9 - Simplicial approximation

Note Title 3/2/2010

Thm: every w.e. is a Homology equiv.

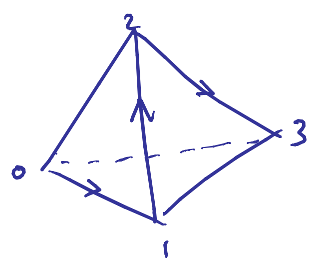
In order to prove this thm, we need to take a brief journey into

Simplicial sets

[UNDERSTAND singular homology]

Represent n-simplex

Δ^n :



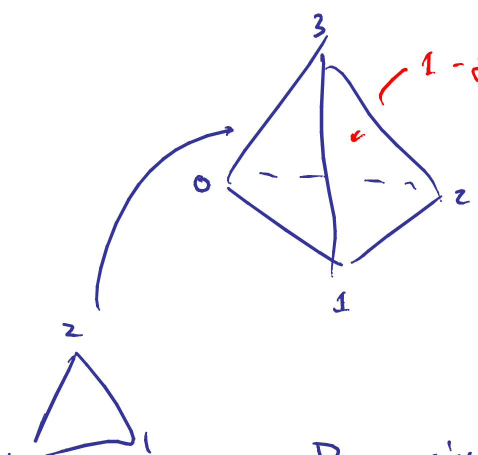
ordering on vertices

$(0,1,2,3)$

$\partial_i : \Delta^{n-1} \longrightarrow \Delta^n$

$0 \le i \le n$

faces determined by which vertices are cut



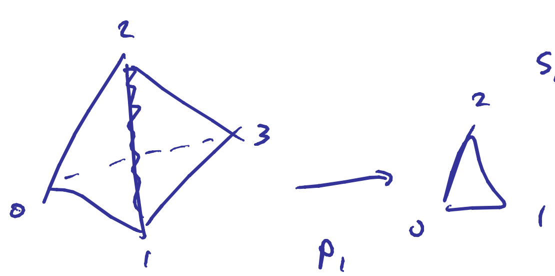
1 - face

$= \quad d_i \quad \begin{array}{l} 0 \longmapsto 0 \\ 1 \longmapsto 2 \\ 2 \longmapsto 3 \end{array}$

$p_i : \Delta^n \longrightarrow \Delta^{n-1}$

$0 \le i \le n-1$

Degeneracies:



$s_i \quad \begin{array}{l} 0 \longmapsto 0 \\ 1 \longmapsto 1 \\ 2 \longmapsto 1 \\ 3 \longmapsto 2 \end{array}$

Δ = category of finite ordered sets, order preserving morphisms

$\underline{0}, \underline{1}, \underline{2}, \ldots$

$\underline{n} = (0, 1, 2, \ldots, n)$

Every morphism is a composite of

$d_i : (0, 1, \ldots, n) \longmapsto (0, 1, \ldots, i-1, i+1, \ldots, n+1)$

$s_i : (0, 1, \ldots, n) \longmapsto (0, 1, \ldots, i, i, \ldots, n-1)$

$\Delta^{[\ ]} : \Delta \longrightarrow Top$ is a functor.

$\underline{n} \longmapsto \Delta^n$

$d_i \longmapsto \partial_i$

$s_i \longmapsto p_i$

sSet = category   objects: functors $\Delta^{op} \longrightarrow \text{Set}$

morphs: natural transformations

$X \in$ sSet, $X_n = X([n])$

$$\begin{tikzcd} X_0 & X_1 \arrow[l, shift left=2, "d_0"'] \arrow[l, shift right=2, "d_1"] & X_2 \arrow[l, shift left=4, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift right=4, "d_2"] & X_3 \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & \cdots \arrow[from=1-1, to=1-2, "s_0"] \arrow[from=1-2, to=1-3, shift left=2, "s_0"] \arrow[from=1-2, to=1-3, shift right=2, "s_1"'] \end{tikzcd}$$

"generalised simplicial complex."

X_n = "set of n-simplices"

(some of them are degenerate)

s_i record   degeneracies

d_i record   face attachments

"simplicial homology"

$$C_*(X_\bullet) = \mathbb{Z}X_0 \xleftarrow{\partial_1} \mathbb{Z}X_1 \xleftarrow{\partial_2} \mathbb{Z}X_2 \xleftarrow{\partial_3} \cdots$$

$$\partial_i = \sum_i (-1)^i d_i \qquad H_*(X_\bullet) := H_*(C_*(X_\bullet))$$

<u>Geometric</u> <u>realization</u>

$$|-|: sSet \longrightarrow Top$$

$$|X_\bullet| = \coprod_n \coprod_{\alpha \in X_n - \bigcup_i Im(s_i)} \Delta^n \Big/ \sim$$ "attach faces via d_i"

<u>Singular</u> <u>functor</u>

$$S_\bullet : Top : \longrightarrow sSet$$

$$S_n X = Map(\Delta^n, X)$$

$$H_*^{sing}(X) = H_*(C_*(S_\bullet X))$$

<u>Check:</u>

$$|-| : sSet \rightleftarrows Top : S_\bullet$$

adjoint functors!

<u>Get:</u> $\Gamma X := |S_\bullet X| \xrightarrow[\gamma]{} X$

Prop: $\quad \Gamma X \longrightarrow X$

is a w.e.

(pf) Surjectivity on π_*

$$S^n \xrightarrow{\alpha} X$$

express as $|\mathbb{S}^n|$

$$\Rightarrow \quad \tilde{\alpha} : \mathbb{S}^n \longrightarrow S_\bullet X$$

$$\Rightarrow \quad |\tilde{\alpha}| : |\mathbb{S}^n| \longrightarrow |S_\bullet X|$$

Injectivity $\quad$ Suppose $\alpha, \beta : S^n \longrightarrow \Gamma X$ $\quad$ ($S^n \simeq |\mathbb{S}^n|$)

H: $\gamma(\alpha) \simeq \gamma(\beta)$

Use "simplicial approximation thm"

$\alpha \simeq \alpha'$, $\beta \simeq \beta'$ for $\tilde{\mathbb{S}}^n$ subdivision of $\mathbb{S}^n$

α', β' simplicial

i.e. $\alpha' = |\alpha''| \quad \beta' = |\beta''|$

$\alpha'', \beta'' : \tilde{\mathbb{S}}^n \longrightarrow S_\bullet X$

$$H : S^n \wedge I \longrightarrow X$$
$$\shortparallel$$
$$|\tilde{S}^n \wedge \mathbb{I}|$$

$$\tilde{H} : \tilde{S}^n \wedge \mathbb{I} \longrightarrow S_\bullet X$$

$$|\tilde{H}| : |\tilde{S}^n \wedge \mathbb{I}| \longrightarrow |S_\bullet X|$$

htpy between α', β'

Lem $\Gamma X \to X$ is a homology equiv.

(Pf) $\Gamma^{fat} X = |S_\bullet X|^{fat}$ do not mod out degenerate simplices

$$\Gamma^{fat} X \xrightarrow[h.e]{} \Gamma X$$

$$C_*^{cell}(\Gamma^{fat} X) = C_*^{sing}(X)$$

iso on H_*

$$\begin{tikzcd}
C_*^{cell}(\Gamma^{fat} X) \arrow[r, "f", "\text{iso on } H_*"'] \arrow[drr, equal] & C_*^{cell}(\Gamma X) \arrow[r] & C_*^{sing}(\Gamma X) \arrow[d, "\gamma_*"] \\
& & C_*^{sing}(X)
\end{tikzcd}$$

← implies the next be iso on homly

$\square$

pf: w.e. $\Rightarrow$ homology iso.

$X \to Y$ w.e.

$$\begin{tikzcd}
\Gamma X \arrow[r, "H_* \cong", "\text{w.e.}"'] \arrow[d] & X \arrow[d, "\text{w.e.}"] \\
\Gamma Y \arrow[r, "H_* \cong", "\text{w.e.}"'] & Y
\end{tikzcd}$$

$\Rightarrow \quad \Gamma X \to \Gamma Y \quad$ w.e.

$\Rightarrow \quad \Gamma X \to \Gamma Y \quad$ h.e.

$\Rightarrow \quad \Gamma X \to \Gamma Y \quad H_*$ - iso

$\Rightarrow \quad X \to Y \quad H_*$-iso $\quad \square$