Math 10360 – Example Set 13A
Section 14.6: Chain Rule

Section 14.6 (Chain Rule): Implicit Differentiation

1. Given that $ze^{x+2y} + z^2 - x - y = 0$. Find $z_x = \dfrac{\partial z}{\partial x}$ and $z_y(1, -1, 0)$.

Topic: Applications of Chain Rule

Linear Approximation of Change in a Function. Section 14.4 (Pg 808).

a. Consider a particle moving from point (a, b) to point $(a+k, b+h)$. If the particle travel at a constant speed and the total duration of the motion is 1 second, find in terms of time t (in seconds), a formula for the position (x, y).

b. Consider a function $f(x, y)$ such that its first partial derivatives exist for all points near (a, b). If (x, y) is a point on the line segment in Q2(a), find a formula for the rate of change of f with respect to t.

c. For a small change in time Δt, let the corresponding change in x from a be Δx, the corresponding change in y from b be Δy, and Δf be the corresponding change in f from $f(a, b)$. Then we have

$$\frac{\Delta f}{\Delta t} \approx \left.\frac{df}{dt}\right|_{t=0}.$$

Prove that

$$\boxed{\Delta f \approx \frac{\partial f}{\partial x}(a, b) \cdot \Delta x + \frac{\partial f}{\partial y}(a, b) \cdot \Delta y}$$

where $\Delta f = f(a + \Delta x, b + \Delta y) - f(a, b)$. This boxed formula is called the Linear Approximation of change in f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

Alternately, for all (x, y) near (a, b) then $x = a + \Delta x$ and $y = b + \Delta y$ for some small Δx and Δy. Then we can estimate $f(x, y)$ by the formula:

$$\boxed{f(x, y) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b)}$$

We call the right hand side the linearization of $f(x, y)$ at (a, b).

2. Using linear approximation, estimate the change in $g(x, y) = xe^{x^2 y}$ when (x, y) changes from $(1, 0)$ to $(0.9, 0.2)$. That is estimate the value $g(0.9, 0.2) - g(1, 0)$. Find also the linearization of $g(x, y)$ at $(1, 0)$.

Math 10360 – Example Set 13B
Section 14.6: Chain Rule
Section 14.8: Linear Approximation

Sensitivity and Elasticity: Let $z = f(x, y)$. Set $x = a$, $y = b$. Then the sensitivity of the quantity z relative to x at (a, b) is measured by

$$\frac{\partial f}{\partial x}(a, b).$$

We call this the **sensitivity coefficient** of f with respect to x at (a, b).

Likewise, the sensitivity coefficient of f with respect to y at (a, b) is $\dfrac{\partial f}{\partial y}(a, b)$.

On the other hand, the elasticity of the quantity z relative to x is the percentage change in z given a 1% change in x from $x = a$ (with no change in $y = b$). This percentage change in z is also called **elasticity coefficient** with respect to x at (a, b). Likewise, the elasticity of the quantity z relative to y is the percentage change in z given a 1% change in y from $y = b$ (with no change in $x = a$). This percentage change in z is also called elasticity coefficient with respect to y at (a, b).

Let $z = f(x_1, x_2, ..., x_n)$. Set $x_1 = a_1$, $x_2 = a_2$, ... $x_n = a_n$. Then the sensitivity of the quantity z relative to x_i at $(a_1, a_2, ..., a_n)$ is measured by

$$\frac{\partial f}{\partial x_i}(a_1, a_2, ..., a_n).$$

We call this the **sensitivity coefficient** of f with respect to x_i.

On the other hand, the **elasticity** of the quantity z relative to x_i is the **linear approximation** of the percentage change in z given a 1% change in x_i from $x_i = a_i$ (with no change in the other variable). This **estimated** percentage change in z is also called **elasticity coefficient** with respect to x_i at $(a_1, a_2, ..., a_n)$. In general the elasticity coefficients of z are dependent on the independent values $x_1, x_2, ..., x_n$.

1. Consider a cylindrical rod with height (length) 100cm and diameter 5cm.

1a. If the measuring instrument has an error of 0.1 cm, estimate using linear approximation the corresponding (propagated) error in the value of the volume if the above measurements are used.

1b. What is sensitivity of the volume of the given rod to its (i) height and (ii) diameter? That is compute the sensitivity coefficients (first partial derivatives) with respect to the height and (independently) the diameter at $(100, 5)$.

1c. Discuss the elasticity (proportional sensitivity) of the volume relative to its dimensions. That is discuss how the percentage change in volume for a 1% change in the height (alone) compared to the percentage change in volume for a 1% change in the diameter (alone).

Math 10360 – Example Set 13C
Section 10.1 Sequences
Section 10.2 Summing an Infinite Series

Introduction to Sequences and Series. A sequence (of numbers) is an ordered listing of numbers.

1. For each sequence below, write down (A) the formula for the general term of the sequence example and (B) limit of the sequence.

1a. $\frac{2}{4 \cdot 5}, \quad \frac{2}{5 \cdot 6}, \quad \frac{2}{6 \cdot 7}, \quad \frac{2}{7 \cdot 8}, \quad \cdots$

(A) General term: $a_n =$ ____________________________ for $n = 1, 2, 3, ...$

(B) Limit of the sequence $\{a_n\}_{n=1}^{\infty}$ is the value a_n approaches as n gets larger and larger unboundedly. Find the limit of the sequence $\{a_n\}_{n=1}^{\infty}$.

Notation: $\lim_{n \to \infty} a_n \stackrel{?}{=}$

1b. $2, \quad -2, \quad 2, \quad -2, \quad \cdots$

(A) General term: $b_n =$ ____________________________ for $n = 1, 2, 3, ...$

(B) Find the limit of the sequence $\{b_n\}_{n=1}^{\infty}$.

Notation: $\lim_{n\to\infty} b_n \stackrel{?}{=}$

2. The sum of a sequence is called a **series**. For example adding the terms of the sequence $\{a_n\}_{n=1}^{\infty}$ in Q1a above gives the series:

$$\sum_{n=1}^{\infty} a_n \stackrel{?}{=}$$

Find the sum (value) of the above series using the following steps

a. Find S_N the Nth partial sum of the given series.

b. Find the value of the given series by taking the limit of its Nth partial sum. Is the given series convergent?

3a. Find the sum of the first 51 terms in the series $\sum_{n=3}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$.

3b. Is the series $\sum_{n=3}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ convergent?

Summary. The sum (value) of a series $\sum_{n=1}^{\infty} a_n$ is found using

The **Nth partial sum** of the series $\sum_{n=1}^{\infty} a_n$ is the sum S_N of the first Nth terms of the series that is the value: $S_N =$ ____

The sum of a series $\sum_{n=1}^{\infty} a_n$ is defined to be the limit of the Nth partial sum S_N. That is the value:

$$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} \left(\rule{8cm}{0.4pt} \right) = \lim_{N \to \infty} S_N$$

(a) If $\lim_{N \to \infty} S_N = S$ then we say that the series coverges to S and write $\sum_{n=1}^{\infty} a_n = S$ (the sum of the infinite series).

(b) If $\lim_{N \to \infty} S_N$ does not exist then we say that the series diverges and the sum of the infinite series $\sum_{n=1}^{\infty} a_n$ does not exist.