# 12 - Classifying spaces

Note Title 3/9/2010

apply to $\tilde{H}^n(-;\pi)$

get $K = K(\pi;n)$

<u>Rmk</u>: This is <u>overkill</u>: Can construct $K(\pi,n)$ manually — assume $\pi$ is abelian, $n\geq 2$

<u>Moore Spaces</u>:

$$[\bigvee_J S^n, \bigvee_I S^n] = \prod_J \underbrace{[S^n, \bigvee_I S^n]}_{\pi_n(\bigvee_I S^n)}$$

$$= \prod_J \bigoplus_I \underbrace{\pi_n(S^n)}_{\mathbb{Z}}$$

$$= \prod_J \mathrm{Hom}(\mathbb{Z}, \bigoplus_I \mathbb{Z})$$

$$= \mathrm{Hom}(\bigoplus_J \mathbb{Z}, \bigoplus_I \mathbb{Z})$$

$$0 \to \bigoplus_J \mathbb{Z} \xrightarrow{\alpha} \bigoplus_I \mathbb{Z} \longrightarrow \pi \to 0$$

$$\bigvee_J S^n \xrightarrow{\tilde{\alpha}} \bigvee_I S^n \longrightarrow C(\alpha) =: M(\pi,n)$$

$$\tilde{H}_k(M(\pi,n)) = \begin{cases} \pi, & k = n \\ 0, & o/w \end{cases}$$

<u>$K(\pi,n)$</u> $\qquad \pi_n M(\pi,n) = \pi$

inductively kill higher htpy grps

$$\begin{tikzcd}
\bigvee_{\pi_{n+1}(K^0)} S^{n+1} \arrow[r] & K^0 = M(\pi,n) \arrow[d] \\
\bigvee_{\pi_{n+2}(K^1)} S^{n+2} \arrow[r] & K^1 \arrow[d] \\
& K^2 \\
& \vdots
\end{tikzcd}$$

$$\varinjlim K^i = K(\pi,n)$$

Suppose $X$ is a $K(\pi,n)$

$$\begin{tikzcd}
\bigvee_J S^n \arrow[r, shift left] \arrow[r, shift right] & \bigvee_I S^n \arrow[r] \arrow[d] & M(\pi,n) \arrow[dl, dashed, "\text{iso on } \pi_n"] \\
& X &
\end{tikzcd}$$

Inductively

$$\begin{tikzcd}
\bigvee_{\pi_{n+i}(K^{i-1})} S^{n+i-1} \arrow[r] & K^{i-1} \arrow[r] \arrow[d] & K^i \arrow[dl, dashed] \\
& X &
\end{tikzcd}$$

get $K(\pi,n) \to X$ w.e. (h.e. if $X$ cw)

<u>Prop</u> $\Omega K(\pi, n+1) \simeq K(\pi,n)$

## Principle $G$-bundles

$G$ = top'l gp

A $G$-space

$$G \times X \xrightarrow[\mu]{\text{continuous}} X \qquad \mu(1, x) = x$$

$$\mu(gh, x) = \mu(g, \mu(h, x))$$

A principle $G$-bundle $/B$

$$\begin{array}{l} E \text{ an } G\text{-space} \\ \downarrow p \\ B = E/G \end{array}$$

s.t. $\forall\ x \in B$

$\exists$ nbhd $U$ of $x$

$$\begin{tikzcd} G \times U \arrow[r, "\approx", "\text{equivariant homeo}"'] \arrow[dr, "\pi_2"'] & p^{-1}(U) \arrow[d, "p"] \\ & U \end{tikzcd}$$

iso:

$$\begin{tikzcd} E \arrow[rr, "\text{equivariant}", "\approx"'] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

Rmk: $E, E'$ prin. $G$-bdls $/B$

any equivariant map $E \to E'$ is an iso.

Thm:

$$\left\{ n\text{-dim'l } \ \text{v.b.'s} \ / \overbrace{X}^{\text{paracompact}} \right\} \cong \left\{ \text{principle } O(n)\text{-bundles} / X \right\}$$
$$\cong [X, BO(n)]$$

Idea:

$X$ = right $G$-space

$Y$ = left $G$-space

$$X \times_G Y = X \times Y \big/ (xg, y) \sim (x, gy)$$

$$\begin{array}{c} P \\ \downarrow \\ X \end{array} \quad \text{principle } O(n) \text{ bundle} \quad \rightsquigarrow \quad \begin{array}{c} P \times_{O(n)} \mathbb{R}^n \\ \downarrow \\ X \end{array} \quad \text{vect bundle}$$

Conversely given $\begin{array}{c} Y \\ \downarrow \\ X \end{array}$ ← given metric. Define $P = \left\{ \begin{array}{c} \text{space of} \\ \text{orthonormal} \\ \text{frames} \end{array} \right\}$