18 - Change of group

Note Title 4/8/2010

Variants

$$H_s(B; H_t(F;R)) \Rightarrow H_{s+t}(E;R)$$

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F' \arrow[d] \\ E \arrow[r, hook] \arrow[d] & E' \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

(maps of fibred cx's)

$$H_s(B; H_t(F,F')) \Rightarrow H_{s+t}(E,E')$$

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F \arrow[d] \\ E' \arrow[r] \arrow[d] & E \arrow[d] \\ B' \arrow[r, hook, "\text{subcx}"'] & B \end{tikzcd}$$

(relative CW filtration)

$$H_s(B,B'; H_t(F)) \Rightarrow H_{s+t}(E,E')$$

Naturality

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r, "p"] \arrow[d, "\tilde{f}"] & B \arrow[d, "f"] \\ F' \arrow[r] & E' \arrow[r, "p'"'] & B' \end{tikzcd}$$

$\Longrightarrow$ map of spectral sequences

$$E^n_{s,t}\left(\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}\right) \longrightarrow E^n_{s,t}\left(\begin{smallmatrix} E' \\ \downarrow \\ B' \end{smallmatrix}\right)$$

on E^2
$$H_s(B, H_t(F)) \xrightarrow[f_*]{} H_s(B', H_t(F'))$$

on E^∞:

$$\begin{array}{ccc} H_n(E) & \xrightarrow{\tilde{f}_*} & H_n(E') \\ \cup & & \cup \\ F_s H_n(E) & \xrightarrow{\tilde{f}_*} & F_s H_n(E') \end{array}$$

$\Longrightarrow$ get
$$\begin{array}{ccc} Gr_s H_{s+t}(E) & \xrightarrow{Gr_s \tilde{f}_*} & Gr_s H_{s+t}(E') \\ \| & & \| \\ E^\infty_{s,t}(p) & & E(p') \end{array}$$

Thm

(i) If $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is a princ. G-bundle w/ B = CW cx

$\pi_* E = 0$

$\Longrightarrow B \simeq BG \quad (E \simeq EG)$

(ii) $\pi_* EG = 0$

(i) Suppose $\begin{matrix} P \\ \downarrow \\ X \end{matrix}$ is a princ. G-bundle

Claim: $\exists$ equivariant $P \to E$

Inductively:

$$\begin{tikzcd}
\coprod G \times S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & P^{(n-1)} \arrow[d] \arrow[ddr, bend left] & \\
\coprod G \times D^n \arrow[r] \arrow[drr, dashed, bend right] & P^{(n)} \arrow[dr, dashed] & \\
& & E
\end{tikzcd}$$

G-maps

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & P^{(n-1)} \arrow[d] \\ \coprod D^n \arrow[r, dashed] & E \end{tikzcd}$$

maps

(exist since E has trivial htpy grps)

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ P/G \arrow[r] \arrow[d, phantom, "\shortparallel"] & E/G \arrow[d, phantom, "\shortparallel"] \\ X & B \end{tikzcd}$$

(ii) Bar construction

Our goal:

$$H^* BU(n)$$

Need $\quad U(n-1) \to U(n) \to S^{2n-1}$

Claim $\rightsquigarrow \quad S^{2n-1} \longrightarrow BU(n-1) \to BU(n)$

fiber sequences

To achieve this we digress

to discuss "functoriality of BG

in G"

Change of Group $H \to G$ homomorphism of top'l gps

$$\Rightarrow \left\{ \begin{array}{c} \text{principle} \\ H\text{-bundles}/X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{principle} \\ G\text{-bundles}/X \end{array} \right\}$$

$$\begin{array}{ccc} P & & G \times_H P \\ \downarrow & \longmapsto & \downarrow \\ X & & X \end{array}$$

(note locally: $G \times_H H \times D^n \cong G \times D^n$)

Get: $BH \longrightarrow BG$

More explicitly:

$H \hookrightarrow G$ sub-Lie gp

$\Rightarrow$ G is an H-free CW cx

$\Rightarrow$ EG is an H-free CW cx (contr'ble)

$\Rightarrow$ $BH = EG/H$

$$\begin{tikzcd}
G/H \arrow[r] & EG/H \arrow[r, "\text{fibration}"] \arrow[d, phantom, "\simeq" description] & EG/G \arrow[d, phantom, "\simeq" description] \\
 & BH & BG
\end{tikzcd}$$

To show this agrees, we must show univ G-bundle pulls back over EG/H to give $G \times_H EG$

$$\begin{tikzcd}
G \times_H EG \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & EG \arrow[d] \\
EG/H \arrow[r] & EG/G
\end{tikzcd}$$

$$EG/H \times_{EG/G} EG = \{(Hx, y) \mid gx = y\} \qquad x \mapsto hx$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$G \times_H EG \ni (g, g^{-1}y) \qquad\qquad (gh^{-1}, hg^{-1}y)$$

this confirms the two maps $BH \to BG$ are the same

$$\begin{tikzcd}
\Omega BH \arrow[r] \arrow[d, phantom, "\shortparallel" description] & \Omega BG \arrow[r] \arrow[d, phantom, "\shortparallel" description] & G/H \arrow[r] & BH \arrow[r] & BG \\
H & G
\end{tikzcd}$$