)^2$$

where $0 \le y \le 1$ and t is in years. If the initial fraction of species in the lake is assumed to be at 1, solve for $y(t)$. Explain from the model why the number of species is decreasing. How long will it take for the species of fish to completely disappear?

(See 50 years left for sea fish: http://news.bbc.co.uk/2/hi/science/nature/6108414.stm)