HOMEWORK 1

ASSIGNED: 2/6/2014, DUE 2/11/2014

Here are a few exercises in category theory, to acclimate you with the definitions. Many of the problems, to verify every last detail (such as naturality) explicitly would require a large amount of tedious writing. I ask that you supply enough detail so that you feel satisfied with the validity of the statements.

1. *Yoneda lemma.* Let $\mathcal{C}$ be a category. We may consider the category $\mathrm{Funct}(\mathcal{C}^{op}, \mathit{Sets})$ whose objects are contravariant functors $\mathcal{C} \to \mathit{Sets}$ and whose morphisms are natural transformations, ignoring the caveat that the collection of natural transformations between two functors may not form a set. We have seen that objects $Z \in \mathcal{C}$ give rise to contravariant functors

$$\begin{aligned} F_Z : \mathcal{C} &\to \mathit{Sets} \\ X &\mapsto \mathrm{Map}_{\mathcal{C}}(X, Z) = F_Z(X). \end{aligned}$$

We have also seen that morphisms $f : Z_1 \to Z_2$ give rise to natural transformations

$$f_* : F_{Z_1} = \mathrm{Map}_{\mathcal{C}}(-, Z_1) \to \mathrm{Map}_{\mathcal{C}}(-, Z_2) = F_{Z_2}.$$

We thus have a functor

$$\mathcal{Y} : \mathcal{C} \to \mathrm{Funct}(\mathcal{C}, \mathit{Sets})$$

given by $\mathcal{Y}(Z) = F_Z$. This functor is called the *Yoneda embedding.*

Prove *Yoneda's lemma*: the map

$$\mathrm{Map}_{\mathcal{C}}(Z_1, Z_2) \to \mathrm{Nat}(F_{Z_1}, F_{Z_2})$$

is a bijection. Here, $\mathrm{Nat}(F_{Z_1}, F_{Z_2})$ is the collection of natural transformations. In particular, F_{Z_1} and F_{Z_2} are naturally isomorphic functors if and only if Z_1 and Z_2 are isomorphic.

2. *Adjoint functors.* Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A pair of covariant functors

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

are said to form an *adjoint pair* (F, G) if there is a natural isomorphism

$$\eta : \mathrm{Map}_{\mathcal{D}}(F(-), -) \xrightarrow{\cong} \mathrm{Map}_{\mathcal{C}}(-, G(-))$$

between functors from $\mathcal{C}^{op} \times \mathcal{D} \to \mathit{Sets}$. Such an isomorphism η is called an *adjunction.* We say that F is *left adjoint* to G, and that G is *right adjoint* to F.

- **(a):** Show that if G' is also right adjoint to F, then there is a natural isomorphism $G \cong G'$ (hint: you can use the Yoneda lemma).
- **(b):** Show that if F' is also left adjoint to G, then there is a natural isomorphism $F \cong F'$ (hint: deduce this from (a) by being sneaky).

(c): Let S be a set. Show that there is an adjunction

$$\mathrm{Map}(X \times S, Y) \cong \mathrm{Map}(X, \mathrm{Map}(S, Y)).$$

3. *Adjoint functor formulation of limit, colimit* Let I be a small category, and suppose that $\mathcal{C}$ is a category which has all limits and colimits. Show that the pairs

$$\mathrm{const} : \mathcal{C} \leftrightarrows \mathcal{C}^I : \varprojlim$$
$$\varinjlim : \mathcal{C}^I \leftrightarrows \mathcal{C} : \mathrm{const}$$

are adjoint pairs. Here

$$\mathrm{const} : \mathcal{C} \to \mathcal{C}^I$$

is the functor which assigns to $X \in Ob\mathcal{C}$ the "constant diagram"

$$(\mathrm{const}X)(i) = X$$

with all arrows in the diagram identity morphisms. (Note that $\varprojlim$ and $\varinjlim$ may be regarded as functors - this follows from the fact that the universal property implies that if limits and colimits exist, then they are unique up to unique isomorphism.)

4. *Limit preservation properties of adjoint functors* (a) Suppose that I is a small category, and that

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

are a pair of adjoint functors. Suppose that both $\mathcal{C}$ and $\mathcal{D}$ have all limits and colimits. Show that for any diagram $X \in \mathcal{D}^I$, there is an isomorphism

$$\varprojlim_{i \in I} G(X(i)) \cong G(\varprojlim_{i \in I} X(i))$$

and for any diagram $Y \in \mathcal{C}^I$, there is an isomorphism

$$\varinjlim_{i \in I} F(Y(i)) \cong F(\varinjlim_{i \in I} Y(i)).$$

(Note you should only need to prove one of these - the other should be deduced using opposite categories)

(b) Show that if $\mathcal{C}$ has limits and colimits, then $\mathcal{C}^I$ has all limits and colimits, and these are formed "pointwise": i.e. if J is a small category, and $X \in (\mathcal{C}^I)^J$ is a J-shaped diagram of I-shaped diagrams, then

$$(\varprojlim_{j \in J} X(j))(i) \cong \varprojlim_{j \in J}(X(j)(i))$$
$$(\varinjlim_{j \in J} X(j))(i) \cong \varinjlim_{j \in J}(X(j)(i))$$

In the above equations, the limit/colimit on the LHS is taken in the category $\mathcal{C}^I$, whereas the limit/colimit on the RHS is taken in the category $\mathcal{C}$.

(c) Deduce from (a) and (b) (and problem 3) that if I and J are small categories, $\mathcal{C}$ has all limits and colimits, and

$$Z : I \times J \to \mathcal{C}$$

is a functor, that we have

$$\varinjlim_{i\in I}\varinjlim_{j\in J} Z(i,j) \cong \varinjlim_{(i,j)\in I\times J} Z(i,j) \cong \varinjlim_{j\in J}\varinjlim_{i\in I} Z(i,j)$$
$$\varprojlim_{i\in I}\varprojlim_{j\in J} Z(i,j) \cong \varprojlim_{(i,j)\in I\times J} Z(i,j) \cong \varprojlim_{j\in J}\varprojlim_{i\in I} Z(i,j)$$