Math 10360 – Example Set 10A

Application to Autonomous Equations: $\dfrac{dy}{dt} = F(y)$

1. An outbreak of zombies was discovered in the rural City of Sweet Water. The all embracing Mayor Dumass of the city believes that humans and zombies could coexist and wanted laws set up to stop (the still human) people in the city from an all out hunting spree. Renown scientist Dr Madd who survived a recent zombie attack has called for a City Council meeting to convince Mayor Dumass and all administrators of the serious dangers of an unchecked zombie population. Recent estimates give the number of zombies at one thousand and the total city population (both zombies and humans) at forty thousand. If zombie infection occurs at a rate of 0.1 thousand per day per one thousand human per one thousand zombie, how would scientitst Dr Madd convince the City Council that they need to act fast and hit hard on the zombie population before it is too late?

Population Modeling.
A population y of a single species with unrestricted growth is given by the differential equation

$$\frac{dy}{dt} = ky.$$

Here t is the time variable and k is the growth constant. If k is positive, we know that y grows exponentially with rate k. If k is negative, we know that y decays exponentially with rate k.

But the exponential growth model is unrealistic for population of single species (like fish) in the real world because resources for the population, like food and room to grow, are limited. In reality we will need to consider growth models that accounts for a maximal possible population. In 1845, Pierre Verhulst constructed a growth model in which the growth rate decreases to zero as the population reaches its maximal possible size. This model is commonly know as the logistic model and is given by the differential equation, for some fixed $k, N > 0$:

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{N}\right).$$

2a. (9.1/9.2) Write down an initial value problem for a population (in thousands) of fish growing exponentially with growth constant 0.5 and initial population 5 thousand. What is unrealistic about this model?

2b. (9.4) A population of fish grows with growth constant (intrinsic growth rate) of 0.5 in a lake of carrying capacity 10 thousand is modeled by the logistic differential equation

$$\frac{dp}{dt} = 0.5p\left(1 - \frac{p}{10}\right)$$

where p is its population measured in thousands. If the initial population is 5 thousand, find a formula for $p(t)$. What is the value of $\lim_{t\to\infty} p(t)$?

Math 10360 – Example Set 10B

1. Solve the initial value problem: $xy' + y = e^{2x}$ and $y(1) = 0$. Hint: Take a good look at the left-handside.

Remark: The equation in Q2 is a special type of differential equation call a linear first order equation. In general, it takes the form:

$$y' + A(x)y = B(x).$$

What are $A(x)$ and $B(x)$ in Q2? How do we solve these equations in general?

2. Solve the following equations using integrating factors:

a. $y' - (\tan x)y = 1; y(0) = 3 \qquad -\pi/2 < x < \pi/2$

b. $x\dfrac{dy}{dx} + 5y = \dfrac{e^x}{x^3}$

Math 10360 – Example Set 10C

1a. A tank contains 800 L of fresh water. Brine that contains 0.05 kg of salt per liter enters the tank at a tank at a rate of 40 L/min. Brine is drained from the tank at the same rate of 40 L/min. Find an expression for the amount of salt in the tank at any time t.

1b. What if the brine in Q1a. is drained out at slower rate of 10 L/min? Find an expression for the amount of salt in the tank at any time t in this case.

2. A tank with a capacity of 400 liters is full of a mixture of water and chlorine with a concentration of 0.05 grams of chlorine per liter. Chlorinated water with concentration of 0.01 grams of chlorine per liter is pumped into the tank at a rate of 4 liters per second. The mixture is kept stirred and is pumped out at a rate of 5 liters per second. Find the amount of chlorine in the tank as a function of time.