Math 10360 – Example Set 14A

Geometric Sequence and Series

Definition (Geometric Sequence): A sequence $\{c_n\}_{n=1}^{\infty}$ is said to be geometric if the ratio of its consecutive terms $\frac{c_{n+1}}{c_n}$ is a ____________________.

We call such a sequence a **geometric sequence** with first term $c_1 = c$ and common ratio r. We list out the first four terms of the geometric sequence:

The Nth term c_N of the sequence is given by __________________________.

Definition (Geometric Series): We called an infinite sum (series) of the form

$$c + cr + cr^2 + cr^3 + \cdots + cr^{n-1} + \cdots$$

a geometric series with first term c and common ratio r.

(Finite Sums of a Geometric Series): The sum formula for the first N terms of a geometric series:

$$c + cr + cr^2 + cr^3 + \cdots + cr^{N-1} = \frac{c(1-r^N)}{1-r}$$

Proof: Consider first the finite sum $S_N = c + cr + cr^2 + \cdots + cr^{N-1}$

$$S_N = c + cr + cr^2 + cr^3 + \cdots + cr^{N-1}$$

$$rS_N = \quad cr + cr^2 + cr^3 + \cdots + cr^{N-1} + cr^N$$

$$S_N - rS_N \stackrel{?}{=}$$

So $(1-r)S_N \stackrel{?}{=}$ $\qquad \Rightarrow S_N \stackrel{?}{=}$

Take limit $N \to \infty$ of the **finite sum** of a geometric series to find the sum (to infinity) of a geometric series with first term c and common ratio r.

Summary: Sum formulas for Geometric Series:

Consider a geometric series with first term c and common ratio r.

The sum formula for the first N terms of a geometric series:

$$c + \ cr \ + \ cr^2 \ + \ cr^3 \ + \ \cdots \ + \ cr^{N-1} = \underline{\hspace{6cm}}$$

If $|r| < 1$ then the geometric series is **convergent** is sum is given by

$$c + cr + cr^2 + cr^3 + \cdots + cr^{n-1} + \cdots = \underline{\hspace{6cm}}$$

If $|r| \geq 1$, then the geometric series is divergent.

1. What are the common ratio and first term of the geometric series $\displaystyle\sum_{n=3}^{\infty} \frac{2^{2n}}{3^n}$. What is the 20th partial sum? What is the sum (to infinity) of the series.

Geometric Series and its Applications

2. Rewrite each of the following repeated decimals as a fraction.

a. $0.\bar{9} = 0.999\cdots \stackrel{?}{=}$

b. $3.0\,\overline{12} = 3.0121212\cdots \stackrel{?}{=}$

3. A drug is designed so that 60% remains in the body at the end of each 24 hour period (one day). If 30 mg of the drug is given daily to a patient find (A) the amount of drug in the body after 10 days before the next dose is given, (B) the approximate amount of drug in the body after a very long time assuming measurement is done **before** the next dose is given. (C) Estimate also the **range** of the approximate amount of drug in the body whenever we take a measurement after taking a drug for a long time.

Math 10360 – Example Set 14B

1. Consider the sequence $\{c_n\}_{n=1}^{\infty}$ given by the iterative formula $c_{n+1} = (n+1)c_n$ for $n \geq 1$ and $c_1 = 1$. (a) Is the sequence $\{c_n\}$ geometric? (b) Find a formula for the general term c_n.

Testing Convergence of General Series.

Recall that a **geometric series** $\sum c_n$ with common ratio $\dfrac{c_{n+1}}{c_n} = r$. Then we have:

(a) $\sum c_n$ converges if ________________________.

(b) $\sum c_n$ diverges if ________________________.

For general series which are **NOT** geometric we can apply the Ratio Test.

Ratio Test Let $\sum a_n$ be a series with no zero terms. Consider the value ρ given by

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then we have the following:

(a) $\sum a_n$ converges (in fact, absolutely) if ________________________.

(b) $\sum a_n$ diverges if ________________________ or ________________________.

(c) The Ratio Test is inconclusive if ________________________.

2. Determine if the following series are convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-7)^n}{n^5}$ **b.** $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$ **c.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+2}$

Introduction to Power Series

A **Power Series** can be thought about as a polynomial with infinitely many terms or arbitrarily high degree. Here are some examples:

(1) $$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

(2) $$\sum_{k=1}^{\infty} \frac{1}{k+1}(x-2)^k = \frac{1}{2}(x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3}(x-2)^3 + \cdots + \frac{1}{n+1}(x-2)^n + \cdots$$

A general power series has the form:

$$a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

where the coefficients $a_0, a_1, a_2, a_3, \ldots$ is a sequence of real numbers.

We call this a power series centered at $x = c$. Fill in the blanks below.

(1) $$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

(1) is called a power series centered at $x =$ ________.

(2) $$\sum_{k=1}^{\infty} \frac{1}{k+1}(x-2)^k = \frac{1}{2}(x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3}(x-2)^3 + \cdots + \frac{1}{n+1}(x-2)^n + \cdots$$

(2) is called a power series centered at $x =$ ________.

3. Find the values of x for which each of the following power series is convergent. You may ignore the discussion if the power series is convergent at the end-points of the interval found. What is the radius of convergent?

a. $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^3}$ **b.** $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k}}{2k+1}$ **c.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Math 10360 Example Set 14C
Sections 10.7 & 10.8 Taylor Polynomials & Taylor Series

A function $f(x)$ is said to be **analytic** if it has a (convergent) power series representation for each c i.e.

$$(1)\ f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots \qquad \text{for } -r < (x-c) < r.$$

where the coefficients a_i and radius of convergent r are to be determined. We can this series **The Taylor Series of the function** $f(x)$ centered at $x = c$

For the special case of $c = 0$, we get:

$$(2)\ f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \qquad \text{for } -r < x < r.$$

We call (2) the **Maclaurin Series** for $f(x)$ or the Taylor Series for $f(x)$ centered at $x = 0$.

The geometric series summation formula give us an example of the Taylor series of $f(x) = \dfrac{1}{1-x}$ center at $x = 0$:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1.$$

We will discuss in the next few lessons how to find the Taylor Series and its partial sums the Taylor Polynomials. The interval of convergence of Taylor Series are found using the Ratio Test.

1. (Formula for Taylor Series) Using repeated differentiation, show that the coefficients a_0, a_1, a_2, ... a_n, ... in the Taylor series for $f(x)$ centered at $x = c$:

$$a_0 = f(c), \qquad a_n = \frac{f^{(n)}(c)}{n!}$$

This gives us the following formula for the Taylor series for $f(x)$ centered at $x = c$.

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

The partial sum up till the degree N term of the Taylor series for $f(x)$ centered at $x = c$ are called the **Nth Taylor Polynomial** of $f(x)$ centered at $x = c$. This polynomial is given by

$$T_N(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(N)}(c)}{N!}(x-c)^N.$$

Remark: The partial sums of the Taylor series is often used to estimate the value of $f(x)$. Specifically, we have:

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(N)}(c)}{N!}(x-c)^N$$

for x near to c.

Here are some special cases of Taylor polynomial:

(a) The 1st Taylor polynomial for $f(x)$ centered at c is $T_1(x) =$ ______________________________ .

(Linear Approximation of $f(x)$ at $x = c$)

(b) The 2nd Taylor polynomial for $f(x)$ centered at c is $T_2(x) =$ ______________________________ .

(c) The 3rd Taylor polynomial for $f(x)$ centered at c is $T_3(x) =$ ______________________________ .

We have the following theorem:

$\boxed{\textbf{Theorem.}}$ If $f(x)$ is analytic then there exists some r such that for some interval $c - r < x < c + r$ containing c, we have:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

(a) For $c - r < x < c + r$ (especially for those x near c), we have the approximation

$$f(x) \approx T_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f^{(2)}(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

(b) The accuracy of the approximation in (a) improves as n increases. More specifically,

$$f(x) = \lim_{n\to\infty} T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Remark: For the special case where $x = 0$, the Taylor series for the function $f(x)$ centered at $x = 0$ is also call the Maclaurin series for $f(x)$. This is simply the power series representation of $f(x)$ in x:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

1. Find the 3rd Taylor polynomial for the function $\ln(x+2)$ centered at -1, and estimate $\ln(0.8)$.