Solutions: 18.100A exam 1 from Spring 2007

Note Title 3/8/2009

1.) Let $\varepsilon > 0$

For $n > \frac{1}{\varepsilon}$, we have

$$\left|1 - \left(1 + \frac{1}{n^2}\right)\right| = \frac{1}{n^2} \leq \frac{1}{n} < \varepsilon$$

Therefore $1 + \frac{1}{n^2} \approx_\varepsilon 1$ for $n \gg 0$.

2.) A bounded sequence must posess a convergent subsequence by the Bolzano-Weierstass thm. Let L be the limit of such a subsequence. Then by the cluster point theorem, L must be a cluster point of the original sequence.

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3. (a) $\frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2} \longrightarrow 0$ and is strictly decreasing.

By Cauchy's thm,

$\sum (-1)^n \frac{n+1}{n^2}$ converges

(b) $1+\frac{1}{n^2} \longrightarrow 1 \neq 0$ (by problem 1)

The series diverges by the n^{th} term test.

(c) We have

$$0 \leq \left|\frac{(\sin n)^n}{2^n}\right| \leq \frac{1}{2^n}$$

Since $\sum \frac{1}{2^n}$ converges (geometric series)

$\Longrightarrow$ comparison thm $\sum \left|\left(\frac{\sin n}{2}\right)^n\right|$ converges

$\Longrightarrow$ Absolute convergence thm $\sum \left(\frac{\sin n}{2}\right)^n$ converges

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4.) (contradiction) Suppose not. Then we have

$L \geq a$ for all $a \in A$.

But then $M \leq L$ (sup-2)

Contradiction.

We deduce that there must be some $a \in A$ so that $L < a$

5) We study the ratios (assuming $x \neq 0$, where the thing obviously converges)

$$\left| \frac{\frac{x^{n+1}}{(2n+2)!}}{\frac{x^n}{(2n)!}} \right| = \frac{|x|}{(2n+1)(2n+2)} \longrightarrow 0 < 1$$

(Regardless of x).

Therefore, by the Ratio Test, the series converges absolutely for all x.

$\Longrightarrow$ Radius of convergence $= \infty$.