15 Serre Spectral Sequence

Note Title 4/1/2010

$F \to E \to B$ Serre fiber sequence

B connected

Serre spectral Sequence:

$$E^2_{s,t} = \underbrace{H_s(B; H_t(F))}_{\text{local coef!}} \Rightarrow H_{s+t}(E)$$

[Do you know what this means]

$\pi_1 B = 0 \Rightarrow$ no issues

<u>Construction</u>

$$\begin{array}{l} E = \varinjlim E^{[s]} \\ \downarrow \\ B = \varinjlim B^{[s]} \end{array} \qquad \underbrace{C_*^{sing}(E) = \varinjlim C_*^{sing}(E^{[s]})}_{\text{filtered cx.}}$$

SS $\leftarrow$ filtered cx

$$\Rightarrow \quad E^1_{s,t} = \underbrace{H_{s+t}\left(C_\bullet^{sing}(E^{[s]}) / C_\bullet^{sing}(E^{[s-1]})\right)}_{H_{s+t}(E^{[s]}, E^{[s-1]})} \Rightarrow \underbrace{H_{s+t}(C_\bullet^{sing} E)}_{H_{s+t} E}$$

All that's left is to identify E^2_{st} ...

Aside on twisted cohomology

X, connected CW cx.

Coefficient system $=$ functor $\pi_{oid} X \longrightarrow Ab$

$$x \longmapsto M_x$$

$$x \xrightarrow{\gamma} y \qquad \gamma_* : M_x \to M_y$$

M is determined by $M_x \circlearrowleft \pi_1(X,x)$

Pick $\forall y$, $x \xrightarrow[\gamma_y]{} y$

these give isos $M_x \xrightarrow[\cong]{} M_y$

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma_y"'] & y \end{tikzcd} \qquad \begin{tikzcd} M_x \arrow[r, bend left, "\gamma"] \arrow[dr, dashed, "(\gamma_y^{-1}\gamma)_*"'] & M_y \arrow[d, phantom, "\wr\wr ?"] \\ & M_x \end{tikzcd}$$

in $\pi_1(X,x)$

Note: $f : X \to Y \implies$ $M =$ coef sys on Y
$f^*M =$ coef sys on X

$$f^*M : \pi_{oid} X \xrightarrow[f_*]{} \pi_{oid} Y \xrightarrow[M]{} Ab$$

Two different perspectives:

$\forall$ s-cells e_i^s of X, pick $x_i \in$ interior e_i^s

$$e_i^s : D^s \longrightarrow X$$

$(e_i^s)^* M$ = coef sys on D^s $\longleftrightarrow$ ab gp $M_{e_i^s} = M_{x_i}$

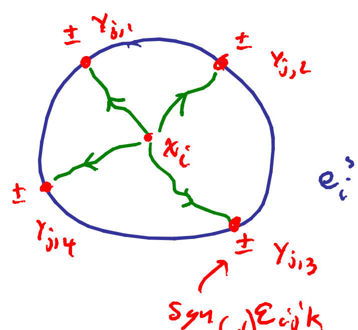
$$\begin{tikzcd}
\partial D_i^s \arrow[r, "\partial e_i^s"] \arrow[rrr, bend right, "\alpha_{ij}"'] & X^{[s-1]} \arrow[r] & \bigvee S_j^{s-1} \arrow[r] & S_j^{s-1}
\end{tikzcd}$$

$\alpha_{ij} \simeq \alpha'_{ij}$ map w/ regular value x_j'

$M_{x_j'} \cong M_{e_j^{s-1}}$

$(\alpha'_{ij})^{-1}(x_j') = \{y_{j,k}\} \subset \partial D_i^s$

pick $\gamma_{i,j,k}$



Form: "twisted cellular chain cx"

$$\cdots \to \bigoplus_{i \in s\text{-cells}} M_{x_i} \xrightarrow{\partial} \bigoplus_{j - (s-1)\text{-cells}} M_{x_j} \to \cdots$$

$m \in M_{x_i}$

$$\partial(m) = \sum_j \sum_k (\gamma_{i,j,k})_* (m)$$

$(-1)^{\varepsilon_{ijk}}$

Alternatively:

$$\begin{array}{c}\tilde{X} \\ \downarrow \\ X\end{array} \leftarrow \text{universal cover}$$

Regard M as π_1-module

$$s\text{-cells}(\tilde{X}) \circlearrowleft \pi_1(X) \qquad \text{free action}$$

$$C_*^{cell}(\tilde{X}) = \text{free } \mathbb{Z}[\pi_1(X)]\text{-module}$$

trivial π_1-module

$$C_*^{cell}(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z} \cong C_*^{cell}(X).$$

$$H_*\left(C_*^{cell}(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} M\right) = H_*^{twisted}(X; M)$$

Note: $M = \mathbb{Z} \Rightarrow$ regular H_*

Trivializations, and π_1-action:

For simplicity, assume $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is a **fibration**

[Slightly more complicated for Serre fibration]

Lemma:

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near end] & E \arrow[d, "p"] \\ I^s \arrow[r] & B \end{tikzcd}$$

$$\begin{tikzcd} F' \times I^s \arrow[rr, dashed, "w.e."] \arrow[dr, "\pi_2"'] & & E' \arrow[dl, "p'"] \\ & I^s & \end{tikzcd}$$

F' = fiber over $(0,\dots,0)$
$\cap$
I^s

(Pf)

LES

$$\begin{tikzcd}
F'\times\{0\} \arrow[r, "\sim"] \arrow[d, "\sim"'] & E' \arrow[d] \\
F'\times I' \arrow[r] \arrow[ur, dashed] & I'
\end{tikzcd}$$

so this is equiv.

"weak parallel transport" $b \xrightarrow[\gamma]{} b'$

$$\begin{tikzcd}
E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near end] & E \arrow[d] \\
I \arrow[r, "\gamma"'] & B
\end{tikzcd}$$

get:

$$\begin{tikzcd}
F_b\times\{0\} \arrow[r] \arrow[d] & E' \arrow[d] \\
F_b\times I \arrow[r] \arrow[ur, dashed, "\sim"] & I
\end{tikzcd}$$

gives $\quad F_b \xrightarrow[\sim]{\gamma_*} F_{b'}$

Different perspective

$$\begin{tikzcd} & F_{b'} \arrow[d, "\sim"] \\ F_b \arrow[r, "\sim"'] & E' \end{tikzcd} \qquad \begin{tikzcd} & H_* F_{b'} \arrow[d, "\cong"] \\ H_* F_b \arrow[ur, dashed, "\gamma_*"] \arrow[r, "\cong"'] & H_* E' \end{tikzcd}$$

this map independent of choice of lift

<u>Lem:</u> $\gamma \simeq \gamma'$ rel $\partial I \implies \gamma_* = \gamma'_* : H_* F_b \longrightarrow H_* F_{b'}$

(pf)
$$\begin{tikzcd} h^* E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & E \arrow[d] \\ I \times I \arrow[r, "h"'] & B \end{tikzcd}$$

$$\begin{tikzcd} & \gamma^* E \arrow[d, "\simeq_H"] & \\ F_b \arrow[ur, "\simeq_H"] \arrow[dr, "\simeq_H"'] \arrow[rr, bend left=80, "\gamma_*"] \arrow[rr, bend right=80, "\gamma'_*"'] & h^* E & F_{b'} \arrow[ul, "\simeq_H"'] \arrow[dl, "\simeq_H"] \\ & (\gamma')^* E \arrow[u, "\simeq_H"'] & \end{tikzcd}$$

apply H_*
