Hw 5 Solutions

Note Title 3/18/2009

$\boxed{8.1\ (1)}$

(b) Ratio test:

$$\left|\frac{\dfrac{x^{n+1}}{2^{n+1}\sqrt{n+1}}}{\dfrac{x^n}{2^n\sqrt{n}}}\right| = \frac{|x|}{2}\sqrt{\frac{n}{n+1}} \xrightarrow[n\to\infty]{} \frac{|x|}{2} < 1$$

$$\Updownarrow$$

$$|x| < 2$$

$$\boxed{R = 2}$$

(d) ratio test:

$$\left|\frac{\dfrac{(-1)^{n+1}x^{2(n+1)}}{4^{n+1}((n+1)!)^2}}{\dfrac{(-1)^n x^{2n}}{4^n (n!)^2}}\right| = \frac{|x|^2}{4(n+1)^2} \longrightarrow 0 < 1$$

for any x

$$\Longrightarrow \boxed{R = \infty}$$

(5)

Ratio test

$$\frac{\left|\frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}\right|}{\left|\frac{n^n x^n}{n!}\right|} = \left(\frac{n+1}{n}\right)^n \left(\frac{n+1}{n+1}\right) |x|$$

$$= \left(1+\frac{1}{n}\right)^n |x|$$

$$\longrightarrow e|x| < 1$$

see example 1.4

$$\Longrightarrow \boxed{R = 1/e}$$

8.4 (1)

(a) (i) $a_n = 1$

$$\Rightarrow S_n = \underbrace{1 + 1 + \cdots + 1}_{(n+1)} = n+1$$

$$f(x) = \sum a_n x^n = \sum x^n = \frac{1}{1-x}$$

for $|x| < 1$
(geometric series)

$$\sum S_n x^n = \sum_{n=0}^{\infty} (n+1)\, x^n$$

partial sum!

$$S_n' = 1 + 2x + 3x^2 + \cdots\cdots + (n+1)x^n$$

So

$$x S_n' = x + 2x^2 + 3x^3 + \cdots + (n+1)x^{n+1}$$

$$= S_n' - \underbrace{(1 + x + x^2 + \cdots + x^n)}_{\frac{1-x^{n+1}}{1-x}} + (n+1)x^{n+1}$$

So

$$(x-1)S_n' = -\frac{1-x^{n+1}}{1-x} + (n+1)x^{n+1}$$

$$\Longrightarrow \quad S_n' = \frac{1-x^{n+1}}{(1-x)^2} + \frac{(n+1)x^{n+1}}{x-1}$$

$\downarrow$ $n \to \infty$ if $|x| < 1$

$$\frac{1}{(1-x)^2}$$

$$\frac{f(x)}{1-x} = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} \quad \checkmark$$

(ii) $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} x^n = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

$$= \frac{1}{1-\frac{x}{2}}$$

$$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n}$$

$$= 2 - \frac{1}{2^n}$$

$$\sum S_n x^n = \sum \left(2 - \frac{1}{2^n}\right) x^n$$

$$= \sum 2x^n - \sum \left(\frac{x}{2}\right)^n$$

$$= \frac{2}{1-x} - \frac{1}{1-\frac{x}{2}}$$

$|x| < 1$

$$= \frac{2\left(1-\frac{x}{2}\right) - (1-x)}{(1-x)\left(1-\frac{x}{2}\right)} = \frac{1}{(1-x)\left(1-\frac{x}{2}\right)}$$

$$\frac{f(x)}{1-x} = \frac{1}{(1-x)\left(1-\frac{x}{2}\right)}$$

✓

(b) Consider

$$\left(\sum x^n\right)\left(\sum a_n x^n\right)$$

Series for the product has terms:

$$b_n = a_n x^n + x \cdot a_{n-1} x^{n-1} + x^2 \cdot a_{n-2} x^{n-2} + \cdots$$
$$+ \cdots + x^n \cdot a_0$$

$$= s_n x^n$$

So by the multiplication thm for series:

$$\frac{f(x)}{1-x} = \left(\sum x^n\right)\left(\sum a_n x^n\right) = \sum s_n x^n$$

as long as both converge

i.e. $|x| < 1$. $\square$

Finally, suppose $\sum a_n$ converges.

$\Longrightarrow$ $\sum a_n x^n$ converges for $x = 1$

$\Rightarrow$ radius of convergence is ≤ 1

$\Rightarrow$ $\sum a_n x^n$ converges for $|x| < 1$
