# HOMEWORK 4

DUE: TUESDAY, MARCH 4

*Turn in problems 3 and 4*

1. Show that if $f : X \to Y$ is a fibration, and $Y$ is based, then the canonical map

$$f^{-1}(*) \to F(f)$$

is a homotopy equivalence.

2. A *Serre fibration* is a map $f : X \to Y$ satisfying a restricted form of the homotopy lifting property For all $n \geq 0$ and all $g, h$ making the outer square commute

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{g} & X \\ \downarrow & \nearrow & \downarrow f \\ I^{n+1} & \xrightarrow[h]{} & Y \end{array}$$

there exists a dotted arrow as above making the diagram commute. The notion of Serre fibration is often times more convenient than the notion of fibration.

Suppose that $Y$ is pointed. Show that the canonical map $f^{-1}(*) \to F(f)$ is a weak equivalence. Deduce that Serre fibrations have long exact sequences of homotopy groups.

3. (Path-loop fibration) Let $X$ be a pointed space.

(a) Show that the evaluation map

$$ev_1 : \underline{\mathrm{Map}}_*(I, X) \to X$$

is a Serre fibration, with fiber $\Omega X$. (Note: it is actually a fibration.) This fiber sequence is called the path-loop fibration.

(b) Show that if $p : E \to X$ is a Serre fibration with contractible total space $E$, and fiber $F$, then there is a weak equivalence $F \to \Omega X$. (Hint: one approach is to compare with the LES of the path-loop fibration.)

A *locally trivial bundle* is a map $p : E \to B$ so that for every $b \in B$, there is a neighborhood $U$ of $b$, a space $F$, and a homeomorphism making the following diagram commute

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\approx} & U \times F \\ & {}_{p|_U}\searrow \quad \swarrow{}_{\pi_1} & \\ & U & \end{array}$$

where $\pi_1$ is the projection on the first component.

4. Show that all locally trivial bundles are Serre fibrations.

5. Let $H$ be a closed sub-Lie group of a compact Lie group $G$. Show that $G \to G/H$ is a locally trivial bundle with fiber $H$. (Note: I think that the assumption that $G$ is compact is not necessary, but might make the problem easier.)