Math 10360 – Example Set 16A
Section 14.8

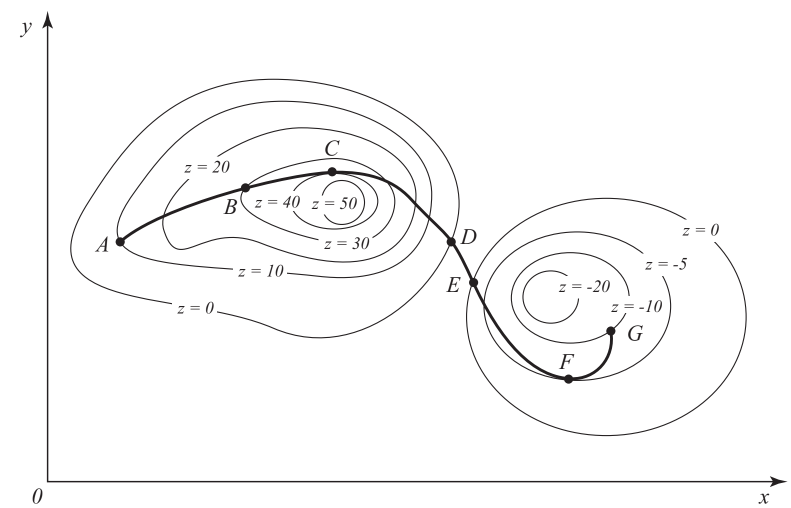
Topic: Optimization with a Constraint Using Lagrange Multipliers

Idea: Recall that for a continuous function $y = f(x)$ on a closed and bounded interval $a \leq x \leq b$, we optimize $f(x)$ with the following two facts:

(a) $f(x)$ must attain it minimum and maximum for some values of x in the interval $a \leq x \leq b$.

(b) The minimum and maximum of $f(x)$ occurs at the end points (i) $x = a, b$ or at (ii) critical points in $a < x < b$.

The range $a \leq x \leq b$ of values of x is the **constraint** on which $f(x)$ is optimized. However for multivariable functions the constraint may be some complicated relation satisfied by the independent variables. For example, in the hiking exercise you did you are reading the highest point on your path above sea level and the lowest point on your path below sea level. In that context, the constraint is the hiking path on the $xy-$plane while the function you are optimizing is the height function.



Let the height function be given by $z = f(x, y)$ and the equation of the (projected) path be $g(x, y) = 0$.

From geometric considerations, we see that the constraint curve and the contour curve at a possible

min or max must share the same ______________________________.

Therefore the critical points on the path are given by the equations:

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = \lambda \dfrac{\partial g}{\partial x}(x,y) & (1) \\ \dfrac{\partial f}{\partial y}(x,y) = \lambda \dfrac{\partial g}{\partial y}(x,y) & (2) \\ g(x,y) = 0 & (3) \end{cases}$$

where x, y, and λ are to be determined. Here (x, y) are the critical points. Note that Equation (3) ensures the solution is on the constraint curve. The first two equations are called Lagrange Multipliers.

If the constraint path is closed and bounded (either a closed loop or curve including end points without self crossings) then the function $f(x, y)$ must attain minimum and maximum at some points on the path.

1a. The height of the slanted roof a house is given $h(x, y) = 2x+4y+20$. A spider on the roof is observed from the top view crawling on the closed path $x^2 + y^2 = 4$. What is the minimum and maximum height attained by the spider?

In this context, the function we need to optimize is the height $h(x, y) = 2x+4y+20$ with constraint $x^2+y^2-4 = 0$.

Can you roughly draw a picture to depict the path of the spider on the roof showing where the graph of $x^2 + y^2 = 4$ is in relation to the actual path of the spider? Use Lagrange multipliers to find the minimum and maximum heights of the spider.

1b. How would you change your answer if the spider only crawled on the path that tracks the upper semi-circular part of the curve $x^2 + y^2 = 4$?

Extra Example

2. The Cobb-Douglas production P function for a certain item is given by

$$P(K, L) = 5K^{0.3}L^{0.7}$$

where K is the capital input and L is the labour input both in millions of dollars. Find the marginal of P relative to capital and the marginal of P relative to labour when $K = 6$ and $L = 8$. Find also the elasticity of P relative to K and the elasticity of P relative to L when $K = 6$ and $L = 8$.