2-compactly generated spaces

Note Title 2/3/2010

Started by proving Set, Top

are complete, cocomplete,

and giving "formulas" for

$\varinjlim X$, $\varprojlim X$

Problem w/ Top:

c.o. top

product top

$$\underline{\mathrm{Map}}(X \times Y, Z) \longrightarrow \underline{\mathrm{Map}}(X, \underline{\mathrm{Map}}(Y, Z))$$

is <u>not</u> surjective

[is always homeo onto its image]

The solution: compactly generated spaces

<u>Def</u> a top'l space X is a k-space if "closed sets are detected by maps of compact Hausdorff's in"

<u>i.e.</u>

$$\left(C \subseteq X \text{ is closed} \right) \Leftrightarrow \left(\begin{array}{l} \forall \text{ compact Haus } K \\ g : K \longrightarrow X \\ g^{-1}(C) \text{ is closed} \end{array} \right)$$

<u>e.g.</u> locally cpt $\Rightarrow$ k-space

so CW cx's are k-spaces.

<u>k-ification</u> Let $k\mathrm{Top} \subset \mathrm{Top}$ be subcat of k-spaces

There is an adjoint pair (forget, k)

$$\text{forget}: k\mathrm{Top} \rightleftarrows \mathrm{Top} : k$$

$k(X)$ has same underlying set as X

<u>But</u> $C \subseteq k(X)$ is closed iff $\forall$ Comp. Haus K $\forall$ cont $g: K \to X$ $g^{-1}(C)$ is

Note! $k(k(X)) = k(X)$ [$k(X)$ has <u>more</u> closed sets]

$X \in \mathrm{Top}_k$, $Y \in \mathrm{Top}$

$$\mathrm{Map}_{\mathrm{Top}}(X, Y) \longleftarrow \mathrm{Map}_{\mathrm{Top}_k}(X, kY)$$

because $k(Y)$ has more closed sets

<u>Iso:</u> is obvious

Top_k is complete and cocomplete

$$\varinjlim^{\mathrm{Top}_k} X_i = \varinjlim^{\mathrm{Top}} X_i$$

$$\varprojlim^{\mathrm{Top}_k} X_i = k \varprojlim^{\mathrm{Top}} X_i$$

follows from adjunction (similar to the problem)

In particular

$$X \times_k Y = k(X \times Y)$$

$X, Y \in k\mathrm{Top}$

<u>Note</u>: X locally cpt, Y hausdorff k-space

$$\Rightarrow X \times_k Y = X \times Y$$

$$\underline{\mathrm{Map}}_k(X, Y) := k\, \underline{\mathrm{Map}}(X, Y)$$

<u>Def</u>: A space X weak <u>hausdorff</u>

if $\forall$ cpt Haus K

cts $g : K \longrightarrow X$

$g(K)$ is closed

Clearly Hausdorff $\Rightarrow$ weak hausdorff

$\boxed{\text{Three reasons to prefer weak hausdorff:}}$

(1) cpt hausdorf subspaces closed
(in particular, points are closed)
[also, continuous images of cpt hausdorfs are cpt hausdorf]

(2) If $A \underset{\text{closed}}{\hookrightarrow} X$, X weak Hausdorff

$\Longrightarrow$ X/A weak hausdorff [Not true of hausdorff]

(3) Works well w/ notion of k-space

- X k-space: X weak hausdorf
$\Longleftrightarrow$ $X \overset{\Delta}{\hookrightarrow} X \times_k X$ closed

- X weak haus: X k-space
$\Longleftrightarrow$ (C closed $\Leftrightarrow$ $C \cap K$ closed $\forall$ cpt hausdorf $K \subseteq X$)

Def: X is cptly generated if
X is weak hausdorff k-space

Prop: Top_{CG} is complete & cocomplete

Idea:

$$\text{wHaus}: \mathrm{Top}_k \longrightarrow \mathrm{Top}_{CG} : \mathcal{U}$$

$\text{wHaus}\, X = X/\text{smallest closed equiv relation}$

$$\varinjlim_i{}^{CG} X_i = w\mathrm{Haus}\left(\varinjlim_i X_i\right)$$

$$\varprojlim_i{}^{CG} X_i = \varprojlim_i{}^{\mathrm{Top}_k}(X_i) = k \varprojlim_i{}^{\mathrm{Top}} X_i$$

Thm: $X, Y, Z \in CG$

$$\mathrm{Map}(X \times_k Y, Z) \xrightarrow{\cong} \mathrm{Map}(X, \underline{\mathrm{Map}}_k(Y, Z))$$

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given

$$X \times_k Y \xrightarrow{f} Z$$

get

$$X \xrightarrow{\eta} \underline{\mathrm{Map}}_k(Y, X \times_k Y) \xrightarrow{f_*} \underline{\mathrm{Map}}_k(Y, Z)$$

$$x \longmapsto (y \mapsto (x, y))$$

Note η is continuous:

$$\begin{tikzcd}
\underline{\mathrm{Map}}_k(Y, X \times_k Y) \arrow[r, phantom, "\cong"] & k\,\underline{\mathrm{Map}}(Y, X \times Y) \\
 & X \arrow[u, "\text{cts by adjointness}"']
\end{tikzcd}$$

Other way

give $\quad X \xrightarrow[g]{} \underline{Map}_k(Y,Z)$

get

need this to be continuous

$$X \times_k Y \xrightarrow[g \times Y]{} \underline{Map}_k(Y,Z) \times_k Y \xrightarrow[ev]{} Z$$

This is where k-spaces are needed!

$$(f\ ,\ y) \longmapsto f(y)$$

K cpt haus

$$K \xrightarrow{\alpha} \underline{Map}_k(Y,Z) \times_k Y \longrightarrow Z \qquad \text{cts ?}$$

$U \subseteq Z$ open

Need $\alpha^{-1} ev^{-1} U$ to be open

pick $x_0 \in K$, $ev\,\alpha(x_0) \in U$ $\qquad \alpha(x_0) = (f_0, z_0) \qquad \alpha = (\alpha_1, \alpha_2)$

i.e. $f_0(z_0) \in U$

Need to find open $V \ni x_0$

s.t. $ev\,\alpha(V) \subseteq U$

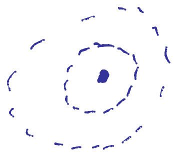
$\alpha = \alpha_1 \times \alpha_2$

f_0 continuous, K compact Hausdorff $\Longrightarrow$ $\alpha_2(K)$ cpt Haus.

(thus normal)

[closed sets can be separated]

$\exists \quad \alpha_2(x_0) \in N \subseteq \alpha_2(K)$ nbhd



$f_0^{-1}(U)$

$\bar{N} \subseteq f_0^{-1}(U)$

closure in $\alpha_2(K)$

Here you need separation axiom

$f_0(\bar{N}) \subseteq U$

$W(\bar{N}, U) \times N$ open

$V = \alpha^{-1}\big(\text{———} \big)$ open in K

$\underline{\text{and}}$ $\quad (f, x) \in W(\bar{N}, U) \times N$

$f(x) \in U$.

$$\Longrightarrow \alpha^{-1}\big(W(\bar{N}, U) \times N\big) \subset (ev \circ \alpha)^{-1}(U)$$

$\square$

Consequence:

$\underline{\text{Prop}}$:

$$[X, Y] = \pi_0 \, \underline{Map}(X, Y)$$

$\underline{\text{pf}}$

$$\text{homotopies} \longleftrightarrow \text{paths in } \underline{Map}(X, Y)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\{ X \times I \longrightarrow Y \} \longleftrightarrow \{ I \longrightarrow \underline{Map}(X, Y) \}$$

$\square$