Solutions to Exam 2, spring 2007

Note Title 4/14/2009

1.) Take $\varepsilon = \min(f(x_0) - a,\ b - f(x_0))$

Then: $f(x) \approx_{\varepsilon} f(x_0)$ for $x \approx x_0$

since f is continuous

$$\Longrightarrow f(x) \in \underset{(a,b)}{\underset{\cap|}{(f(x_0)-\varepsilon,\ f(x_0)+\varepsilon)}}$$

for $x \approx x_0$.

2.) $\sin(2\pi n) = 0 \qquad n \in \mathbb{Z}$

$\sin(2\pi n + \pi) = 1 \qquad n \in \mathbb{Z}$

$\sin x$ is continuous.

$\underset{IVT}{\Longrightarrow}$ $\exists\ x_n \in [2\pi n,\ 2\pi n + \pi]$

such that $\sin(x_n) = 1/3$

$\{x_n\}$ $n \in \mathbb{Z}$ is an infinite set.

3.) Let $b = f(0)$

For $x > 0$, apply MVT to $[0, x]$

get
$$f(x) - f(0) = f'(x)(x-0)$$
$$\parallel \qquad\qquad \parallel$$
$$f(x) - b \qquad mx$$

$$\Longrightarrow f(x) = mx + b$$

For $x < 0$, apply MVT to $[x, 0]$

get
$$f(0) - f(x) = f'(x)(0-x)$$
$$\parallel \qquad\qquad \parallel$$
$$b - f(x) \qquad -mx$$

$$\Longrightarrow f(x) = mx + b.$$

$$f(x) = (1+x)^{-2}$$

$$f'(x) = -2(1+x)^{-3}$$

$$f''(x) = 2\cdot 3\,(1+x)^{-4}$$

$$f'''(x) = -2\cdot 3\cdot 4\,(1+x)^{-5}$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^n (n+1)!\,(1+x)^{-(n+2)}$$

$$T_n(x) = \sum_{k=0}^{n} (-1)^k (k+1)\, x^k$$

$$R_n(x) = (-1)^{n+1}(n+2)\overbrace{(1+c_n)^{-(n+2)}}^{\text{smaller than } 1 \text{ since } c_n > 1}\, x^{n+1}$$

for some $c_n \in (0, x)$

$$|R_n(x)| \leq (n+2)\, x^{n+1}$$

Now, $\sum (n+2)\, x^{n+1}$ converges by ratio test!

$$\left| \frac{(n+3)\, x^{n+2}}{(n+2)\, x^{n+1}} \right| = \left(\frac{n+3}{n+2}\right) x \xrightarrow[n\to\infty]{} x < 1$$

$$\underset{\text{n^{th} term test}}{\Longrightarrow} \quad (n+2)\, x^{n+1} \xrightarrow[n\to\infty]{} 0$$

$$\underset{\text{squeeze thm}}{\Longrightarrow} \quad |R_n(x)| \xrightarrow[n\to\infty]{} 0$$

$$\Longrightarrow \quad T_n(x) \longrightarrow f(x)$$

for $x \in [0, 1)$

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5. Let K be a cluster point of S.

Then for $\varepsilon = 1/n$, there are ∞'ly many elements $x \in S$ such that $x \approx_{1/n} K$

Pick one; call it x_n

Then

$$|x_n - K| < \frac{1}{n} \longrightarrow 0$$

$$\Longrightarrow \quad x_n \longrightarrow K$$

Sequential compactness implies that there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$

So that

$$x_{n_i} \longrightarrow L \in S$$

But, by the subsequence thm,

$$x_{n_i} \longrightarrow K$$

so $L = K$, and $K \in S$.