Homotopy Excision, Hurewicz Thm

Note Title 3/2/2010

Homotopy excision

Problem: π_* does not satisfy excision.

(if it did, $\pi_n X \xrightarrow[\cong]{} \pi_{n+1}(\Sigma X)$)

yet $\pi_2(S^1) = 0$

$\pi_3(S^2) \neq 0$

But π_* does satisfy excision through a range....

Def

$f : X \to Y$ is an

<u>n-equivalence</u> if

(n-1)-connected

$\pi_k(X, x) \longrightarrow \pi_k(Y, f(x))$

is $\begin{cases} \text{iso} & k < n \\ \text{epi} & k = n \end{cases}$

Rmk f is n connected

$\iff \forall \, x \in X$

$$\begin{tikzcd} F(f)_x \arrow[r] & X \arrow[r] & Y \\ * & x \arrow[u, phantom, "\in" rotate=90] & f(x) \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$F(f)_x$ is n connected

n-connected $= \pi_{\leq n} = 0$

0-connected $=$ path connected

1-connected $=$ simply connected.

Thm (Htpy excision)

$$f : A \longrightarrow X \quad m\text{-connected}$$
$$g : A \longrightarrow Y \quad n\text{-connected}$$

one of these is a relative CW cx

$$\begin{tikzcd}
F(f) \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "g'"] \\
F(f') \arrow[r] & Y \arrow[r, "f'"'] & Z
\end{tikzcd}$$

$(n+m)$ connected

(pf) Hatcher & May. (we may redo it using SSS)

Cor: (Htpy) $f : A \longrightarrow X$, A m-connected, f n-connected

- $F(f) \longrightarrow \Omega C(f)$ $(n+m)$-connected
- f cofibration

  $\pi_k(X, A) \longrightarrow \pi_k(X/A)$

  iso $k \leq n+m+1$

  epi $k = n+m+2$
- A m-connected, $X = pt$ $\Rightarrow$ $A \longrightarrow *$ m-connected $\Rightarrow$ $A \longrightarrow \Omega\Sigma A$ $2m$-connected

$$\underset{n-1 \text{ connected}}{\pi_k(S^n)} \longrightarrow \pi_k(\Omega S^{n+1}) \qquad \begin{array}{ll} \text{iso} & k \le 2n-2 \\ \text{epi} & k = 2n-1 \end{array}$$

$$\pi_2(S^2) = \mathbb{Z} \qquad \text{(Hopf fib)}$$

$$\underset{2}{\pi_k(S^2)} \xrightarrow[\substack{\text{iso} \\ k \le 2}]{\cong} \underset{3}{\pi_{k+1}(S^3)} \xrightarrow[\substack{\text{iso} \\ k+1 \le 4}]{\cong} \underset{4}{\pi_{k+2}(S^4)} \xrightarrow[\substack{\text{iso} \\ k+2 \le 6}]{\cong} \underset{5}{\pi_{k+3}(S^5)}$$

Thm $\quad \pi_n(S^n) \cong \mathbb{Z}$

$$\begin{array}{ccccccc} \pi_3(S^2) & \xrightarrow{3 \overset{?}{\le} 2} & \pi_4(S^3) & \xrightarrow[\cong]{4 \overset{?}{\le} 4} & \pi_5(S^4) & \xrightarrow[\cong]{5 \overset{?}{\le} 6} & \cdots \\ \| & & & & & & \\ \mathbb{Z} & & & & \mathbb{Z}/2 & & \text{---} \end{array}$$

turns out this is

$\mathbb{Z}/2$

More generally: fix k

$$\pi_k(S^0) \to \pi_{k+1}(S^1) \to \pi_{k+2}(S^2) \to \pi_{k+3}(S^3) \to$$

iso $k+1 \le 0$ $\qquad$ iso $k+2 \le 2$ $\qquad$ iso $k+3 \le 6$

eventually stabilizes. $\qquad \pi_k^s = \varinjlim \pi_{k+n}(S^n)$

$$k+n \le 2n-2 \iff n \ge k+2$$

$$\pi_0^s = \mathbb{Z}$$

e.g. $\quad \pi_1^s = \pi_4(S^3) = \mathbb{Z}/2$

Hurewicz Homo:

$$h: \pi_1(X) \longrightarrow \tilde{H}_1(X)$$

abelianization if X is path connected.

More generally:

get: $\quad h: \pi_k(X) \longrightarrow \tilde{H}_k(X)$

$f: S^k \to X \qquad \tilde{H}_k(S^k) \xrightarrow[f_*]{} \tilde{H}_k(X)$

$\qquad\qquad\qquad\qquad c_k \longmapsto h([f])$

$f, g \in \pi_k X$

use $\quad f+g: S^k \xrightarrow[\text{pinch}]{} S^k \vee S^k \xrightarrow[f \vee g]{} X$

to get h is a homo.

Relative Hurewicz: $\quad i: A \hookrightarrow X$

$$\alpha: (D^k, \partial D^k, *) \longrightarrow (X, A, *)$$

get $\quad H_k(D^k, \partial D^k) \xrightarrow{\alpha_*} H_k(X, A)$

$\qquad\qquad c_k \longmapsto h(\alpha)$

$$h: \pi_k(X, A) \longrightarrow H_k(X, A)$$

Thm: (Hurewicz)

X $n-1$ connected, $n \geq 2$

$\Longrightarrow$ $\pi_k X \longrightarrow \tilde{H}_k(X)$ is iso $k = n$

epi $k = n+1$

(pf deferred)

Thm: $f : X \to Y$ H_*-iso

X, Y simply connected

$\Longrightarrow$ f is a w.e.

(pf)

$$F(f) \longrightarrow X \longrightarrow Y$$

$\boxed{\text{WLOG: } X, Y \text{ well pointed}}$

(to get LES of $C(f)$)

X, Y simply connected

$$\begin{tikzcd} \pi_2 X \arrow[r] \arrow[d, "\cong"] & \pi_2 Y \arrow[d, "\cong"] \\ \tilde{H}_2 X \arrow[r, "\cong"'] & \tilde{H}_2 Y \end{tikzcd}$$

$\Longrightarrow$ $F(f)$ 1-connected

f 1-conn'd

$$F(f) \xrightarrow[\wr]{} \Omega C(f)$$

$1+1 = 2$ connected

$\Rightarrow \quad C(f) \quad$ 2-connected

$\Rightarrow \quad \pi_3 C(f) \xrightarrow[\cong]{} \tilde{H}_3 C(f) = 0$

$\Rightarrow \quad C(f)$ is 3 connected

$\vdots$

$\Rightarrow \quad C(f)$ is π_* - acyclic

so $\quad \pi_2 F(f) \xrightarrow[\cong]{} \pi_2 \Omega C(f)$

$\Rightarrow \quad F(f) \quad 2$ - connected

$\Rightarrow \quad F(f) \longrightarrow \Omega C(f)$

$(2+1) = 3$ connected

$\Rightarrow \quad F(f) \quad 3$ connected

$\vdots$

$\Rightarrow \quad F(f) \quad \pi_* \quad$ acyclic

LES $\Rightarrow \quad f$ is π_* iso

(w.e.)

<u>Cor:</u> $f: X \to Y$ simply connected CW cx's

H_* iso $\Longrightarrow$ h.e.

(pf of Hurewicz)

<u>Check:</u> get a map of LES:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k(A) \arrow[r] \arrow[d, "h"] & \pi_k(X) \arrow[r] \arrow[d, "h"] & \pi_k(X,A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \tilde{H}_k(A) \arrow[r] & \tilde{H}_k(X) \arrow[r] & \tilde{H}_k(X,A) \arrow[r] & \cdots
\end{tikzcd}$$

X $(n-1)$ connected

Arg for cellular approx

$$\Longrightarrow \quad \underset{\text{CW cx w/ no cells below dim } n}{\tilde{X}} \xrightarrow[\text{w.e.}]{} X$$

Since v.e. $\Rightarrow$ H_* - iso

WLOG X is a CW cx with no cells below dim n.

<u>Case 1</u> $X = \bigvee S^n$

$\overset{HW}{\Longrightarrow}$ $\pi_n(S^n \vee S^n) \cong \pi_n(S^n) \oplus \pi_n(S^n)$

$$\oplus \underbrace{\pi_{n+1}(S^n \times S^n, S^n \vee S^n)}$$

$\downarrow$ 2n-1 connected

$\pi_{n+1}(S^n \wedge S^n)$

$n+1 \leq 2n-1 \iff 2 \leq n$ ✓

$\Rightarrow$ $\pi_{n+1}(S^n \times S^n, S^n \vee S^n) = 0$

this takes care of finite wedges.

use $\underrightarrow{\lim}$ to any infinite wedges.

$X^{[n]}$ = wedge of spheres .... done. $\quad n+1 > m$

$$\begin{tikzcd}
\pi_k X^{[n]} \arrow[r] & \pi_k X^{[n+1]} \arrow[r] & \pi_k\left(X^{[n+1]}, X^{[n]}\right) \arrow[d] \\
& & \tilde{H}_k\left(X^{[n+1]}, X^{[n]}\right)
\end{tikzcd}$$

$X^{[n]}$ $(m-1)$-connected

HTZ excision: $X^{[n]} \to X^{[n+1]}$ $(m-1)$-connected (epi on π_m because of dim attaching cells)

$$\pi_k\left(X^{[n+1]}, X^{[n]}\right) \to \pi_k\left(X^{[n+1]}/X^{[n]}\right)$$

iso for $k \leq 2m-1$

epi for $k = 2m$

$$X^{[n+1]}/X^{[n]} = \bigvee S^{n+1}$$

$$H_k\left(X^{[n+1]}, X^{[n]}\right) \xrightarrow{\cong} \tilde{H}_k\left(X^{[n+1]}/X^{[n]}\right)$$

iso for $k \leq m+1$ $\Longleftarrow$ [iso for $k \leq m+1$ $\Rightarrow$ iso for $k \leq m+1$]

$k = n$

iso iso iso iso iso

$$
\begin{tikzcd}
\pi_{k+1}(X^{n+1}, X^{n}) \arrow[r] \arrow[d] & \pi_k(X^{[n]}) \arrow[r] \arrow[d] & \pi_k X^{[n+1]} \arrow[r] \arrow[d] & \pi_k(X^{[n+1]}, X^{[n]}) \arrow[r] \arrow[d] & \pi_{k-1} X^{[n]} \arrow[d] \\
\tilde{H}_{k+1}(X^{n+1}, X^{n}) \arrow[r] & \tilde{H}_k(X^{[n]}) \arrow[r] & \tilde{H}_k(X^{[n+1]}) \arrow[r] & H_k(X^{[n+1]}, X^{[n]}) \arrow[r] & \tilde{H}_{k-1} X^{[n]}
\end{tikzcd}
$$

epi epi iso iso

$k = n+1$

∎